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Application of Adomian polynomials for solving nonlinear integro-differential equations

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Abstract

In this study, the nonlinear integro-differential equation (NIDE) of the second kind is resolved using the Adomian decomposition method (ADM). The term non-linearity can be dealt with easily if used techniques of Adomian polynomials. The existence of at least one positive continuous solution to the nonlinear integro-differential equation is ensured by sufficient conditions. Both the Arzelà-Ascoli theorem and the Tychonoff fixed point principle are used in this method. These types of equations are solved using the Adomian decomposition method and the repeated trapezoidal method. The method presented at the end of the article has been tested on many examples and has proven its efficiency after discussing the results.

Keywords: Nonlinear integro-differential equation, Adomian decomposition method, repeated trapezoidal method, Tychonoff fixed point theorem, fixed point theorem.

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1. Introduction

One of the most well-known mathematical equations is the integro-differential equation, which is used in many disciplines including computer science [25], medicine [20], dynamics [26, 33], biology [8], physics [9, 24], etc. Their precise form varies depending on the scientific task being investigated. Many of these equations have already been studied in earlier papers on related topics. Among them, the equations with a weakly singular kernel in the network studies [10, 21], in the non-linear Fredholm form [4, 31], in the COVID-19 researches [29], in the non-linear Volterra-Fredholm form [28, 32], in the non-linear Volterra form [16, 17, 30], in the linear Fredholm form [2, 19, 27], and others.

Sometimes most of the non-linear integral differential equations are difficult to obtain the exact solution, so we resort to the use of numerical methods. Since integro-differential equations are a relatively new area of mathematics, there are only a few techniques for solving them.

The authors have recently used a variety of techniques to solve the second kind integro-differential equations and integral equations of Fredholm and Volterra, both linear and nonlinear [14, 18].

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In this work, we take into account the second kind of continuous kernel nonlinear integro-differential equation with continuous kernels with respect to t. To solve nonlinear integro-differential equations, we used Adomian polynomials. This approach has several advantages, including the ability to solve various linear and nonlinear equations analytically and efficiency when dealing with these types of equations, see [6, 9]. In addition to the repeated trapezoidal method [13], which is used to solve these kinds of equations, the results obtained from the preceding two methods are compared, where the Tychonoff fixed point method is used to discuss and prove the existence of at least one solution to the nonlinear integro-differential equation.

Here, we'll look at how to solve a nonlinear integro-differential equation of the following form using a modified form of the Adomian decomposition method:

$$\Psi(t) = f(t) + \lambda \int_0^1 k(t,\tau)\vartheta(\tau, \Psi'(\tau))d\tau + \lambda \int_0^t \xi(t,\tau)\upsilon(\tau, \Psi'(\tau))d\tau, \tag{1.1}$$

$\Psi(t)$ is an unknown function in Banach space and continuous with their derivative with respect to t in the space $C^1([0,1])$, where $[0,1]$ is the domain of integration with respect to the t and it's called the potential function of the mixed integral equation. The constant parameter λ may be complex and has many physical meanings, the known function $f(t)$ is continuous, its derivatives with respect to t, and the kernels $k(t,\tau), \xi(t,\tau)$ are positive and continuous in $C^1([0,1])$.

By differentiating equation (1.1) with respect to t we can now create an integral equation that is equivalent to the integro-differential equation (1.1), giving us

$$\Psi'(t) = f'(t) + \lambda\xi(t) + \lambda \int_0^1 k'(t,\tau)\vartheta(\tau, \Psi'(\tau))d\tau + \lambda \int_0^t \xi'(t,\tau)\upsilon(\tau, \Psi'(\tau))d\tau.$$

Let's assume

$$\Phi(t) = \Psi'(t), \quad g(t) = f'(t) + \lambda\xi(t).$$

Then the last integral equation becomes

$$\Phi(t) = g(t) + \lambda \int_0^1 k'(t,\tau)\vartheta(\tau, \Phi(\tau))d\tau + \lambda \int_0^t \xi'(t,\tau)\upsilon(\tau, \Phi(\tau))d\tau. \tag{1.2}$$

The equation (1.2) is called nonlinear integro-differential equation. A type of functional equation known as an integro-differential equation has associated derivatives and integral of an unknown function. These equations bear the names of the top mathematicians who first researched them, including Volterra and Fredholm. The two types of equations that are most frequently encountered are Fredholm and Volterra equations, see [15]. There is, formally, they only differ in that the region of integration in the Fredholm equation is constant whereas the region in the Volterra equation is variable. Equations that combine differential and integral terms are known as integral-differential equations.

The organization of this work is as follows. In the following section, we provide the preliminary and auxiliary results about the fixed point theorems. Section 3 discusses the existence of at least one integral equation solution (1.2) by using Tychonoff fixed point theorem. In Section 4, we give some definitions and properties of the Adomian polynomials. In Section 5, we describe the method for approximating solution of nonlinear integro-differential equation, in Section 6, examples are provided to illustrate how our findings can be applied. Finally, final remarks are deduced.

2. Preliminaries of analytical

Following are some definitions and fixed-point theorems that are utilised in the study and on which the existence results in this section will be based.

