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# Arf numerical semigroups with low multiplicity via Gröbner basis

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## Abstract

In this paper, the Gröbner basis over RF-matrices of Arf numerical semigroups are presented. The Arf properties ideals for the RF-matrices obtained by RF-Relations are provided and the aforementioned concepts are associated through the Gröbner basis of the Arf numerical semigroup. Moreover, we prove that if we have a minimal presentation (or a Gröbner basis of the ideal associated to the semigroup), then this will be a system of generators of the subgroup of $\mathbb{Z}^p$ with the equation $n_1x_1 + n_2x_2 + \cdots + n_px_p = 0$, where $\{n_1, n_2, \ldots, n_p\}$ are the generators of numerical semigroup $G$.

**Keywords:** Arf numerical semigroups, multiplicity, presentation, RF (ow factoritazion)-matrices, Gröbner basis.

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## 1. Introduction

Frobenius and Sylvester first discussed the historical development of numerical semigroups in the 19th century. They termed this problem the ''Frobenius coin problem" and ''Frobenius linear diophant equation." Initially, they questioned what was the largest amount of money that cannot be obtained using coins that possess no common divisor?". In brief, "$a$ and $b$ are natural numbers, $p$ and $q$ are greater than 1 and prime between them; what is the largest integer that cannot be expressed as a linear combination $ap + bq$?" This number is represented by $F(G)$[20].

The Frobrenius problem was identified by Brauer [6], with many researchers and scientists examining the topic between 1958 and 1978 [10]. Numerical semigroups started to receive increasing research attention during the late 20th century, primarily because of their use in algebraic geometry. Research into numerical semigroups serves to identify non-negative solutions of non-homogeneous linear equations with positive integer coefficients. Therefore, in the relevant literature, numerical semigroups have been a persistent and classic problem.

When Du Val [15] presented his research on the Jacobian algorithm and the multiplicity sequence of an algebraic branchat at Istanbul University, he questioned whether his geometric approach contained an algebraic expression. The characters that Patrick Du Val mentioned in his speech can be written

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algebraically, according to Cahit Arf, who was in attendance at the time. Moreover, he demonstrated how to algebraically type and calculate the characters seven days after the lecture. These characters were eventually referred to be Arf Characters of a Curve after the results were published in [6]. Arf's plan was to determine the Arf ring closure of the curve ring's coordinate and its ARF semigroup. Lipman made reference to this in his work at [12]. The Arf characters form the minimal generators in this semigroup.

A maximal embedding dimension exists for Arf numerical semigroups [19]. The smallest element in the generator and the total number of elements in the generator must both be identical for there to be a maximal embedding size. We may determine the multiplicity of the numerical semigroup from the smallest element in the generator. Moreover, famous mathematician Karakaº identified various Arf numerical semigroups in his work [14].

In this work, the low multiplicity of Arf numerical semigroups and RF-matrices have been presented. Subsequently, the Gröbner basis and minimal presentaitons were discussed.

## 2. Preliminaries

Most of the definitions are found in [1], [8], [10], [11], and [18].

**Definition 2.1.** Let $\mathbb{N}$ be the set of natural numbers and $G \subseteq \mathbb{N}$. If G is closed under the addition in $\mathbb{N}$ and $0 \in G$ and $\mathbb{N}\backslash G$ is finite, then G is called a numerical semigroup. For all $n_1, n_2, \ldots, n_p \in G$,

$$G = \langle n_1, n_2, \ldots, n_p \rangle = \left\{ \sum_{i=1}^{p} a_i n_i : a_i \in \mathbb{N} \right\},$$

and $(n_{1,} n_2, \ldots, n_p) = 1 \Leftrightarrow \mathbb{N}\backslash G$ isfinite.

**Definition 2.2.** The $\mathbb{N} \backslash G$ is the *gaps* set of the numerical semigroup and denoted by $H(G)$. The number of elements of the set $H(G)$ tells us the *genus* of the numerical semigroup G and is represented by $h(G)$.

**Definition 2.3.** Let be $B \subset G$. For every $g \in G$, if $g$ is expressed as a linear combination of the elements of the set $B = \{b_1, b_2, \ldots, b_n\}$, then B is called the set of *generators* of G and is denoted by $G = \langle B \rangle$. If G cannot be generated by any proper subset of B, then B is called a *minimal generator system*.

**Definition 2.4.** The number of elements of the minimal generator system mentioned in Definition 2.3 is referred to as *embedding dimension* of the G numerical semigroup and is denoted by $e(G)$.

**Definition 2.5.** The smallest element of the generator set of the numerical semigroup G is called the *multiplicity* of G and is expressed by $m(G)$.

**Definition 2.6.** If $m(G) = e(G)$, then G is called the semigroup with *maximal embedding dimension*.

**Definition 2.7.** The largest element of the gaps set of the numerical semigroup is called the *frobenius number*.

**Definition 2.8.** The smallest integer $x$ given as $x + n \in G$ and $n \in \mathbb{N}$ for the numerical semigroup is referred to as *conductor* of G and is expressed by $I(G)$,

$$I(G) = F(G) + 1.$$

**Definition 2.9.** Let $x$ be an integer such that $x \in G$. If $g \in G \diagdown \{0\}$ and $x + g \in G$, then $x$ is called a *pseudo-Frobenius number* of G. The set of all pseudo-Frobenius numbers of the numerical semigroup is denoted by $PF(G)$.

**Example 2.10.** Let $G = \{0, 4, 7, 8, 9, 11, \rightarrow\}$.

1. $0 \in G$.

2. $G$ is closed under the addition operation, that is $t, k \in G \Rightarrow t + k \in G$.
3. $\mathbb{N} \backslash G = \{1, 2, 3, 5, \ 6, 10\}$ is finite so $G$ is a numerical semigroup.
4. Gaps set and genus: $H(G) = \{1, 2, 3, 5, 6, 10\}$ and $h(G) = 6$.
5. Minimal generator set and embedding dimension: $G = \langle 4, 7, 9 \rangle$ and $e(G) = 3$.
6. Multiplicity: $m(G) = 4$.
7. Since $m(G) \neq e(G)$, it is not a maximal embedding dimension.
8. Frobenius number: $F(G) = 10$.
9. Conductor: $I(G) = 11$.
10. Pseudo-Frobenius number and type: $PF(G) = \{5, 10\}$ and $t(G) = 2$.

