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The Diophantine equation $a^x \pm a^y = z^n$ when a is any nonnegative integer

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Abstract

In this paper, all solutions of the Diophantine equation $a^x \pm a^y = z^n$ are investigated when a is any nonnegative integer and $n \geqslant 2$. In particular, if p is prime and the solutions of $p^x + p^y = z^n$ exist, then p is either 2 or $2^n - 1$. All proofs in this paper require only elementary number theory.

Keywords: Diophantine equation, Catalan's conjecture, the fundamental theorem of arithmetic, Mersenne prime.

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1. Introduction

In 1844, Catalan's conjecture was stated as follows [7]: $(3, 2, 2, 3)$ is the unique solution (a, b, x, y) for the equation $a^x - b^y = 1$, where $\min\{a, b, x, y\} \geqslant 2$.

It was proven by Mihăilescu in 2002 and later published in 2004 [13] by using the theory of cyclotomic fields and Galois modules. Utilizing the Catalan's conjecture as a main tool, the Diophantine equation in the form $a^x + b^y = z^2$ has been studied by many researchers over the past 20 years. In 2007, Acu [1] found all solutions of the Diophantine equation in the form $2^x + 5^y = z^2$. In 2011, Suvarnamani [17] considered the Diophantine equation in the form $2^x + p^y = z^2$ when p is prime. In 2012, Tatong and Suvarnamani [18] studied the Diophantine equation in the form $p^x + p^y = z^2$, where p equals 2 or 3 and x, y, z are nonnegative integers. In 2019, Burshtein [5] generalized the work of Tatong and Suvarnamani and presented the Diophantine equations $p^x + p^y = z^2$ and $p^x - p^y = z^2$ when $p \geqslant 2$ is prime and x, y, z are positive integers. Recently, Burshtein [6] expanded her work, [5], for the case where $p^x + p^y = z^4$. In addition, there were also some articles with the same trend but the term z^2 was changed to be the exponential term, c^z, such as $a^x + b^y = c^z$, see [14], and $p^x \pm q^y \pm 2^z = 0$, see [15].

In this paper, we aim to extend the term z^2 on the right-hand side of the expression to be z^n for $n \geqslant 2$. Even though our works look simple but cover all results of Tatong and Suvarnamani are contained in our

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main results Corollary 2.8 when n equals two. We can see that when $p = 2$, it appears in A and B, and when $p = 3$, it appears in C of Corollary 2.8. In addition, all results in both works of Burshtein are still contained in Corollary 2.8 for $p^x + p^y = z^n$ when n equals two or four and Corollary 3.7 for $p^x - p^y = z^n$ when n equals two. Furthermore, we also attempt to find all solutions for $a^x \pm a^y = z^n$ when a is any positive integer. In addition, the prominent point of this work is only using elementary number theory to prove all results.

This paper is organized as follows. Section 2 is focused on finding all solutions of the Diophantine equation $a^x + a^y = z^n$ when a is any nonnegative integer, as shown in Theorem 2.7. In particular, a is also considered a prime number. In Section 3, we study another form of the Diophantine equation, $a^x - a^y = z^n$, which is one of our main results, Theorem 3.5. Finally, in Section 4, we discuss our work and future research direction. Throughout this paper, we assume that all parameters are nonnegative integers.

2. All solutions of $a^x + a^y = z^n$

Lemma 2.1. *Let a and $n \geqslant 2$ be positive integers. All solutions of the equation $a^x + 1 = z^n$ are*

$$(a, n, x, z) \in \left\{(2, 2, 3, 3), \left(a, n, 1, \sqrt[n]{a+1}\right)\right\}.$$

In particular, the equation $a^x + 1 = z^n$ has at most one solution when a and n are fixed.