Definition 2.1 (Convex set [23]). A set $B \subset M$ is said to be a convex set if $\forall \beta \in [0,1]$ and $\forall \phi, \psi \in B$, $\beta\phi + (1-\beta)\psi \in B$.

Theorem 2.2 (Banach's Fixed Point Theorem [10]). *If M is a Banach space and $T : M \to M$ is a contraction mapping, then T has a unique fixed point in M.*

Theorem 2.3 (Tychonoff's Fixed Point Theorem [12]). *Let M be a locally convex linear space, a (???) complete, and B_r is a closed convex subset of M. Suppose that $T : B_r \to B_r$ be continuous and $T(M) \subset M$. If the closure of $T(M)$ is compact, then T has a fixed-point in M.*

Theorem 2.4 (Arzelà-Ascoli Theorem [22]). *Assume M be a compact metric space and $C^1(M)$ is the Banach space of real valued continuous functions normed by*

$$\|\Phi\| = \max_{a \leqslant t \leqslant b} |\Phi(t)| + \max_{a \leqslant t \leqslant b} |\Phi'(t)|.$$

If $G = \{g_n\}$ is a sequence in $C^1(M)$, that is, equi-continuous and uniformly bounded, then the closure of G is compact.

3. Existence of a solution of a nonlinear integral equation (1.2)

Here, we discuss the existence of at least one solution of Eq (1.2). Integral equation (1.2) can be expressed in the integral operator form as

$$(W\Phi)(t) = g(t) + \lambda \int_0^1 k'(t,\tau)\vartheta(\tau, \Phi(\tau))d\tau + \lambda \int_0^t \xi'(t,\tau)\upsilon(\tau, \Phi(\tau))d\tau.$$

We make the following assumptions in order to discuss whether there is at least one solution of equation (1.2).

(i) $g : I \to R$ is a continuous function on I.
(ii) $k' : I \times I \to R$ is continuous, such that $|k'(t,\tau)| < k_1$, k_1 is a positive constant.
(iii) $\xi' : I \times I \to R$ is continuous, so that $|\xi'(t,\tau)| < k_2$, k_2 is a positive constant.
(iv) The function $\vartheta : I \times R \to R$ satisfies Lipschitz condition with Lipschitz constant m_1, and there exists a nondecreasing function $m_2 : R \to R$ in which $|\vartheta(\tau, \Phi(\tau))| \leqslant m_2(|\Phi|)$.
(v) The function $\upsilon : I \times R \to R$ satisfies Lipschitz condition with Lipschitz constant l_1, and there exists a function $l_2 : R \to R :\ |\upsilon(\tau, \Phi(\tau))| \leqslant l_2(|\Phi|)$.
(vi) The inequality
$$\|g\| + \lambda k[m_2(r) + l_2(r)] \leqslant r; \ (k = \max\{k_1, k_2\}).$$

We can now state the fundamental existence theorem.

Theorem 3.1. *Under the conditions* (i)-(vi)*, Eq.* (1.2) *has at least one solution $\Phi = \Phi(t)$, which belongs to the space $C^1([0,1])$ and is nondecreasing and nonnegative on the interval $[0,1]$.*

Proof. Here B_r represent the subset of the space $C^1([0,1])$ as defined:

$$B_r = \{\Phi \in C^1([0,1]) : |\Phi(t)| \leqslant r \ \text{ for } \ t \in [0,1]\}.$$

It has been demonstrated in [12] that the space $C^1([0,1])$ is a complete locally convex linear space, it's obvious that the set B_r is bounded, closed, and nonempty, but we shall demonstrate that the B_r set is convex. Assume Φ_1, $\Phi_2 \in B_r$ and $\beta \in [0,1]$, then we have

$$\|\beta\Phi_1 + (1-\beta)\Phi_2\| \leqslant \beta\|\Phi_1\| + (1-\beta)\|\Phi_2\| \leqslant \beta r + (1-\beta)r \leqslant \beta r + r - \beta r = r.$$

Then $\beta\Phi_1 + (1-\beta)\Phi_2 \in B_r$, from the previous, we get that this is a convex set. Consider the following definition of the operator W in the space $C^1([0,1])$:

$$(W\Phi)(t) = g(t) + \lambda \int_0^1 k'(t,\tau)\vartheta(\tau,\Phi(\tau))d\tau + \lambda \int_0^t \xi'(t,\tau)\upsilon(\tau,\Phi(\tau))d\tau.$$

To show this, the space B_r is transformed into itself by the operator W. Let's $\Phi \in B_r$ for that, then

$$|(W\Phi)(t)| \leqslant \left| g(t) + \lambda \int_0^1 k'(t,\tau)\vartheta(\tau,\Phi(\tau))d\tau \right| + \left| \lambda \int_0^t \xi'(t,\tau)\upsilon(\tau,\Phi(\tau))d\tau \right|.$$

We obtain following by using the properties of the norm

$$|(W\Phi)(t)| \leqslant \|g(t)\| + \lambda \int_0^1 |k'(t,\tau)||\vartheta(\tau,\Phi(\tau))|d\tau + \lambda \int_0^t |\xi'(t,\tau)||\upsilon(\tau,\Phi(\tau))|d\tau.$$

Using conditions (i)-(vi), we obtain

$$|(W\Phi)(t)| \leqslant \|g\| + \lambda k[m_2(r) + l_2(r)], \leqslant r; \quad (k = \max\{k_1, k_2\}).$$

According to the estimate presented above and condition (vi), then $(W\Phi)(t) \in B_r$ implies $WB_r \subset B_r$.