**Definition 2.11.** [8] Let $G$ be a numerical semigroup. If the numerical semigroup $G$ satisfies the property below,

$$\forall m, n, t \in G : m \geqslant n \geqslant t \Longrightarrow m + n - t \in G,$$

then $G$ is called *Arf numerical semigroup*.

The Arf property on $G$ is equivalent to the following lemma.

**Lemma 2.12.** *[14] Let $G$ be a numerical semigroup, then*

$$\forall m, n \in G : m \geqslant n \Longrightarrow 2m - n \in G.$$

*Proof.* See [4]. □

Combining Lemma 2.12 and Definition 2.11 results in that $G$ is an Arf numerical semigroup if and only if $2x - y \in G$ and $\forall x, y \in G$ with $I > x > y$.

**Example 2.13.** Let $G = \langle 3, 10, 11 \rangle = \{0, 3, 6, 9, \rightarrow\}$.

$$\begin{aligned}
&m = 3, n = 3 \Longrightarrow 2.3 - 3 = 3 \in G, && m = 6, n = 3 \Longrightarrow 2.6 - 3 = 9 \in G,\\
&m = 6, n = 6 \Longrightarrow 2.6 - 6 = 6 \in G, && m = 9, n = 3 \Longrightarrow 2.9 - 3 = 15 \in G,\\
&m = 9, n = 6 \Longrightarrow 2.9 - 6 = 12 \in G, && m = 9, n = 9 \Longrightarrow 2.9 - 9 = 9 \in G.
\end{aligned}$$

If $m \geqslant 9$, $2m - n \in G$ will always be $m - n \geqslant 8$. In this case, $G$ is an Arf numerical semigroup.

**Proposition 2.14.** *[9] The embedding dimension of all Arf numerical semigroups is equal to the multiplicity.*

The above assertion means Arf numerical semigroups are maximal embedding dimensional. Note that all maximal embedding dimensional numerical semigroups are Arf numerical semigroups.

**Example 2.15.** Let $G = \langle 3, 4, 8 \rangle = \{0, 3, \ 4, 7, 8, 9, 10 \rightarrow\}$, since $m(G) = e(G) = 3$, it has a maximal embedding dimension but not Arf numerical semigroup.

$$m = 3, n = 3 \Longrightarrow 2.3 - 3 = 3 \in G, \qquad m = 4, n = 3 \Longrightarrow 2.4 - 3 = 5 \notin G.$$

Notice that $G$ is not an Arf numerical semigroup because 5 is not an element of $G$.

## 3. RF-matrices of Arf numerical semigroups with low multiplicity

RF-matrices were first introduced by Moscariello in 2016. Ayşe Çalışkan in Master's thesis determined RF-matrices in 2020. The RF-matrices of Arf numerical semigroups with low multiplicity are found with GAP program [7].

**Definition 3.1.** Let $\mathfrak{f} \in \mathsf{PF}(G)$. An $e \times e$ matrix $A = (a_{ij})$ is an RF-matrices of $\mathfrak{f}$, if $a_{ii} = -1$, $a_{ij} \in \mathbb{N}$ if $i \neq j$, and for every $i = 1, \dots, e$,

$$\sum_{j=1}^{e} a_{ij} n_j = \mathfrak{f}.$$

**Proposition 3.2.** *The size of the RF-matrices is determined by the number of elements in the minimal generator system of the numerical semigroup.*

**Example 3.3.** Let $G = \langle 4, 21, 22, 23 \rangle = \{0, 4, 8, 12, 16, 20, \to\}$. We determine the RF-matrices of the Arf numerical semigroup G. In order to find the RF-matrices, we calculate the elements of the pseudo-Frobenius set. In this case,

$$\mathsf{PF}(G) = \{k \in G | k + g \in G\},$$

gaps of G; $H(G) = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$ and $F(G) = 19 \Longrightarrow \mathsf{PF}(G) = \{17, 18, 19\}$.

For $f = 19 \in \mathsf{PF}(G)$, $19 = a_{11}.4 + a_{12}.21 + a_{13}.22 + a_{14}.23$ and for $a_{11} = -1$,

$$19 = -1.4 + a_{12}.21 + a_{13}.22 + a_{14}.23,$$

the first row of the RF-matrices: $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$.

If $f = 19 \in \mathsf{PF}(G)$, $19 = a_{21}.4 + a_{22}.21 + a_{23}.22 + a_{24}.23$ and for $a_{22} = -1$,

$$19 = -1.4 + a_{22}.21 + a_{23}.22 + a_{24}.23,$$

the second row of the RF-matrices: $\begin{bmatrix} 10 & -1 & 0 & 0 \end{bmatrix}$.

If $f = 19 \in \mathsf{PF}(G)$, we obtain $19 = a_{31}.4 + a_{32}.21 + a_{33}.22 + a_{34}.23$ and for $a_{33} = -1$,

$$19 = a_{31}.4 + a_{32}.21 + -1.22 + a_{34}.23,$$

the third row of the RF-matrices: $\begin{bmatrix} 5 & 1 & -1 & 0 \end{bmatrix}$.

If $f = 19 \in \mathsf{PF}(G)$, we acquire $19 = a_{41}.4 + a_{42}.21 + a_{43}.22 + a_{44}.23$ and for $a_{44} = -1$,

$$19 = a_{41}.4 + a_{42}.21 + a_{43}.22 + -1.23,$$

the fourth row of the RF-matrices: $\begin{bmatrix} 0 & 2 & 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 & 1 & -1 \end{bmatrix}$.