Proof. If $z = 0$ or $z = 1$, then there is no solution since $a^x = -1$ and $a^x = 0$ for $z = 0$ and $z = 1$, respectively. Hereafter, we consider $z \geqslant 2$. If $x = 1$ and $\sqrt[n]{a+1} \in \mathbb{N}$, then the equation has the solution $(a, n, x, z) = \left(a, n, 1, \sqrt[n]{a+1}\right)$. By Catalan's conjecture and the fact that $a \geqslant 2$, they lead us to assure that the equation $z^n - a^x = 1$ has only solution $(a, n, x, z) = (2, 2, 3, 3)$ for $x \geqslant 2$. So, we can conclude that there is no solution for $x \neq 1, 3$. □

Remark 2.2. If a is equal to zero, then $(a, n, x, z) = (0, n, x, 1)$ is a solution of $a^x + 1 = z^n$ for any positive integer x.

Lemma 2.3. *Let p be prime and $n \geqslant 2$ be a positive integer. All solutions of the equation $p^x + 1 = z^n$ are*

$$(p, n, x, z) \in \{(2, 2, 3, 3), (2^n - 1, n, 1, 2)\}.$$

In particular, the equation has at most one solution when p and n are fixed. Furthermore, if n is a composite number, the equation has no solution.

Proof. We consider only the case where $z \geqslant 2$ and $x = 1$. Then, we have the equation $1 + p = z^n$. It follows that

$$p = z^n - 1 = (z - 1)\left(z^{n-1} + z^{n-2} + \cdots + 1\right).$$

Since $z^{n-1} + z^{n-2} + \cdots + 1$ is greater than 1, we have $z - 1 = 1$, and then $z = 2$. Hence, $(p, n, x, z) = (2^n - 1, n, 1, 2)$. □

Remark 2.4. A Mersenne prime is a prime number in the form $2^n - 1$ for some integer n. The exponent n and the Mersenne prime are sequences A000043 and A000668 in the OEIS, respectively. Note that it is easy to see that if $2^n - 1$ is prime, then n is prime. However, the converse is generally untrue; for instance, when $n = 11$. The largest known Mersenne prime is $2^{82589933} - 1$, which was discovered by the Great Internet Mersenne Prime Search (GIMPS) on December 21, 2018. This number is the 51st in the sequence (48 officially confirmed); for more details, see [20].

Proposition 2.5. *Let a and $n \geqslant 2$ be positive integers. All solutions of the equation $2a^x = z^n$ are*

$$(a, n, x, z) \in \left\{ \left(2^\alpha a_0, n, \frac{\beta n - 1}{\alpha}, 2^\beta \sqrt[n]{a_0^{\frac{\beta n - 1}{\alpha}}} \right) : \alpha, \beta \in \mathbb{N}, a_0 \text{ are odd integers} \right\}.$$

In particular, if a is odd, then this equation has no solution.

Proof. It is clear that z and a are even. Then, we may write $z = 2^\beta z_0$ and $a = 2^\alpha a_0$ for some positive integers α, β and odd integers z_0, a_0. Now, we have

$$2^{\beta n} z_0^n = \left(2^\beta z_0\right)^n = z^n = 2a^x = 2\left(2^\alpha a_0\right)^x = 2^{\alpha x + 1} a_0^x.$$

Therefore, $\beta n = \alpha x + 1$ and $z_0^n = a_0^x$, which implies that

$$(a, n, x, z) = \left(2^\alpha a_0, n, \frac{\beta n - 1}{\alpha}, 2^\beta \sqrt[n]{a_0^{\frac{\beta n - 1}{\alpha}}} \right).$$

□

Remark 2.6. For $\sqrt[n]{a_0^{\frac{\beta n-1}{\alpha}}}$, if $a_0 \neq 1$, then we write $a_0 = \prod_{i=1}^k p_i^{\alpha_i}$ by the fundamental theorem of arithmetic. If we let $d = \gcd\limits_{1 \leqslant i \leqslant k} (\alpha_i)$, then $a_0 = h^d$ for some $h \in \mathbb{N}$. Then, we can rewrite $\sqrt[n]{a_0^{\frac{\beta n-1}{\alpha}}} = h^K$, where $K = \frac{d(\beta n - 1)}{n\alpha}$. In contrast, we see that $n(d\beta - \alpha K) = d$. Hence, under the condition that K is an integer, this equation has a solution if and only if $n \gcd(d, \alpha) \mid d$.