Now, Assume that the fix arbitrarily $\delta > 0$ and $t_1, t_2 \in [0,1]$ such that $|t_2 - t_1| \leqslant \delta$, $t_2 \geqslant t_1$. Then, taking into consideration in mind our hypotheses, we get

$$\begin{aligned}|(W\Phi)(t_2) - (W\Phi)(t_1)| \leqslant & |g(t_2) - g(t_1)| + \lambda \int_0^1 |k'(t_2,\tau) - k'(t_1,\tau)||\vartheta(\tau,\Phi(\tau))d\tau| \\ & + \lambda \int_0^{t_2} |\xi'(t_2,\tau) - \xi'(t_1,\tau)||\upsilon(\tau,\Phi(\tau))d\tau|,\end{aligned}$$

Using the conditions (i)-(vi) and the norm's properties, we are able to

$$|(W\Phi)(t_2) - (W\Phi)(t_1)| \leqslant |g(t_2) - g(t_1)| + \lambda m_2(|\Phi|)|k'(t_2,1) - k'(t_1,1)| + \lambda l_2(|\Phi|)|\xi'(t_2,1) - \xi'(t_1,1)|t_2.$$

As a result, considering our hypotheses and the previously mentioned facts, we determine the following formula:

$$|(W\Phi)(t_2) - (W\Phi)(t_1)| \to 0 \quad \text{as} \quad |t_2 - t_1| \to 0.$$

The function WB_r is therefore equi-continuous on $[0,1]$. By using Theorem 2.4, we may infer that is WB_r compact. Now that all of the conditions of the Tychonoff fixed point theorem have been met, integral equation (1.2) has at least one solution, which is $\Phi \in C^1([0,1])$. This completes the proof. □

Example 3.2. We will discuss the following example and appling Theorem 3.1, then check the results. Consider the integral equation:

$$\Psi(t) = t - 0.001\left(\frac{1}{3} + t^2\right) - 0.1\left(\frac{t^2}{2} + t^3\right) + 0.001\int_0^1 (t^2+\tau^2)\Psi'^2(\tau)d\tau + 0.1\int_0^t (t^2+\tau)\Psi'^3(\tau)d\tau, \tag{3.1}$$

by differentiating equation (3.1) with respect to t, we get

$$\Psi'(t) = 1 - 0.002t - 0.001(t+3t^2) + 0.001(t^2+t) + 0.1\int_0^1 (2t+\tau^2)\Psi'^2(\tau)d\tau + 0.001\int_0^t (2t+\tau)\Psi'^3(\tau)d\tau.$$

Assume that

$$\Phi(t) = \Psi'(t), \quad g(t) = 1 - 0.002t - 0.001(t+3t^2) + 0.001t(t+1).$$

Then the last integral equation becomes

$$\Phi(t) = g(t) + 0.001\int_0^1 (2t+\tau^2)\Phi^2(\tau)d\tau + 0.001\int_0^t (2t+\tau)\Phi^3(\tau)d\tau. \tag{3.2}$$

Comparing this example to equation (1.2) and conditions (i)-(vi), we have $g(t) = 1 - 0.002t - 0.001(t + 3t^2) + 0.001t(t+1)$, which is continuous on $[0,1]$ with norm $\|g(t)\| = 0.985$, the kernel $k'(t,\tau) = (2t+\tau^2)$, which is continuous with respect to t and τ. Also, we have $|k'(t,\tau)| = \left|2t+\tau^2\right| \leqslant 3$, $(k_1 = 3)$ and $\xi'(t,\tau) = (2t+\tau)$, where $|\xi'(t,\tau)| = |2t+\tau| \leqslant 3$, $(k_2 = 3)$, the function $\vartheta(\tau,\Phi(\tau)) = \Phi^2(\tau)$, which satisfies the condition (iv) with $|\vartheta(\tau,\Phi(\tau))| = |\Phi^2(\tau)| \leqslant |\Phi(\tau)|$. Then, we obtain $m_2(r) = r$ and $\upsilon(\tau,\Phi(\tau)) = \Phi^3(\tau)$, which satisfies the condition (v) with $|\upsilon(\tau,\Phi(\tau))| = |\Phi^3(\tau)| \leqslant |\Phi(\tau)|$. Then, we obtain $l_2(r) = r$. Additionally, let's consider inequality

$$0.985 + 0.006r \leqslant r. \tag{3.3}$$

We can confirm that the function exists using conventional ways $\rho(r) = (0.985 - 0.994r)$ attains its minimum at the point $r_0 = 1$ and $\rho(r_0) = (0.985 - 0.994(1)) \leqslant 0$. So, the number r_0 is a positive solution of the inequality (3.3) and therefore the Theorem 3.1 is true.