The minimal generator set of the numerical semigroup G has 4 elements, so the RF-Matrix is a matrix of type $4 \times 4$:

$$\mathsf{RF}(19) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 10 & -1 & 0 & 0 \\ 5 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}, \qquad \mathsf{RF}(19) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 10 & -1 & 0 & 0 \\ 5 & 1 & -1 & 0 \\ 5 & 0 & 1 & -2 \end{bmatrix}.$$

*Remark* 3.4. RF-matrices can be written for each element in the pseudo-Frobenius set. Notice that such matrices are not usually unique.

**Lemma 3.5.** *Let* $f, f' \in \mathsf{PF}(G)$ *and* $f + f' \notin G$. *Let* $\mathsf{RF}(f) = A = (a_{pq})$ *and* $\mathsf{RF}(f') = B = (b_{pq})$. *In this case,* $a_{pq} = 0$ *or* $b_{pq} = 0$ *for each* $p \neq q$. *In particular, if* $\mathsf{RF}(\frac{F(G)}{2}) = (a_{pq})$, *then* $a_{pq} = 0$ *or* $b_{pq} = 0$ *for every* $p \neq q$.

*Proof.* See [2]. □

## 4. RF-relations

**Definition 4.1.** The ideal $I_G$ is called the defining ideal of the ring $F[G]$. $I_G$ is a homogeneous ideal and is generated by binomials.

**Definition 4.2.** For vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, if then $a_p \geqslant 0$, $a_p = (0, \dots, 0, a_p, 0, \dots, 0)$ and $a^- = a^+ - a$ so $a = a^+ - a^-$.

**Lemma 4.3.** *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the row vectors of RF(f)-matrices and take $a_{pq} = a_p - a_q$ for $\forall p, q1 \leqslant p < q \leqslant n$. Then, $\phi = X^{a^+_{pq}} - X^{a^-_{pq}} \in I_G$ for $p < q$.*

*Proof.* See [2]. □

*Remark* 4.4. $PF'(G) = PF(G) - F(G)$.

**Definition 4.5.** Let $G = \langle g_1, \dots, g_2 \rangle$ be a numerical semigroup and $f \in PF'(G)$ be the binomial relation. Then $\phi = X^{a^+_{pq}} - X^{a^-_{pq}} \in I_G$ is called RF(f)-relation, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are row vectors of RF(f) and $1 \leqslant p < q \leqslant$.

**Lemma 4.6.** *Assume that $f \in PF'(G)$ and $p < q$ such that $u, v$ monomials are $x_p/u$ and $x_q/v$, $\forall k$, $\phi_1, \phi_2, \dots, \phi_G \in I_G$ and $\phi_k = u - v$. In this case, $I_G$ is generated by RF-relations.*

*Proof.* See [2]. □

**Example 4.7.** Let $G = \langle 3, 4, 5 \rangle = \{0, 3, \rightarrow\}$. We consider the ideals of the numerical semigroup G using the RF-relations method. For RF-matrices $PF(G) = \{1, 2\}$,

$$PF'(G) = PF(G) - F(G) \Longrightarrow PF'(G) = \{1\}.$$

For $f = 1 \in PF'(G)$, we get

$$RF(1) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Row vectors of the RF(1)-matrices are

$$a_1 = (-1, 1, 0), \quad a_2 = (0, -1, 1), \quad a_3 = (2, 0, -1).$$

Take the difference of the first and second rows in the RF(1)-matrices

$$a_{12} = a_1 - a_2 = (-1, 2, -1),$$

$a^+_{12} = (0, 0, 2)$ and $a^-_{12} = (1, 0, 1) \Longrightarrow x^{a^+_{12}} = x_2^2$ and $x^{a^-_{12}} = x_1x_3$. In this case, $\phi_{12} = x_2^2 - x_1x_3 \in I_G$ binomial generator is obtained.

Similarly, we consider the binomial generators obtained by $a_{13}$ and $a_{23}$:

$$a_{13} = a_1 - a_3 = (-3, 1, 1),$$

$a^+_{13} = (0, 1, 1)$ and $a^-_{13} = (3, 0, 0) \Longrightarrow x^{a^+_{13}} = x_2x_3$ and $x^{a^-_{13}} = x_1^3$, so

$$\phi_{13} = x_2x_3 - x_1^3 \in I_G, \qquad a_{23} = a_2 - a_3 = (-2, -1, 2),$$

$a^+_{23} = (0, 0, 2)$ and $a^-_{23} = (2, 1, 0) \Longrightarrow x^{a^+_{23}} = x_3^2$ and $x^{a^-_{23}} = x_1^2x_2$, hence $\phi_{13} = x_3^2 - x_1^2x_2 \in I_G$ binomial generators are obtained. Thus, 3 minimal generator of the $I_G$ is $I_G = \langle x_2^2 - x_1x_3, x_2x_3 - x_1^3, x_3^2 - x_1^2x_2 \rangle$.

## 5. Arf numerical semigroup with low multiplicity

**Definition 5.1.** [14] Let $G$ be an Arf numerical semigroup with multiplicity $m$. Then $G$ is the minimal generator set $(Ap(G,m) \setminus \{0\} \cup \{m\})$.

Let $d \in G$ be non-zero element. We define the Apery set of $G$ with respect to $d$, denoted by $Ap(G,d)$, as follows

$$Ap(G,d) = \{g \in G \mid g - d \notin G\}.$$

It is well known that

$$Ap(G,d) = \{\omega(0) = 0, \omega(1), \dots, \omega(d-1)\},$$

where $\omega(i) = \min\{g \in G \mid \mathrm{mod}\, d = i\}$ *for each* $i = \{1, \dots, d-1\}$.

Let $a \in \mathbb{Z}$. We say that $a$ is a pseudo-Frobenius number of $G$ if $a \notin G$ and $a + g \in G$ for all non-zero element $a \in G$. The set of pseudo-Frobenius numbers of $G$ is denoted by $PF(G)$, and the cardinality of $PF(G)$ is called the type of $G$, denoted by $t(G)$. For $a, b \in \mathbb{Z}$, we say that $a \leqslant_G w$ if $b - a \in G$, which defines a partially ordered relation.