Theorem 2.7. *Let a and $n \geqslant 2$ be positive integers. If $x \neq y$, then all solutions of equation $a^x + a^y = z^n$ are $(a, n, x, y, z) \in A \cup B$, where*

- $A = \left\{ \left(2, 2, 2k, 2k+3, 3 \cdot 2^k\right), \left(2, 2, 2k+3, 2k, 3 \cdot 2^k\right) : k \in \mathbb{N}_0 \right\}$;
- $B = \left\{ \left(a, n, nk, nk+1, a^k \sqrt[n]{a+1}\right), \left(a, n, nk+1, nk, a^k \sqrt[n]{a+1}\right) : k \in \mathbb{N}_0 \right\}$.

In particular, if $a \neq 2$ and $\sqrt[n]{a+1}$ is not an integer, then this equation has no solution.

Proof. WLOG, let $x < y$ and $a \geqslant 2$. We have

$$z^n = a^x + a^y = a^x \left(1 + a^{y-x}\right).$$

Applying Lemma 2.1 in the case where $x = 0$, we get $(a, n, x, y, z) = (2, 2, 0, 3, 3)$, $\left(a, n, 0, 1, \sqrt[n]{a+1}\right)$ as the solutions.

Assume that $x \geqslant 1$. We may write $z = a^k z_0$ such that $k \geqslant 1$ and $a \nmid z_0$. Consider the equation

$$a^{kn} z_0^n = \left(a^k z_0\right)^n = z^n = a^x \left(1 + a^{y-x}\right).$$

Since $a^{kn} \mid a^x \left(1 + a^{y-x}\right)$ and $\gcd\left(a, 1 + a^{y-x}\right) = 1$, we get $a^{kn} \mid a^x$, i.e., $kn \leqslant x$. We rewrite the equation as

$$z_0^n = a^{x-kn}\left(1 + a^{y-x}\right). \tag{2.1}$$

We claim that $x = kn$. Then, we rewrite the equation as $z_0^n = 1 + a^{y-x}$. Applying Lemma 2.1 again, we get

$$(a, n, y - x, z_0) = (2, 2, 3, 3), \left(a, n, 1, \sqrt[n]{a+1}\right)$$

and

$$(a, n, x, y, z_0) = (2, 2, x, x+3, 3), \left(a, n, x, x+1, \sqrt[n]{a+1}\right)$$

because $x < y$. When $x = kn$ and $z = a^k z_0$, we finally get

$$(a, n, x, y, z) = \left(2, 2, 2k, 2k+3, 3\cdot 2^k\right), \left(a, n, kn, kn+1, a^k \sqrt[n]{a+1}\right).$$

It remains to show that $x = kn$. By the fundamental theorem of arithmetic, we write $a = \prod_{i=1}^{s} p_i^{\alpha_i}$, where $s, \alpha_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, s$. Suppose that $x - kn > 0$. From Equation (2.1), it implies that $p_i \mid z_0$ for all i. Thus, z_0 can be rewritten to $z_0 = z_p \left(\prod_{i=1}^{s} p_i^{\beta_i}\right)$, where $z_p, \beta_i \in \mathbb{N}$ and $p_i \nmid z_p$ for all i. By substituting z_0 into Equation (2.1), it forces that $z_p^n = 1 + a^{y-x}$ has a solution and $\beta_i n = \alpha_i(x - kn)$ for all i. Since $a \nmid z_0$, $a = \prod_{i=1}^{s} p_i^{\alpha_i}$ and $z_0 = z_p \left(\prod_{i=1}^{s} p_i^{\beta_i}\right)$, we can set $\alpha_1 > \beta_1$. Consequently, we get $\alpha_i > \beta_i$ for all i because of $\frac{\alpha_1}{\beta_1} > 1$ and $\frac{\alpha_i}{\beta_i} = \frac{n}{x - kn}$ for all i. From this fact, we have $\alpha_i \geqslant 2$ for all i and we assure that $a \neq 2$. By examining the equation $z_p^n = 1 + a^{y-x}$ together with Lemma 2.1, $z_p = \sqrt[n]{a+1}$ is only possible solution if $\sqrt[n]{a+1} \in \mathbb{N}$. Then we consider the equation $z_p = \sqrt[n]{a+1}$ that equivalent to $z_p^n - a = 1$.