Theorem 3.1 and the previously mentioned facts lead us to the conclusion that Eq. (3.2) has at least one solution $\Phi = \Phi(t)$ nondecreasing, continuous, and defined in $[0,1]$.

4. Preliminaries of numerical

The definitions and properties of the Adomian polynomials are provided in this section. Consider the general functional equation:

$$\Phi = g + N_1\Phi + N_2\Phi, \tag{4.1}$$

where N_1, N_2 are a nonlinear operators, g is a known function, and we are looking for a solution Φ that satisfies (4.1). We suppose that equation (4.1) has only one solution for every g.

$$\Phi = \sum_{k=0}^{\infty} \Phi_k, \tag{4.2}$$

and decomposing N_1 and N_2, which represent the nonlinear operators

$$N_1\Phi = \sum_{k=0}^{\infty} Y_k, \qquad N_2\Phi = \sum_{k=0}^{\infty} Z_k, \tag{4.3}$$

respectively, where Y_k, Z_k are polynomials (called Adomian polynomials) of $\{\Phi_0, \Phi_1, \dots, \Phi_k\}$ [5, 7] given by

$$Y_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_1\left(\sum_{j=0}^{\infty}\lambda^j\Phi_j\right)\right]_{\lambda=0}, \ k = 0,1,2,\dots; \quad Z_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_2\left(\sum_{j=0}^{\infty}\lambda^j\Phi_j\right)\right]_{\lambda=0}, \ k = 0,1,2,\dots.$$

The proofs of the convergence of the series $\sum_{k=0}^{\infty}\Phi_k, \sum_{k=0}^{\infty}Y_k$ and $\sum_{k=0}^{\infty}Z_k$ are given in [1]. Using equations (4.2) and (4.3) in (4.1), yields

$$\sum_{k=0}^{\infty}\Phi_k = g + \sum_{k=0}^{\infty}Y_k + \sum_{k=0}^{\infty}Z_k.$$

Thus, the following can be obtained

$$\Phi_0 = g, \quad \Phi_{k+1} = Y_k(\Phi_0,\Phi_1,\dots,\Phi_k) + Z_k(\Phi_0,\Phi_1,\dots,\Phi_k); \quad k = 0,1,2,\dots.$$

As a result, once Y_k, Z_k are known, all of the components of ϕ may be determined. We then identify the n-terms approximate to the solution Φ by

$$\Psi_k[\Phi] = \sum_{j=0}^{k} \Phi_j, \quad \text{with} \quad \lim_{k\to\infty} \Psi_k[\Phi] = \Phi.$$

5. Description of the method

Both the Adomian decomposition method and the Adomian polynomials may be utilized to tackle (1.2) in addition to dealing with nonlinear terms $\vartheta(\tau,\Phi(\tau)), \upsilon(\tau,\Phi(\tau))$. First, we express the linear term $\Phi(t)$ at the left side by an infinite series of components provided by

$$\Phi(t) = \sum_{k=0}^{\infty} \Phi_k(t), \tag{5.1}$$

where the components $\Phi_k; k \geqslant 0$ will be calculated recursively, whereas, an infinite series of the Adomian polynomials Y_k, Z_k will be used to represent the nonlinear variables $\vartheta(\tau,\Phi(\tau)), \upsilon(\tau,\Phi(\tau))$, respectively, at the right side of Equation (1.2),

$$\vartheta(\tau,\Phi(\tau)) = \sum_{k=0}^{\infty} Y_k(\tau), \qquad \upsilon(\tau,\Phi(\tau)) = \sum_{k=0}^{\infty} Z_k(\tau), \tag{5.2}$$

where $Y_k, Z_k, k \geqslant 0$ are defined by

$$Y_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\vartheta\left(\sum_{j=0}^{\infty}\lambda^j\phi_j\right)\right]_{\lambda=0}, \ k=0,1,2,\ldots; \quad Z_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\upsilon\left(\sum_{j=0}^{\infty}\lambda^j\phi_j\right)\right]_{\lambda=0}, \ k=0,1,2,\ldots,$$

where the so-called Adomian polynomials Y_k, Z_k can be assessed for all forms of nonlinearity. In other words, let the nonlinear function is $\vartheta(\tau,\Phi(\tau)), \upsilon(\tau,\Phi(\tau))$, hence the Adomian polynomials are given by

$$\begin{aligned} Y_0 &= \vartheta(\Phi_0), & Z_0 &= \upsilon(\Phi_0),\\ Y_1 &= \Phi_1\vartheta'(\Phi_0), & Z_1 &= \Phi_1\upsilon'(\Phi_0),\\ Y_2 &= \Phi_2\vartheta'(\Phi_0) + \frac{1}{2}\Phi_1^2\vartheta''(\Phi_0), & Z_2 &= \Phi_2\upsilon'(\Phi_0) + \frac{1}{2}\Phi_1^2\upsilon''(\Phi_0). \end{aligned}$$

Substituting (5.1) and (5.2) into (1.2), we will get

$$\sum_{k=0}^{\infty}\Phi_k(t) = g(t) + \int_0^1 k'(t,\tau)\sum_{k=0}^{\infty} Y_k(\tau)d\tau + \int_0^t \xi'(t,\tau)\sum_{k=0}^{\infty} Z_k(\tau)d\tau.$$