**Definition 5.2.** [9] Let $G$ be a numerical semigroup and $d \neq 0$ be an element of $G$. Then

$$PF(G) = \{b - d \mid b \in Maximals_{\leqslant_G}(Ap(G,d))\}.$$

*Remark* 5.3. Arf numerical semigroup $G$ with multiplicity $m(G) = m$, the set $(Ap(G,m) \cup \{m\}) \setminus \{0\} = \{m, \omega(0) = 0, \omega(1), \dots, \omega(m-1)\}$ forms the minimal generating system for $G$. Moreover, if $G$ is minimally generated by $\{n_1 < n_2 < \cdots < n_e\}$, then

$$PF(G) = \{n_2 - n_1, n_3 - n_1, \dots, n_e - n_1\}.$$

**Arf numerical semigroup with multiplicity one:** The Arf numerical semigroup with a multiplicity of 1 is just the set of natural numbers.

### 5.1. *Arf numerical semigroups of multiplicity two*

**Proposition 5.4.** *[14] Any numerical semigroup with a multiplicity of* $2$ *is also an Arf numerical semigroup. The conductor* $I(G) = I$ *and the Arf numerical semigroup with multiplicity two is expressed by* $G = \langle 2, I+1 \rangle$.

**Proposition 5.5.** *[13]* $G = \langle 2, I+1 \rangle$ *generated by with multiplicity* $2$ *Arf numerical semigroup. The RF-matrices that can be written with the pseudo-Frobenius set* $PF(G) = \{I-1\}$ *with* $I$ *as conductor,*

$$RF(I-1) = \begin{bmatrix} -1 & 1 \\ I & -1 \end{bmatrix}.$$

*Proof.* We know that since the $G$ Arf numerical semigroup is a doubled numerical semigroup, the conductor $I$ will be an even positive integer. The set $Ap(G,2) = \{0, I+1\}$, is the Apery set of the numerical semigroup $G$. By Remark 5.3, the pseudo-Frobenius set $PF(G) = \{I-1\}$ is obtained. Then we have the following factorizations for $I-1$,

$$I - 1 = -1.2 + a_{12}.(I+1), \quad I - 1 = a_{21}.2 - 1.(I+1).$$

Using the definition of row-factorization matrices, rows $a_{12} = 1$ and $a_{21} = I$ are obtained. Therefore $RF(I-1)$ is found as desired. □

**Example 5.6.** We consider $\mathrm{I} = 6$. In this case the Arf numerical semigroup G is obtained as follows.

$$\mathrm{G} = \langle 2, \mathrm{I}+1 \rangle = \langle 2, 7 \rangle = \{0, 2, 4, 6, \rightarrow\}$$

and the pseudo-Frobenius set,

$$\mathsf{PF}(\mathrm{G}) = \{\mathrm{I}-1\} = \{5\},$$

$f = 5 \in \mathsf{PF}(\mathrm{G})$ we obtain RF-matrices as,

$$\mathsf{RF}(\mathrm{I}-1) = \mathsf{RF}(5) = \begin{bmatrix} -1 & 1 \\ \mathrm{I} & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

*Remark* 5.7. Arf numerical semigroups with multiplicity equal to or greater than 3 cannot be written explicitly by the conductor alone. For this, the genus is needed to determine Arf numerical semigroups. Assuming the Arf property, the Arf numerical semigroup is completely set by the multiplicity and the conductor.

### 5.2. *Arf numerical semigroups of multiplicity three*

**Proposition 5.8.** *[14] Let* $\mathrm{I}$ *be an integer such that* $\mathrm{I} \geqslant 3$ *and* $\mathrm{I} \not\equiv 1 \pmod 3$. *Then the Arf numerical semigroup* G *with a multiplicity 3 and a conductor* I *can be written as one of the following* ($\mathrm{I} \not\equiv 0$ *or* $2 \pmod 3$)*:*

1. $\mathrm{G} = \langle 3, \mathrm{I}+1, \mathrm{I}+2 \rangle$ *if* $\mathrm{I} \equiv 0\,(\mathrm{mod} 3)$*;*
2. $\mathrm{G} = \langle 3, \mathrm{I}, \mathrm{I}+2 \rangle$ *if* $\mathrm{I} \equiv 2\,(\mathrm{mod} 3)$.

**Proposition 5.9.** *[13] Let* $\mathrm{G} = \langle 3, \mathrm{I}+1, \mathrm{I}+2 \rangle$ *be an Arf numerical semigroup with multiplicity* 3 *and conductor* $\mathrm{I} \geqslant 3$. *The RF-matrices that can be written with the pseudo-Frobenius set* $\mathsf{PF}(\mathrm{G}) = \{\mathrm{I}-2, \mathrm{I}-1\}$ *are*

$$\mathsf{RF}(\mathrm{I}-2) = \begin{bmatrix} -1 & 1 & 0 \\ \frac{\mathrm{I}-3}{3} & -1 & 1 \\ \frac{2\mathrm{I}}{3} & 0 & -1 \end{bmatrix} \quad \textit{and} \quad \mathsf{RF}(\mathrm{I}-1) = \begin{bmatrix} -1 & 0 & 1 \\ \frac{2\mathrm{I}}{3} & -1 & 0 \\ \frac{\mathrm{I}}{3} & 1 & -1 \end{bmatrix}.$$

*Proof.* See [13]. □

**Example 5.10.** Let $\mathrm{I} = 9$.

$$\mathrm{G} = \langle 3, \mathrm{I}+1, \mathrm{I}+2 \rangle = \langle 3, 10, 11 \rangle, \quad \mathsf{PF}(\mathrm{G}) = \{\mathrm{I}-2, \mathrm{I}-1\} = \{7, 8\}.$$