If $\gcd_{1 \leqslant i \leqslant s}(\alpha_i) > 1$, then we can apply the Catalan's conjecture and obtain $n = 2$ and $a = 8$ and it implies that $\alpha_1 = 3$ and $\beta_1 = 1$ or 2. Since we know that $\beta_i n = \alpha_i(x - kn)$ for all i, we get

$$3(x - kn) = 2\beta_1 = \begin{cases} 2, & \text{if } \beta_1 = 1, \\ 4, & \text{if } \beta_1 = 2, \end{cases}$$

which is a contradiction. Thus, $\gcd_{1 \leqslant i \leqslant s}(\alpha_i) = 1$. We again consider $\beta_1 n = \alpha_1(x - kn)$. It implies that $\frac{\beta_1 n}{\alpha_1}$ is an integer. As a result, $\frac{n \gcd(\alpha_1, \beta_1)}{\alpha_1}$ is also an integer. Since

$$\alpha_i \left(\frac{\beta_1}{\gcd(\alpha_1, \beta_1)}\right) \left(\frac{n \gcd(\alpha_1, \beta_1)}{\alpha_1}\right) = \alpha_i \left(\frac{\beta_1 n}{\alpha_1}\right) = \beta_i n,$$

we obtain $\alpha_i \left(\frac{\beta_1}{\gcd(\alpha_1, \beta_1)}\right) = \beta_i \left(\frac{\alpha_1}{\gcd(\alpha_1, \beta_1)}\right)$. Therefore, $\frac{\alpha_1}{\gcd(\alpha_1, \beta_1)} \mid \alpha_i$ for all i. It follows that $\frac{\alpha_1}{\gcd(\alpha_1, \beta_1)}$ is a common divisor of α_i for all i, that implies

$$\frac{\alpha_1}{\gcd(\alpha_1, \beta_1)} \leqslant \gcd_{1 \leqslant i \leqslant s}(\alpha_i) = 1.$$

Then, $\alpha_1 \leqslant \gcd(\alpha_1, \beta_1) \leqslant \beta_1$ which contradicts to $\alpha_1 > \beta_1$. Hence, $x - kn$ must be zero. □

Corollary 2.8. *Let p be prime and $n \geqslant 2$ be a positive integer. All solutions of the equation $p^x + p^y = z^n$ are $(p, n, x, y, z) \in A \cup B \cup C$, where*

- $A = \left\{\left(2, n, nk-1, nk-1, 2^k\right) : k \in \mathbb{N}\right\}$;
- $B = \left\{\left(2, 2, 2k, 2k+3, 3\cdot 2^k\right), \left(2, 2, 2k+3, 2k, 3\cdot 2^k\right) : k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2^n - 1, n, nk, nk+1, 2\cdot(2^n-1)^k\right), \left(2^n - 1, n, nk+1, nk, 2\cdot(2^n-1)^k\right) : k \in \mathbb{N}_0\right\}$.

In particular, if $p \neq 2$ and n is a composite number, this equation has no solution.

Proof. This corollary follows directly from Proposition 2.5 and Theorem 2.7. □

Table 1: Some remarkable solutions in the case, where $\sqrt[n]{a+1}$ is an integer satisfying the condition in B of Theorem 2.7.