The recursive relation is presented via the Adomian decomposition method,

$$\begin{aligned} \Phi_0(t) &= g(t),\\ \Phi_1(t) &= \int_0^1 k'(t,\tau)Y_0(\tau)d\tau + \int_0^t \xi'(t,\tau)Z_0(\tau)d\tau,\\ \Phi_2(t) &= \int_0^1 k'(t,\tau)Y_1(\tau)d\tau + \int_0^t \xi'(t,\tau)Z_1(\tau)d\tau. \end{aligned} \tag{5.3}$$

The recursive relation is often provided by

$$\Phi_{k+1}(t) = \int_0^1 k'(t,\tau)Y_k(\tau)d\tau + \int_0^t \xi'(t,\tau)Z_k(\tau)d\tau, \quad k=0,1,2,\ldots. \tag{5.4}$$

When the source term and the required initial conditions can both be functions $g(t)$, the initial solution is crucial, because choosing initial solution (5.3) results in noisy oscillation during the iteration process every t, according to the modified decomposition method, the operate $g(t)$ specified in (5.3) should be divided into two parts:

$$g(t) = g_1(t) + g_2(t).$$

Instead of iteration procedure (5.3)-(5.4), we suggest the following modification

$$\begin{aligned}
\Phi_0(t) &= g_1(t),\\
\Phi_1(t) &= g_2(t) + \int_0^1 k'(t,\tau)Y_0(\tau)d\tau + \int_0^t \xi'(t,\tau)Z_0(\tau)d\tau,\\
&\vdots\\
\Phi_{k+1}(t) &= \int_0^1 k'(t,\tau)Y_k(\tau)d\tau + \int_0^t \xi'(t,\tau)Z_k(\tau)d\tau; \quad k = 0,1,2,\ldots.
\end{aligned}$$

We then define the k-terms approximate to the solution $\Phi(t)$ by

$$\Psi_k[\Phi(t)] = \sum_{j=0}^{k} \Phi_j(t), \quad \text{with} \quad \lim_{k\to\infty} \Psi_k[\Phi(t)] = \Phi(t).$$

The series solution that was found in this study converges to the exact solution.

5.1. A test of convergence

In fact, on every interval the inequality $\|\Phi_{j+1}\|_2 < \beta\|\Phi_j\|_2$ is required to hold for $j = 0,1,\ldots,k$, wherever $0 < \beta < 1$ may be a constant and k is that the maximum order of the approximate used in the computation. Of course, this is often only a necessary condition for convergence, as a result of it might be necessary to compute $\|\Phi_j\|_2$ for each $j = 0,1,\ldots,k$ so as to conclude that the series is convergent.

6. Applications of Adomian decomposition method

In the following two examples, the Adomian decomposition approach for solving integro-differential equations is demonstrated. The maximum error is established as: in order to demonstrate the high precision of the results obtained by using the present method to solve our problem (1.1) in comparison to the exact solution,

$$R_k = \|\Phi_{\text{Exact}}(t) - \Psi_k[\Phi(t)]\|_\infty,$$

where $k = 1,2,\ldots$ represents the number of iterations.

Example 6.1. Consider the nonlinear integro-differential equation:

$$\Psi(t) = 0.790274t^2 - 0.0008t^7 + 0.1\int_0^1 t^2\tau e^{\Psi'(\tau)}d\tau + 0.1\int_0^t t^2\tau^2\Psi'^2(\tau)d\tau, \tag{6.1}$$

by differentiating equation (6.1) with respect to t, hence we get

$$\Psi'(t) = 1.58055t - 0.0056t^6 + 0.1t^4 + 0.2\int_0^1 t\tau e^{\Psi'(\tau)}d\tau + 0.2\int_0^t t\tau^2\Psi'^2(\tau)d\tau.$$

Assume that

$$\Phi(t) = \Psi'(t), \quad g(t) = 1.58055t - 0.0056t^6 + 0.1t^4.$$

Then the last integral equation becomes

$$\Phi(t) = g(t) + 0.2\int_0^1 t\tau e^{\Phi(\tau)} d\tau + 0.2\int_0^t t\tau^2 \Phi^2(\tau) d\tau. \tag{6.2}$$

The exact solution for this problem is

$$\Psi(t) = t^2.$$

Applying ADM to equation (6.2), we obtain

$$\begin{aligned} \Phi_0(t) &= g(t), \\ &\vdots \\ \Phi_{k+1}(t) &= 0.2\int_0^1 t\tau Y_k(\tau) d\tau + 0.2\int_0^t t\tau^2 Z_k(\tau) d\tau, \end{aligned}$$

where Y_k and Z_k are Adomian polynomials of the nonlinear terms e^{Φ} and Φ^2, respectively and the solution will be,

$$\Phi(t) = \sum_{k=0}^{r} \Phi_i(t).$$

Under some conditions, this series solution converges. The absolute error of the Repeated Trapezoidal (RT) solution and the ADM solution are compared in Table 1. We can observe and determine the variations that happen between the approximate solution and the exact solution for each of the two methods ADM and RT for various values of r as well as for various values of $\triangle\tau$.