We obtain the RF-matrices by taking the elements of the pseudo-Frobenius set as

$$\mathsf{RF}(\mathrm{I}-2) = \mathsf{RF}(7) = \begin{bmatrix} -1 & 1 & 0 \\ \frac{\mathrm{I}-3}{3} & -1 & 1 \\ \frac{2\mathrm{I}}{3} & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 1 \\ 6 & 0 & -1 \end{bmatrix},$$

$$\mathsf{RF}(\mathrm{I}-1) = \mathsf{RF}(8) = \begin{bmatrix} -1 & 1 & 0 \\ \frac{2\mathrm{I}}{3} & -1 & 1 \\ \frac{\mathrm{I}}{3} & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 6 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}.$$

**Proposition 5.11.** *Let* $\mathrm{G} = \langle 3, \mathrm{I}, \mathrm{I}+2 \rangle$ *be an Arf numerical semigroup with multiplicity* 3 *and conductor* $\mathrm{I} > 3$. *The RF-matrices that can be written with the pseudo-Frobenius set* $\mathsf{PF}(\mathrm{G}) = \{\mathrm{I}-3, \mathrm{I}-1\}$ *are as follows:*

$$\mathsf{RF}(\mathrm{I}-3) = \begin{bmatrix} -1 & 1 & 0 \\ \frac{\mathrm{I}-5}{3} & -1 & 1 \\ \frac{2\mathrm{I}-1}{3} & 0 & -1 \end{bmatrix} \quad \textit{and} \quad \mathsf{RF}(\mathrm{I}-1) = \begin{bmatrix} -1 & 0 & 1 \\ \frac{2\mathrm{I}-1}{3} & -1 & 0 \\ \frac{\mathrm{I}+1}{3} & 1 & -1 \end{bmatrix}.$$

*Proof.* See [13]. □

**Example 5.12.** Let $G = \langle 3, I, I+2\rangle$ be an Arf numerical semigroup. Then $I(G) = 14$, $PF(G) = \{11, 13\}$, and

$$RF\,(I-3) = RF\,(11) = \begin{bmatrix} -1 & 1 & 0 \\ \frac{I-5}{3} & -1 & 1 \\ \frac{2I-1}{3} & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -1 & 1 \\ 9 & 0 & -1 \end{bmatrix},$$

$$RF\,(I-1) = RF\,(13) = \begin{bmatrix} -1 & 0 & 1 \\ \frac{2I-1}{3} & -1 & 0 \\ \frac{I+1}{3} & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 9 & -1 & 1 \\ 5 & 0 & -1 \end{bmatrix}.$$

*5.3. Arf numerical semigroups of multiplicity four*

**Proposition 5.13.** *[14] The Arf numerical semigroup with conductor $I$ and multiplicity 4 can be written as one of the followin ($I \equiv 0, 2$ or $3\,(\mathrm{mod} 4)$):*

1. $G = \langle 4, 4a+2, I+1, I+3\rangle$, *if* $I \equiv 0\,(\mathrm{mod}4)$, *for* $a \in \left\{1, \ldots, \frac{I}{4}\right\}$;
2. $G = \langle 4, 4a+2, I+1, I+3\rangle$, *if* $I \equiv 2\,(\mathrm{mod}4)$, *for* $a \in \left\{1, \ldots, \frac{I-2}{4}\right\}$;
3. $G = \langle 4, I, I+2, I+3\rangle$, *if* $I \equiv 3\,(\mathrm{mod}4)$.

**Proposition 5.14.** *Let $G = \langle 4, 4a+2, I+1, I+3\rangle$ be Arf numerical semigroup for some $a \in \left\{1, \ldots, \frac{I}{4}\right\}$ with multiplicity 4 and conductor $I \geqslant 4$. The RF-matrices that can be written with the pseudo-Frobenius set, $PF(G) = \{4a-2, I-3, I-1\}$, are*

$$RF\,(4a-2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2a & -1 & 0 & 0 \\ a-1 & 0 & -1 & 1 \\ a & 0 & 1 & -1 \end{bmatrix}, \qquad RF\,(I-3) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ a-1 & -1 & 0 & 1 \\ \frac{I}{2}-a-1 & 1 & -1 & 0 \\ \frac{I}{2} & 0 & 0 & -1 \end{bmatrix},$$

$$RF\,(I-1) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ \frac{I}{2}-a & 1 & 0 & -1 \end{bmatrix}, \text{ or } \qquad RF\,(I-1) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

*Proof.* See [13]. □

**Example 5.15.** Let $I = 32$ be the conductor of the Arf numerical semigroup $G = \langle 4, 4a+2, I+1, I+3\rangle$ for some $a \in \left\{1, \ldots, \frac{I}{4}\right\}$. Then

$$PF(G) = \{4a-2, I-3, I-1\} = \{18, 29, 31\},$$

$$RF\,(4a-2) = RF\,(18) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2a & -1 & 0 & 0 \\ a-1 & 0 & -1 & 1 \\ a & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 10 & -1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ 5 & 0 & 1 & -1 \end{bmatrix},$$

$$RF\,(I-3) = RF\,(29) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ a-1 & -1 & 0 & 1 \\ \frac{I}{2}-a-1 & 1 & -1 & 0 \\ \frac{I}{2} & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 1 \\ 11 & 1 & -1 & 0 \\ 16 & 0 & 0 & -1 \end{bmatrix},$$

$$RF\,(I-1) = RF\,(31) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ \frac{I}{2}-a & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 4 & -1 & 1 & 0 \\ 16 & 0 & -1 & 0 \\ 12 & 1 & 0 & -1 \end{bmatrix}, \text{ or}$$

$$RF\,(I-1) = RF\,(31) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 4 & -1 & 1 & 0 \\ 16 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