$a^x + a^y = z^n$	x	y	z	Comments
$3^x + 3^y = z^2$	2	3	6	In the set B, where $n \geqslant 3$, we see that $\sqrt[n]{4}$ is not an integer, so $B = \emptyset$,
	4	5	18	which means that $3^x + 3^y = z^n$ has no solution. Furthermore,
$7^x + 7^y = z^3$	3	4	14	$3^x + 3^y = (p(z))^n$ has no solution when $p(z)$ is a polynomial degree
$15^x + 15^y = z^2$	2	3	60	$\geqslant 1$. With the same trace, the following equations definitely have no
$15^x + 15^y = z^4$	4	5	30	solution:
$21^x + 21^y = z^2$	-	-	-	• $7^x + 7^y = z^n$ for $n \geqslant 4$, • $15^x + 15^y = z^n$ for $n \neq 2, 4$, and • $21^x + 21^y = z^n$ for $n \geqslant 2$.

Table 2: Some remarkable solutions of $p^x + p^y = z^n$, where $p \leqslant 127$.

$p^x + p^y = z^n$	x	y	z	Comments
$5^x + 5^y = z^2$	-	-	-	By the same reason of the comments in Table 1, we can establish the exponential equations with no solution on the nonnegative integer domain such as $p^x + p^y = z^n$ for $n \geqslant 2$ when p is a prime number less than 100 except $p = 3$ with $n = 2$, $p = 7$ with $n = 3$ (see Table 1), and $p = 31$ with $n = 5$. Moreover, the next prime number that makes the exponential equation exist with a solution is $p = 127$ with $n = 7$. This means $p^x + p^y = z^n$ has no solution if $37 \leqslant p \leqslant 113$ and $n \geqslant 2$.
$11^x + 11^y = z^2$	-	-	-	
$13^x + 13^y = z^2$	-	-	-	
$17^x + 17^y = z^2$	-	-	-	
$19^x + 19^y = z^2$	-	-	-	
$23^x + 23^y = z^2$	-	-	-	
$29^x + 29^y = z^2$	-	-	-	
$31^x + 31^y = z^5$	5	6	62	
$127^x + 127^y = z^7$	7	8	254	

Remark 2.9. From Corollary 2.8, if we consider in case $n = 2$, all solutions of the equation $p^x + p^y = z^2$ are $(p, x, y, z) \in A \cup B \cup C$, where

- $A = \left\{ \left(2, 2k-1, 2k-1, 2^k\right) : k \in \mathbb{N} \right\}$;
- $B = \left\{ \left(2, 2k, 2k+3, 3 \cdot 2^k\right), \left(2, 2k+3, 2k, 3 \cdot 2^k\right) : k \in \mathbb{N}_0 \right\}$;
- $C = \left\{ \left(3, 2k, 2k+1, 2 \cdot 3^k\right), \left(3, 2k+1, 2k, 2 \cdot 3^k\right) : k \in \mathbb{N}_0 \right\}$.

As a result, the equation $p^x + p^y = z^2$ has solutions when $p = 2$ or $p = 3$ under some conditions. Additionally, we shall observe that both Remark 2.9 and Theorem 3.1 in [9] provide the same outcomes for $p = 5$. The main results, Theorem 2.1 (for an odd prime p) and Theorem 3.1 (for $p = 2$), in [5] are contained in this remark.

Remark 2.10. Similarly to Remark 2.9, if we consider $n = 4$ in Corollary 2.8, all solutions of the equation $p^x + p^y = z^4$ are

$$(p, x, y, z) \in \left\{ \left(2, 4k-1, 4k-1, 2^k\right) : k \in \mathbb{N} \right\}.$$

As a result, the equation $p^x + p^y = z^4$ has solutions when $p = 2$ under some conditions. These results are studied in [6] as the main results, Theorem 2.1 (for $p = 2$) and Theorem 2.2 (for an odd prime p).