Table 1: Absolute error of presented method.

t	Error of ADM (r = 30)	Error of RT ($\triangle\tau = 0.1$)	Error of ADM (r = 20)	Error of RT ($\triangle\tau = 0.01$)
0	2.32154×10^{-40}	1.32587×10^{-25}	5.326146×10^{-36}	6.20205×10^{-27}
0.1	6.36521×10^{-40}	3.25147×10^{-23}	6.251485×10^{-36}	4.36215×10^{-25}
0.2	3.25416×10^{-38}	2.02158×10^{-20}	4.362147×10^{-35}	6.21487×10^{-23}
0.3	5.36258×10^{-35}	8.32147×10^{-19}	4.321586×10^{-31}	4.32105×10^{-22}
0.4	3.21548×10^{-33}	5.65987×10^{-17}	4.232514×10^{-29}	4.36987×10^{-20}
0.5	1.02589×10^{-27}	1.36985×10^{-16}	6.215475×10^{-26}	6.32514×10^{-18}
0.6	4.32154×10^{-25}	1.36751×10^{-14}	6.215478×10^{-22}	9.32154×10^{-17}
0.7	3.01478×10^{-21}	9.32147×10^{-13}	1.302584×10^{-20}	1.36254×10^{-15}
0.8	3.69852×10^{-17}	7.36952×10^{-11}	3.214587×10^{-15}	4.32651×10^{-14}
0.9	5.25874×10^{-15}	4.36921×10^{-10}	6.321548×10^{-14}	8.32514×10^{-12}
1	6.32145×10^{-14}	1.36214×10^{-9}	4.321587×10^{-13}	6.32544×10^{-11}

In Figures 1 and 3, we compared the exact and approximate solutions using the introduced numerical approach ADM with various values of r. In Figures 2 and 4, we compared the approximate and exact solutions using the presented numerical method (RT) with various values of $\triangle\tau$.

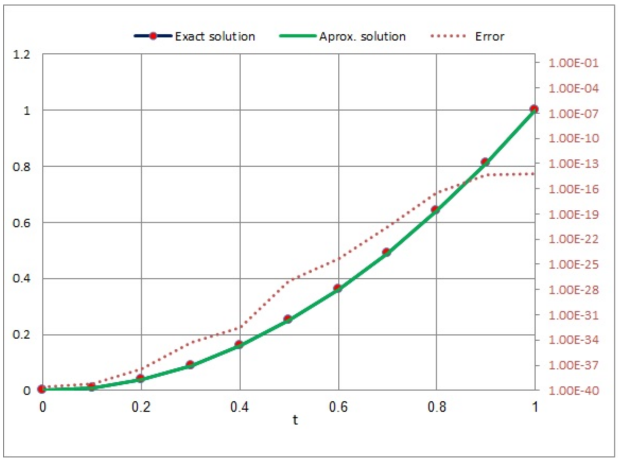

Figure 1: Approximate and exact solution of ADM for $r = 30$.

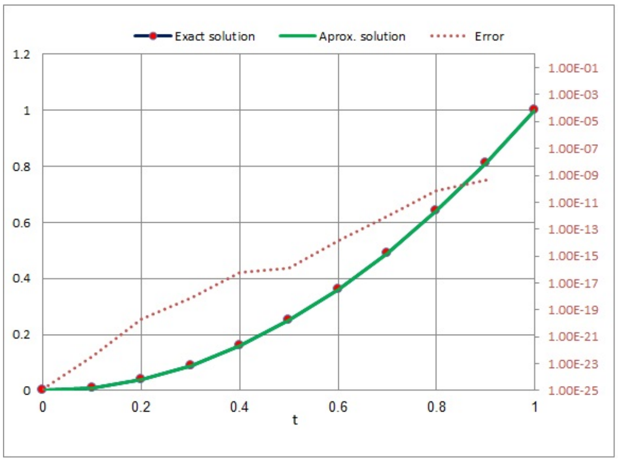

Figure 2: Approximate and exact solution of RT method for $\triangle\tau = 0.1$.

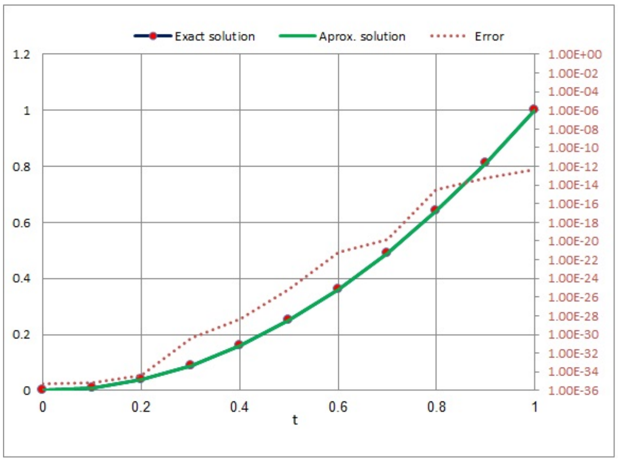

Figure 3: Approximate and exact solution of ADM for $r = 20$.