**Proposition 5.16.** *Let* $G = \langle 4, 4a+2, I+1, I+3\rangle$ *be Arf numerical semigroup for some* $a \in \left\{1, \ldots, \frac{I-2}{4}\right\}$ *with multiplicity* 4 *and conductor* $I > 4$*. The RF-matrices the can be written with the pseudo-Frobenius set* $\mathrm{PF}(G) = \{4a-2, I-3, I-1\}$ *are*

$$\mathrm{RF}(4a-2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2a & -1 & 0 & 0 \\ a-1 & 0 & -1 & 1 \\ a & 0 & 1 & -1 \end{bmatrix}, \qquad \mathrm{RF}(I-3) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ a-1 & -1 & 0 & 1 \\ \frac{I}{2}-a-1 & 1 & -1 & 0 \\ \frac{I}{2} & 0 & 0 & -1 \end{bmatrix},$$

$$\mathrm{RF}(I-1) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ \frac{I}{2}-a & 1 & 0 & -1 \end{bmatrix}, \text{ or } \qquad \mathrm{RF}(I-1) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

*Proof.* See [13]. □

**Example 5.17.** Let the Arf numerical semigroup G with multiplicity 4 and Frobenius number 21 (conductor 22) as

$$G = \langle 4, 4a+2, I+1, I+3\rangle \, \langle 4, 6, 23, 25\rangle \, .$$

Then

$$\mathrm{PF}(G) = \{4a-2, I-3, I-1\} = \{2, 19, 21\},$$

$$\mathrm{RF}(4a-2) = \mathrm{RF}(2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2a & -1 & 0 & 0 \\ a-1 & 0 & -1 & 1 \\ a & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathrm{RF}(I-3) = \mathrm{RF}(19) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ a-1 & -1 & 0 & 1 \\ \frac{I}{2}-a-1 & 1 & -1 & 0 \\ \frac{I}{2} & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 9 & 1 & -1 & 0 \\ 11 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathrm{RF}(I-1) = \mathrm{RF}(21) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ \frac{I}{2}-a & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 11 & 0 & -1 & 0 \\ 10 & 1 & 0 & -1 \end{bmatrix}, \text{ or}$$

$$\mathrm{RF}(I-1) = \mathrm{RF}(21) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ a & -1 & 1 & 0 \\ \frac{I}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 11 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

**Proposition 5.18.** *Let* $G = \langle 4, I, I+2, I+3\rangle$ *be Arf numerical semigroup with multiplicity* 4 *and conductor* $I > 4$*. The RF-matrices that can be written with the pseudo-Frobenius set,* $\mathrm{PF}(G) = \{I-4, I-2, I-1\}$*, and*

$$\mathrm{RF}(I-4) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ \frac{I-7}{4} & -1 & 0 & 1 \\ \frac{I-1}{2} & 0 & -1 & 0 \\ \frac{I-3}{4} & 0 & 1 & -1 \end{bmatrix},$$

$$\mathrm{RF}(I-2) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ \frac{I-1}{2} & -1 & 0 & 0 \\ \frac{I-3}{4} & 0 & -1 & 1 \\ \frac{I+1}{4} & 1 & 0 & -1 \end{bmatrix}, \text{ or } \begin{bmatrix} -1 & 0 & 1 & 0 \\ \frac{I-1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ \frac{I+1}{4} & 1 & 0 & -1 \end{bmatrix},$$

$$\mathrm{RF}\left(\mathrm{I}-1\right)=\begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{I-3}{4} & -1 & 1 & 0 \\ \frac{I+1}{4} & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix},\ or\ \begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{I-3}{4} & -1 & 1 & 0 \\ \frac{I+1}{4} & 1 & -1 & 0 \\ \frac{I+1}{2} & 0 & 0 & -1 \end{bmatrix}.$$

*Proof.* See [13]. □

**Example 5.19.** Let $G$ be the numerical semigroups $\langle 4,19,21,22\rangle$. Then $G$ is an Arf numerical semigroup as in Proposition 5.18 and $I(G)=19$. Thus $PF(G)=\{15,17,18\}$ and

$$\mathrm{RF}\left(\mathrm{I}-4\right)=\begin{bmatrix} -1 & 1 & 0 & 0 \\ \frac{I-7}{4} & -1 & 0 & 1 \\ \frac{I-1}{2} & 0 & -1 & 0 \\ \frac{I-3}{4} & 0 & 1 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 \\ 9 & 0 & -1 & 0 \\ 4 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathrm{RF}\left(\mathrm{I}-2\right)=\begin{bmatrix} -1 & 0 & 1 & 0 \\ \frac{I-1}{2} & -1 & 0 & 0 \\ \frac{I-3}{4} & 0 & -1 & 1 \\ \frac{I+1}{4} & 1 & 0 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 0 & 1 & 0 \\ 9 & -1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ 5 & 1 & 0 & -1 \end{bmatrix},\ \text{or}$$

$$\mathrm{RF}\left(\mathrm{I}-2\right)=\begin{bmatrix} -1 & 0 & 1 & 0 \\ \frac{I-1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ \frac{I+1}{4} & 1 & 0 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 0 & 1 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 5 & 1 & 0 & -1 \end{bmatrix},$$

$$\mathrm{RF}\left(\mathrm{I}-1\right)=\begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{I-3}{4} & -1 & 1 & 0 \\ \frac{I+1}{4} & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 0 & 0 & 1 \\ 4 & -1 & 1 & 0 \\ 5 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix},\ \text{or}$$

$$\mathrm{RF}\left(\mathrm{I}-1\right)=\begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{I-3}{4} & -1 & 1 & 0 \\ \frac{I+1}{4} & 1 & -1 & 0 \\ \frac{I+1}{2} & 0 & 0 & -1 \end{bmatrix}=\begin{bmatrix} -1 & 0 & 0 & 1 \\ 4 & -1 & 1 & 0 \\ 5 & 1 & -1 & 0 \\ 10 & 0 & 0 & -1 \end{bmatrix}.$$

## 6. Generators of Arf numerical semigroups with low multiplicity via Gröbner basis

Buchberger discovered the Gröbner basis in 1965 and then this concept has been of central importance in related algorithmic commutative algebra and algebraic geometry. The aim of the Gröbner basis is to find a generator set such that any polynomial in the ideal set gives a remainder of zero when divided by the generator set.