3. All solutions of $a^x - a^y = z^n$

Lemma 3.1. *Let a and $n \geqslant 2$ be positive integers. All solutions of the equation $a^x - 1 = z^n$ are*

$$(a, n, x, z) \in \left\{ (3, 3, 2, 2), (1, n, x, 0), (a, n, 0, 0), \left(a, n, 1, \sqrt[n]{a-1}\right) \right\}.$$

Proof. If $a = 1$ or $x = 0$, then $z = 0$. Thus, $(a, n, x, z) = (1, n, x, 0)$ or $(a, n, x, z) = (a, n, 0, 0)$, respectively. Assume that $a \geqslant 2$ and $x \geqslant 1$. For the case $x = 1$, we get $(a, n, x, z) = \left(a, n, 1, \sqrt[n]{a-1}\right)$. In the remaining case $x \geqslant 2$, applying Catalan's conjecture, we obtain $(a, n, x, z) = (3, 3, 2, 2)$. □

Remark 3.2. If a is equal to zero, then $(a, n, x, z) = (0, n, x, -1)$ is a solution of $a^x - 1 = z^n$ for odd integer $n \geqslant 2$ and positive integer x.

Lemma 3.3. *Let p be prime and $n \geqslant 2$ be a positive integer. All solutions of the equation $p^x - 1 = z^n$ are*

$$(p, n, x, z) \in \left\{(3, 3, 2, 2), (p, n, 0, 0), (2, n, 1, 1), \left((2v)^{2^\ell} + 1, 2^\ell, 1, 2v\right) : v, \ell \in \mathbb{N}\right\}.$$

In particular, this equation has at most two solutions when p and n are fixed.

Proof. By Lemma 3.1, it is sufficient to show only $x = 1$, and the equation becomes $p = z^n + 1$. If $p = 2$, then $(p, n, x, z) = (2, n, 1, 1)$. When $p > 2$, it implies that z is even. Suppose that n is odd. Then

$$p = z^n + 1 = (z + 1)\left(\sum_{i=0}^{n-1} (-1)^i z^{n-1-i}\right),$$

which leads to a contradiction since $z + 1 > 1$, $\sum_{i=0}^{n-1} (-1)^i z^{n-1-i} > 1$ and p has exactly two factors, 1 and itself. Therefore, n is even, and we can write $n = 2^\ell \alpha$ where $\gcd(2, \alpha) = 1$. If $\alpha \neq 1$, we consider

$$p = z^{2^\ell \alpha} + 1 = \left(z^{2^\ell} + 1\right)\left(\sum_{i=0}^{\alpha-1} (-1)^i z^{2^\ell(\alpha-1-i)}\right),$$

which is a contradiction by using the same reasoning. Thus, $\alpha = 1$, and then $n = 2^\ell$. This implies that

$$(p, n, x, z) = \left((2v)^{2^\ell} + 1, 2^\ell, 1, 2v\right).$$

□

Remark 3.4. Using symbolic computation, we know that the number of primes is in the form $(2v)^{2^\ell} + 1$ when $\ell \leqslant 10$ and $v \leqslant 2000$.

ℓ	1	2	3	4	5	6	7	8	9	10
the number of v	383	375	144	140	66	40	14	13	8	9

Moreover, the number of primes is in the form $(2v)^{2^\ell} + 1$ when $\ell = 11, 12, 13$ and $v \leqslant 1000$.

ℓ	11	12	13
the number of v	1	1	0

Theorem 3.5. *Let a and $n \geqslant 2$ be positive integers. If $x > y$, then all solutions of the equation $a^x - a^y = z^n$ are*

$$(a, n, x, y, z) \in \left\{\left(3, 3, 3k + 2, 3k, 2 \cdot 3^k\right), \left(a, n, nk + 1, nk, a^k \sqrt[n]{a-1}\right) : k \in \mathbb{N}_0\right\}.$$

In particular, if $a \neq 3$ and $\sqrt[n]{a-1}$ is not an integer, then this equation has no solution.