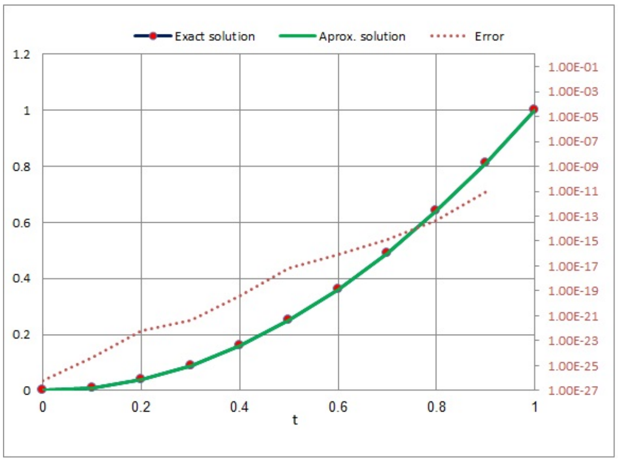

Figure 4: Approximate and exact solution of RT method for $\triangle\tau = 0.01$.

Example 6.2. Consider the following nonlinear integro-differential equation:

$$\begin{aligned}\Psi(t) &= e^{t} - 0.0209726t - 0.00111111(1 - 3t + e^{3t}(-1+6t)) \\ &\quad + 0.01\int_0^1 t\tau\Psi'^2(\tau)d\tau + 0.01\int_0^t (t+\tau)\Psi'^3(\tau)d\tau,\end{aligned} \tag{6.3}$$

differentiating equation (6.3) with respect to t, we get

$$\begin{aligned}\Psi'(t) &= -0.0209726 + e^{t} - 0.00111111(-3 + 6e^{3t} + 3e^{3t}(-1+6t)) + 0.02t \\ &\quad + 0.01\int_0^1 \tau\Psi'^2(\tau)d\tau + 0.01\int_0^t (1+\tau)\Psi'^3(\tau)d\tau.\end{aligned}$$

Assume that

$$\Phi(t) = \Psi'(t), \quad g(t) = -0.0209726 + e^{t} - 0.00111111(-3 + 6e^{3t} + 3e^{3t}(-1+6t)) + 0.02t.$$

Then the last integral equation becomes

$$\Phi(t) = g(t) + 0.01 \int_0^1 \tau\Phi^2(\tau)d\tau + 0.01 \int_0^t (1+\tau)\Phi^3(\tau)d\tau. \tag{6.4}$$

The exact solution for this problem is

$$\Psi(t) = e^t.$$

Applying ADM to equation (6.4), we obtain

$$\begin{aligned}\Phi_0(t) &= g(t),\\ &\vdots\\ \Phi_{k+1}(t) &= 0.01 \int_0^1 \tau Y_k(\tau)d\tau + 0.01 \int_0^t (1+\tau)Z_k(\tau)d\tau,\end{aligned}$$

where Y_k and Z_k are Adomian polynomials of the nonlinear terms Φ^2 and Φ^3, respectively and the solution will be,

$$\Phi(t) = \sum_{k=0}^{r} \Phi_i(t).$$

Under some conditions, this series solution converges. The absolute error of the repeated trapezoidal (RT) solution and the ADM solution are compared in Table 2. We can observe and determine the variations that happen between the approximate solution and the exact solution for each of the two methods ADM and RT for various values of r as well as for various values of $\triangle\tau$. In Figures 5 and 7, we compared the exact and approximate solutions using the introduced numerical approach ADM with various values of r. In Figures 6 and 8, we compared the approximate and exact solution using the presented numerical method (RT) with various values of $\triangle\tau$.

Table 2: Absolute error of presented method.

t	Error of ADM (r = 10)	Error of RT ($\triangle\tau = 0.1$)	Error of ADM (r = 5)	Error of RT ($\triangle\tau = 0.01$)
0	6.323154×10^{-30}	7.365952×10^{-22}	6.325415×10^{-26}	5.365214×10^{-25}
0.1	1.325548×10^{-28}	4.365985×10^{-20}	5.369854×10^{-25}	5.32548×10^{-24}
0.2	3.255487×10^{-25}	5.325614×10^{-18}	2.368545×10^{-22}	1.36254×10^{-23}
0.3	4.365952×10^{-24}	6.025154×10^{-17}	2.369854×10^{-20}	4.36254×10^{-22}
0.4	3.025587×10^{-22}	4.365524×10^{-15}	6.325456×10^{-19}	5.36985×10^{-20}
0.5	1.365985×10^{-19}	5.325149×10^{-13}	6.258745×10^{-17}	1.36652×10^{-18}
0.6	4.454536×10^{-17}	6.352587×10^{-13}	6.215588×10^{-16}	4.36559×10^{-17}
0.7	3.267534×10^{-16}	6.328579×10^{-12}	6.325417×10^{-14}	1.36852×10^{-15}
0.8	3.453455×10^{-15}	8.325514×10^{-11}	3.456287×10^{-13}	4.36855×10^{-14}
0.9	4.356524×10^{-14}	6.320514×10^{-10}	5.326552×10^{-13}	6.32548×10^{-12}
1	7.251587×10^{-12}	4.369585×10^{-9}	5.265569×10^{-11}	3.25564×10^{-11}