Let $F$ be a field and $F[X_1,\ldots,X_n]$ be the $F$-algebra of polynomials with a monomial order $\succ$. For $0\neq \mathfrak{p}\in F[X_1,\ldots,X_n]$, let $Lm_>(\mathfrak{p}), Lt_>(\mathfrak{p}), Lc_>(\mathfrak{p})$ denote the leading monomial. Leading term and leading constant of $\mathfrak{p}$, respectively. We simply write $Lm_>(\mathfrak{p}), Lt_>(\mathfrak{p}), Lc_>(\mathfrak{p})$, when no confusion can occur. For $0\neq \mathfrak{p}$ , $\mathfrak{q}\in F[X_1,\ldots,X_n]$, the S-polynomial of $\mathfrak{p}$ and $\mathfrak{q}$ denoted by $S(\mathfrak{p},\mathfrak{q})$, is the polynomial

$$S(\mathfrak{p},\mathfrak{q}):=\frac{lcm(Lm(\mathfrak{p}),Lm(\mathfrak{q}))}{Lt(\mathfrak{p})}.\mathfrak{p}-\frac{lcm(Lm(\mathfrak{p}),Lm(\mathfrak{q}))}{Lt(\mathfrak{q})}.\mathfrak{q}.$$

Given $H=\{h_1,\ldots,h_t\}\subseteq F[X_1,\ldots,X_n]$ and $\mathfrak{p}\in F[X_1,\ldots,X_n]$, we are saying $\mathfrak{p}$ reduces to zero modulo $H$, denoted by $\mathfrak{p}\longrightarrow_H 0$, if $\mathfrak{p}$ can be written as $\mathfrak{p}=\sum_{i=1}^{t}a_ih_i$, such that $Lm(\mathfrak{p})\geqslant Lm(a_ih_i)$, whenever $a_ih_i\neq 0$, Burchberger's criterion says that, for an ideal $I$ in $F[X_1,\ldots,X_n]$ and a generating set $H=\{h_1,\ldots,h_t\}$ for $I$, $G$ is a Gröbner basis for $I$ iff $S(h_i,h_j)\longrightarrow_H 0$, for every $i\neq j$.

Like groups, we find gröbner basis with Buchberger's theorem [5].

The following lemma is useful for finding $S(\mathfrak{p},\mathfrak{q})$.

**Lemma 6.1.** *[20] Let* $H = \{h_1, \ldots, h_t\} \subseteq F[X_1, \ldots, X_n]$ *and let* $\mathfrak{p}, \mathfrak{q} \in H$ *be non-zero with* $\mathrm{Lc}(\mathfrak{p}) = \mathrm{Lc}(\mathfrak{q}) = 1$ *and* $\gcd(\mathrm{Lm}(\mathfrak{p}), \mathrm{Lm}(\mathfrak{q})) = 1$. *Then,*

1. $S(\mathfrak{p}, \mathfrak{q}) = \mathrm{Lm}(\mathfrak{q}).\mathfrak{p} - \mathrm{Lm}(\mathfrak{p}).\mathfrak{q}$;
2. $S(\mathfrak{p}, \mathfrak{q}) = -(\mathfrak{q} - \mathrm{Lm}(\mathfrak{q})).\mathfrak{p} + (\mathfrak{p} - \mathrm{Lm}(\mathfrak{p})).\mathfrak{q} \longrightarrow_H 0$.

**Definition 6.2.** Let $\prec$ be the lexicographic ordering $F[x_1, x_2, x_3]$ with $x_1 > x_2 > x_3$. Then ${x_1}^{\alpha_0}{x_2}^{\beta_0}{x_3}^{\gamma_0} \prec {x_1}^{\alpha_1}{x_2}^{\beta_1}{x_3}^{\gamma_1}$ if the left most nonzero component of $(\alpha_1, \beta_1, \gamma_1) - (\alpha_0, \beta_0, \gamma_0)$ is positive.

**Definition 6.3.** Let G be a numerical semigroup minimally generated by $\{n_1, n_2, \ldots, n_p\}$. Then the monoid morphism

$$\varphi : \mathbb{N}^p \longrightarrow G, \quad \varphi(a_1, a_2, \ldots, n_p) = \sum_{i=1}^{p} a_i n_i,$$

known as the factorization homomorphism of G, is an epimorphism and consequently G is isomorphic to $\mathbb{N}^p/\mathrm{Ker}\varphi$, where $\mathrm{Ker}\varphi$ is the kernel congruence of

$$\varphi : \mathrm{Ker}\varphi = \{(a, b) \in \mathbb{N}^p \times \mathbb{N}^p | \varphi(a) = \varphi(b)\}.$$

Given $\tau \subset \mathbb{N}^p \times \mathbb{N}^p$, the congruence generated by $\tau$, that is the intersection of all congruences containing $\tau$. The congruence generated by a set is precisely the reflexive, symmetric, transitive closure (equivalence relation), to which we adjoin all pairs $(a + b, b + c)$ whenever $(a + b)$ is in the closure; so that the resulting relation becomes a congruence.

A presentation for G is a generating system of $\mathrm{Ker}\varphi$ as a congruence, and a minimal presentation is a presentation such that none of its proper subsets is a presentation.

**Example 6.4.** A minimal presentation for $G = \langle 2, 3\rangle$ is $\{(3, 0), (0, 2)\}$. This means that G is the numerical semigroup generated by 2 and 3.

*Remark* 6.5. If we have a minimal presentation (or a Gröbner basis of the ideal associated to the semigroup) and take the difference between the relators (or the difference between the exponents in the Gröbner basis of any generator of the ideal of the semigroup), then this will be a system of generators of the subgroup of $\mathbb{Z}^k$ with equation $n_1x_1 + n_2x_2 + \cdots + n_px_p = 0$, where $\{n_1, n_2, \ldots, n_k\}$ are the generators of numerical semigroup G. Thus, we can proceed as follows.