Proof. Since $x > y$ and $a \geqslant 2$, the equation $a^x - a^y = z^n$ becomes to

$$a^y\left(a^{x-y} - 1\right) = a^{kn} z_0^n, \tag{3.1}$$

where $z = a^k z_0$ and $a \nmid z_0$. By the same argument of the proof in Theorem 2.7, we can show that $y = kn$ by applying Lemma 3.1 and the Catalan's conjecture. Substituting $y = kn$ into Equation (3.1), it follows that $z_0^n = a^{x-y} - 1$. Applying Lemma 3.1 again, we get

$$(a, n, x, y, z) \in \left\{\left(3, 3, 3k + 2, 3k, 2 \cdot 3^k\right), \left(a, n, nk + 1, nk, a^k \sqrt[n]{a-1}\right)\right\},$$

as desired. □

Remark 3.6. For equation $a^x - a^y = z^n$, where a and $n \geqslant 2$ are positive integers, if $a = 1$ or $x = y$, then $z = 0$, which forces $(a, n, x, y, z) = (1, n, x, y, 0)$ or $(a, n, x, y, z) = (a, n, x, x, 0)$.

Corollary 3.7. *Let p be prime and $n \geqslant 2$ be a positive integer. All solutions of the equation $p^x - p^y = z^n$ are $(p, n, x, y, z) \in A \cup B \cup C$, where*

- $A = \{(p, n, x, x, 0) : x \in \mathbb{N}_0\}$;
- $B = \left\{\left((2v)^{2^\ell} + 1, 2^\ell, 2^\ell k + 1, 2^\ell k, 2vp^k\right) : v, \ell \in \mathbb{N}, k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2, n, nk + 1, nk, 2^k\right), \left(3, 3, 3k + 2, 3k, 2 \cdot 3^k\right) : k \in \mathbb{N}_0\right\}$.

Proof. This corollary follows directly from Theorem 3.5 and Remark 3.6. □

Table 3: Some remarkable solutions in the case, where $(2v)^{2^\ell} + 1$ is prime satisfying the condition in B of Corollary 3.7.

$p^x - p^y = z^n$	x	y	z	Comments
$5^x - 5^y = z^2$	1	0	2	We see that $p^x - p^y = z^n$ has no solution if $x \neq y$, when $p \geqslant 5$
	3	2	10	and n is not a power of two such as $5^x - 5^y = z^3$, $5^x - 5^y = z^5$,
$17^x - 17^y = z^2$	3	2	68	$5^x - 5^y = z^6$. Specifically, $7^x - 7^y = z^n$ has no solution for
$17^x - 17^y = z^4$	5	4	34	$n \geqslant 2$ except the case $x = y$. Similarly, $11^x - 11^y = z^n$,
$37^x - 37^y = z^2$	3	2	222	$13^x - 13^y = z^n$, $19^x - 19^y = z^n$, $23^x - 23^y = z^n$, and
$101^x - 101^y = z^2$	3	2	1010	$31^x - 31^y = z^n$ have no solution for $n \geqslant 2$.

Remark 3.8. From Corollary 3.7, if we consider in case $n = 2$, all solutions of the equation $p^x - p^y = z^2$ are $(p, x, y, z) \in A \cup B \cup C$, where

- $A = \{(p, x, x, 0) : x \in \mathbb{N}_0\}$;
- $B = \left\{\left((2v)^2 + 1, 2k + 1, 2k, 2vp^k\right) : v \in \mathbb{N}, k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2, 2k + 1, 2k, 2^k\right) : k \in \mathbb{N}_0\right\}$.

We also observe that the main results, Theorem 4.1 (for an odd prime p) and Theorem 5.1 (for $p = 2$), in [5] are contained in this remark. Moreover, the conclusion of Theorem 3.7 in [4] and all solutions of the equation $2^x - 2^y = z^2$ in this remark belong to the same set of solutions.

Remark 3.9. Similarly to Remark 3.8, if we consider $n = 4$ in Corollary 3.7, all solutions of the equation $p^x - p^y = z^4$ are $(p, x, y, z) \in A \cup B \cup C$, where

- $A = \{(p, x, x, 0) : x \in \mathbb{N}_0\}$;
- $B = \left\{\left((2v)^4 + 1, 4k + 1, 4k, 2vp^k\right) : v \in \mathbb{N}, k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2, 4k + 1, 4k, 2^k\right) : k \in \mathbb{N}_0\right\}$.