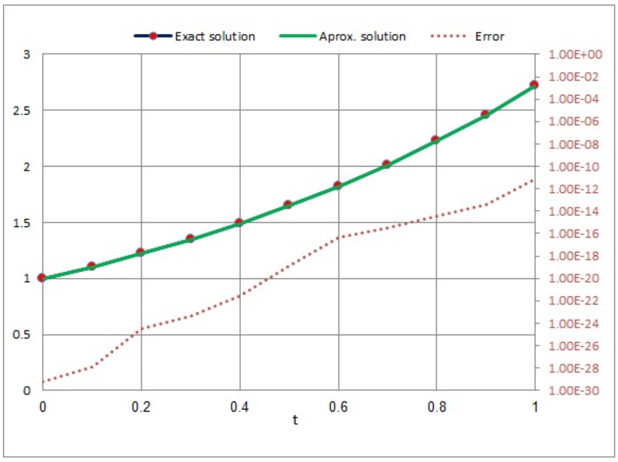

Figure 5: Approximate and exact solution of ADM for $r = 10$.

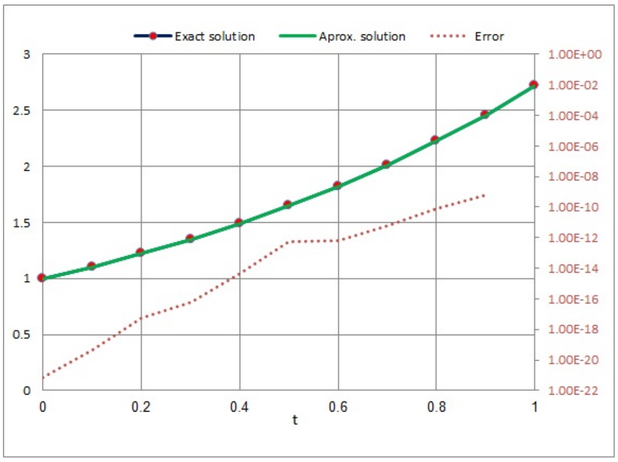

Figure 6: Approximate and exact solution of RT method for $\triangle\tau = 0.1$.

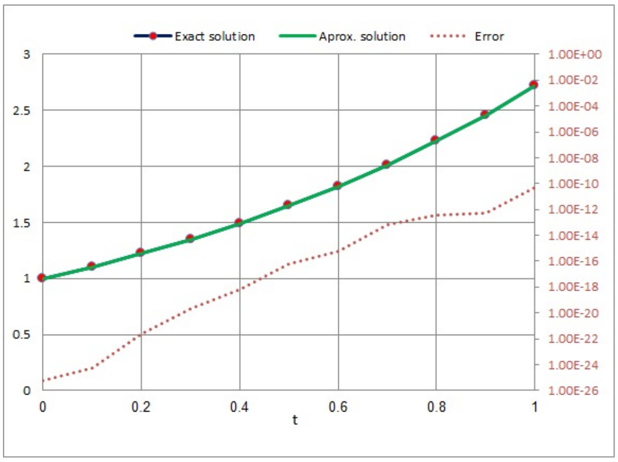

Figure 7: Approximate and exact solution of ADM for $r = 5$.

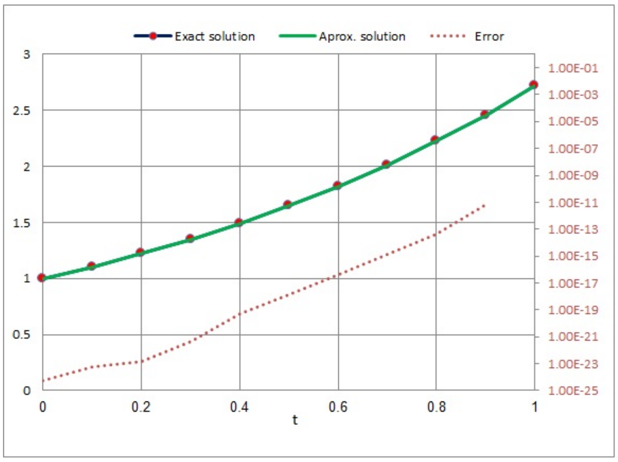

Figure 8: Approximate and exact solution of RT method for $\triangle\tau = 0.01$.

7. Conclusions

The following can be deduced from this work's results and discussion. The space $C^1([0,1])$ contains at least one solution $\Psi = \Psi(t)$ to the equation (1.1). The approximate solution to the nonlinear integro-differential equation has been obtained using the decomposition technique. This approach is very effective at locating analytical and numerical solutions for a variety of nonlinear integro-differential equation classes. It offers numerous realistic series of solutions that quickly converge on solutions for real physical problems.

Both of numerical approaches ADM and RT have a positive relationship between t and error, therefore t was increasing in the interval $[0,1]$, the error values of ADM and RT are also increasing. Through the examples, it can be seen that the approach ADM is more accurate when r is too large and vice versa. The repeated trapezoidal numerical results that were obtained from the illustrative examples leads us to the conclusion that for sufficiently small $\triangle\tau$, we acquire good accuracy.

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Future work

The authors will consider the solution of the principal equation of this paper in two-dimensional problem with a singular kernel.

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