**Example 6.6.** gap> s:=NumericalSemigroup(4,6,23,25);;gap> mp:=MinimalPresentation(s);[ [ [ 0, 0, 0, 2 ], [ 1, 0, 2, 0 ] ], [ [ 0, 0, 1, 1 ], [ 0, 8, 0, 0 ] ], [ [ 0, 0, 2, 0 ], [ 1, 7, 0, 0 ] ], [ [ 0, 1, 0, 1 ], [ 2, 0, 1, 0 ] ], [ [ 0, 1, 1, 0 ], [ 1, 0, 0, 1 ] ], [ [ 0, 2, 0, 0 ], [ 3, 0, 0, 0 ] ] ]gap> mpd:=List(mp,p->p[1]-p[2]);[ [ -1, 0, -2, 2 ], [ 0, -8, 1, 1 ], [ -1, -7, 2, 0 ], [ -2, 1, -1, 1 ], [ -1, 1, 1, -1 ], [ -3, 2, 0, 0 ] ]gap> EquationsOfGroupGeneratedBy(mpd);[ [ [ 4, 6, 23, 25 ] ], [ ], which is telling you that your semigroup is generated by $\{4, 6, 23, 25\}$. In addition,

$$G = \langle 4, 6, 23, 25\rangle = \{0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, \rightarrow\}.$$

We presented the Gröbner basis of the Arf numerical semigroup as follows:

$$\mathrm{PF}(G) = \{2, 19, 21\}, \quad \mathrm{PF}'(G) = \mathrm{PF}(G) - F(G) \Longrightarrow \mathrm{PF}'(G) = \{2\},$$

for RF-matrices $f = 2 \in \mathrm{PF}'(G)$,

$$\mathrm{RF}(2) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

Consider the ideal generator set calculated by the RF-relations

$$I_G = \langle x_2^2 - x_1^3, x_2x_3 - x_1x_4, x_4 - x_1^2x_2x_3, x_1^2x_3 - x_2x_4, x_4^2 - x_1x_2^2\rangle$$

and find Gröbner basis $\mathfrak{P}$. For the present example we used the ordinary lexicographic order $x_1 > x_2 > x_3 > x_4$. With this particular choice we get:

$$\mathfrak{P} = \{-x_1^3 + x_2^2, -x_1x_4 + x_2x_3, -x_1^2x_2x_3 + x_4, x_1^2x_3 - x_2x_4, -x_1x_2^2 + x_4^2, x_2^2x_4 - x_4, x_2^3x_3 - x_2x_3, x_3^2x_4^2 - x_4^2, -x_1x_2x_3^2 + x_2x_4^2, x_2^4 - x_2^2x_3^2, -x_4^3 + x_2x_3, -x_3x_4^4 + x_2x_4, -x_3^2x_4 + x_4, -x_3x_4^7 + x_3x_4, -x_4^7 + x_4\}$$

and find reduced gröbner basis $\mathfrak{RP}$.

$$\mathfrak{RP} = \{x_4^7 - x_4, x_3^2x_4 - x_4, -x_3x_4^4 + x_2x_4, -x_4^3 + x_2x_3, -x_4^6 + x_2^4, x_4^3 + x_1x_4, x_1x_2^2 - x_4^2, -x_3x_4^4 + x_1^2x_3, x_1^3 - x_2^2\}.$$

**Example 6.7.** Let $G = \langle 3, 11, 13\rangle = \{0, 3, 6, 9, 11, \rightarrow\}$. Let us calculate the Gröbner basis of the Arf numerical semigroup.

$$PF(G) = \{8, 10\}, \quad PF'(G) = PF(G) - F(G) \Longrightarrow PF'(G) = \{8\},$$

for $f = 8 \in PF'(G)$,

$$RF(8) = \begin{bmatrix} -1 & 0 & 1 \\ 7 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}.$$

The ideal generator set calculated by the RF-relations is $I_G = \langle x_2x_3 - x_1^8, x_3 - x_1^5x_2, x_1^3x_3 - x_2^2\rangle$ and find Gröbner basis $\mathfrak{P}$. For the present example we used the ordinary lexicographic order $x_1 > x_2 > x_3$. With this particular choice we get:

$$\mathfrak{P} = \{-x_1^8 + x_2x_3, -x_1^5x_2 + x_3, x_1^3x_3 - x_2^2, x_2^2x_3 - x_2^2, -x_2x_3^2 + x_2x_3, x_1^2x_2^3 - x_3^2, -x_1^3x_2^2 + x_2^4, x_3^3 - x_3^2, x_2^5 - x_1x_3^2, x_2^7 - x_1x_2^2, -x_2^6 + x_1x_2x_3, x_2^{13} - x_3^2\}$$

and reduced gröbner basis $\mathfrak{RP}$ is

$$\mathfrak{RP} = \{x_3^3 - x_3^2, x_2x_3^2 - x_2x_3, x_2^2x_3 - x_2^2, x_2^{13} - x_3^2, -x_2^5 + x_1x_3^2, -x_2^6 + x_1x_2x_3, -x_2^7 + x_1x_2^2, x_1^3x_3 - x_2^2, x_1^5x_2 - x_3, x_1^8 - x_2x_3\}.$$

Moreover, we can proceed with GAP program as follows:

```
gap> s:=NumericalSemigroup(3,11,13);
<Numerical semigroup with 3 generators>
gap> mp:=MinimalPresentation(s);
[[[0, 0, 2], [5, 1, 0]], [[0, 1, 1], [ 8, 0, 0 ] ], [ [ 0, 2, 0 ], [ 3, 0, 1 ] ] ]
gap> mpd:=List(mp,p->p[1]-p[2]);
[[ -5, -1, 2], [ -8, 1, 1], [ -3, 2, -1]]
gap> EquationsOfGroupGeneratedBy(mpd);
[[[3, 11, 13]], []]
```

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