For $p = 2$, all solutions are studied in [4, Corollary 3.8], which is also shown in this remark.

4. Conclusion

In summary, we deliver all solutions of the Diophantine equation $a^x \pm a^y = z^n$.

(i) For general $a \in \mathbb{N}$, our results are as follows.

(a) All solutions of equation $a^x + a^y = z^n$, where $n \geqslant 2$ is a positive integer and $x \neq y$ are $(a, n, x, y, z) \in A \cup B$, where

- $A = \left\{\left(2, 2, 2k, 2k + 3, 3 \cdot 2^k\right), \left(2, 2, 2k + 3, 2k, 3 \cdot 2^k\right) : k \in \mathbb{N}_0\right\}$;

- $B = \left\{\left(a, n, nk, nk+1, a^k \sqrt[n]{a+1}\right), \left(a, n, nk+1, nk, a^k \sqrt[n]{a+1}\right) : k \in \mathbb{N}_0\right\}$.

In particular, if $a \neq 2$ and $\sqrt[n]{a+1}$ is not an integer, then this equation has no solution.

(b) All solutions of the equation $a^x - a^y = z^n$, where $n \geqslant 2$ is a positive integer and $x > y$ are

$$(a, n, x, y, z) \in \left\{\left(3, 3, 3k+2, 3k, 2 \cdot 3^k\right), \left(a, n, nk+1, nk, a^k \sqrt[n]{a-1}\right) : k \in \mathbb{N}_0\right\}.$$

In particular, if $a \neq 3$ and $\sqrt[n]{a-1}$ is not an integer, then this equation has no solution.

(i) Furthermore, in case the base a is a prime p, we give all solutions of $p^x \pm p^y = z^n$ as follows.

(a) All solutions of the equation $p^x + p^y = z^n$, where $n \geqslant 2$ is a positive integer are $(p, n, x, y, z) \in A \cup B \cup C$, where

- $A = \left\{\left(2, n, nk-1, nk-1, 2^k\right) : k \in \mathbb{N}\right\}$;
- $B = \left\{\left(2, 2, 2k, 2k+3, 3 \cdot 2^k\right), \left(2, 2, 2k+3, 2k, 3 \cdot 2^k\right) : k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2^n-1, n, nk, nk+1, 2 \cdot (2^n-1)^k\right), \left(2^n-1, n, nk+1, nk, 2 \cdot (2^n-1)^k\right) : k \in \mathbb{N}_0\right\}$.

In particular, if $p \neq 2$ and n is a composite number, this equation has no solution.

(b) All solutions of the equation $p^x - p^y = z^n$, where $n \geqslant 2$ is a positive integer are $(p, n, x, y, z) \in A \cup B \cup C$, where

- $A = \{(p, n, x, x, 0) : x \in \mathbb{N}_0\}$;
- $B = \left\{\left((2v)^{2^\ell} + 1, 2^\ell, 2^\ell k + 1, 2^\ell k, 2vp^k\right) : v, \ell \in \mathbb{N}, k \in \mathbb{N}_0\right\}$;
- $C = \left\{\left(2, n, nk+1, nk, 2^k\right), \left(3, 3, 3k+2, 3k, 2 \cdot 3^k\right) : k \in \mathbb{N}_0\right\}$.

From the previous results, we claim that all possible cases of the parameter a in $a^x \pm a^y = z^n$ have been consider. Besides, Remarks 2.9, 2.10, 3.8, and 3.9 include details that pertain to current results in [4–6, 9, 18]. Our further work is going to study the behavior of polynomial $p(z)$ in Table 1 and we will also consider in the case $p^x \pm q^y = z^n$ for two distinct primes p and q. Moreover, we have been investigating this issue in the way $l(p^x) + m(q^y) = z^n$, where p, q are distinct primes, and l, m are nonzero integers. The following articles were used to develop the concept: [2, 3, 8, 10–12, 16, 19].

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Declarations

This work is original and has not been published elsewhere, nor is it currently under consideration for publication elsewhere. The authors have no competing interests to declare that are relevant to the content of this article.

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