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Generalizations of integral inequalities similar to Hardy inequality on time scales

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Abstract

In this article, we prove some new inequalities similar to Hardy's inequality on time scales. The results as special cases contain integral inequalities similar to Hardy's inequality and contain discrete inequalities. Our main results are established by using chain rule, Holder inequality, and some properties of multiple integrals on time scales. Furthermore, some applications and examples are given to illustrate the investigated results.

Keywords: Time scales, dynamic inequality, Hardy's inequality.

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1. Introduction

In a note published in 1925, Hardy [7] proved that if $\beta > 1$, θ is a nonnegative integrable function on $(0, \xi)$ and θ^{β} is integrable and convergent on $(0, \infty)$, then

$$\int_0^{\infty} \left(\frac{\Lambda(\xi)}{\xi} \right)^{\beta} d\xi \leqslant \left(\frac{\beta}{\beta - 1} \right)^{\beta} \int_0^{\infty} \theta^{\beta}(\xi) d\xi, \tag{1.1}$$

where

$$\Lambda(\xi) = \int_0^{\xi} \theta(\eta) d\eta, \quad \text{for all} \quad \xi > 0.$$

Furthermore, the constant $(\beta/(\beta-1))^{\beta}$ is the best possible.

In 1964, Levinson [9] proved the inequality (1.1) on a finite interval $[\upsilon_1, \upsilon_2]$ as follows. If $\beta > 1$ and $\theta \geqslant 0$ with $0 < \upsilon_1 < \upsilon_2 < \infty$, then

$$\int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\xi} \right)^{\beta} d\xi \leqslant \left(\frac{\beta}{\beta - 1} \right)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\xi) d\xi, \tag{1.2}$$

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where

$$\Lambda(\xi)=\int_{\upsilon_1}^{\xi}\theta(\eta)d\eta,\quad \text{for all}\quad \xi>\upsilon_1.$$

In 2012, Sulaiman [16] proved that if $\beta \geqslant 1$ and $\theta > 0$ on $[\upsilon_1, \upsilon_2] \subseteq (0, \infty)$, then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\left(\frac{\Lambda(\xi)}{\xi}\right)^{\beta}d\xi\leqslant(\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{\theta(\xi)}{\xi}\right)^{\beta}d\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\left(\frac{\theta(\xi)}{\xi}\right)^{\beta}d\xi. \tag{1.3}$$

Also, in the same paper [16] Sulaiman proved that if $0 < \beta < 1$, then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\left(\frac{\Lambda(\xi)}{\xi}\right)^{\beta}d\xi\geqslant\left(\frac{\upsilon_2-\upsilon_1}{\upsilon_2}\right)^{\beta}\int_{\upsilon_1}^{\upsilon_2}(\theta(\xi))^{\beta}d\xi-\frac{1}{\upsilon_2^{\beta}}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}(\theta(\xi))^{\beta}d\xi, \tag{1.4}$$

where

$$\Lambda(\xi)=\int_{\upsilon_1}^{\xi}\theta(\eta)d\eta,\quad \xi\in[\upsilon_1,\upsilon_2].$$

In 2013, Sroysang [15] generalized the two inequalities (1.3) and (1.4) with another additional parameter $\gamma > 0$, as follows: let $\beta \geqslant 1$, and $\theta > 0$ on $[\upsilon_1, \upsilon_2] \subseteq (0, \infty)$. Then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\Lambda^{\beta}(\xi)}{\xi^{\gamma}}d\xi\leqslant(\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\xi)}{\xi^{\gamma}}d\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\xi)}{\xi^{\gamma}}d\xi. \tag{1.5}$$

If $0 < \beta < 1$, then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\Lambda^{\beta}(\xi)}{\xi^{\gamma}}d\xi\geqslant\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\upsilon_2^{\gamma}}\int_{\upsilon_1}^{\upsilon_2}\theta^{\beta}(\xi)\,d\xi-\frac{1}{\upsilon_2^{\gamma}}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\theta^{\beta}(\xi)\,d\xi, \tag{1.6}$$

where

$$\Lambda(\xi)=\int_{\upsilon_1}^{\xi}\theta(\eta)d\eta,\quad \xi\in[\upsilon_1,\upsilon_2].$$

In 2016, Sroysang et al. [17] generalized the two inequalities (1.5) and (1.6), by replacing ξ^{γ} with a non-decreasing function $\phi^{\gamma}(\xi)$, as follows: let $\beta \geqslant 1$, $\gamma > 0$, and $\theta, \phi > 0$ on $[\upsilon_1, \upsilon_2] \subseteq (0, \infty)$, with ϕ be non-decreasing. Then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}d\xi\leqslant(\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)}d\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)}d\xi. \tag{1.7}$$

If $0 < \beta < 1$, then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}d\xi\geqslant\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)}\int_{\upsilon_1}^{\upsilon_2}\theta^{\beta}(\xi)\,d\xi-\frac{1}{\phi^{\gamma}(\upsilon_2)}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\theta^{\beta}(\xi)\,d\xi, \tag{1.8}$$

where

$$\Lambda(\xi)=\int_{\upsilon_1}^{\xi}\theta(\eta)d\eta,\quad \xi\in[\upsilon_1,\upsilon_2].$$

In 2022, Bendaoud et al. [2] established if $\beta < 0$, $\gamma > 0$, and $\theta, \phi > 0$ on $[\upsilon_1, \upsilon_2] \subseteq (0, \infty)$, with ϕ is non-decreasing, then

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}d\xi\leqslant\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)}\int_{\upsilon_1}^{\upsilon_2}\theta^{\beta}(\xi)\,d\xi-\frac{1}{\phi^{\gamma}(\upsilon_2)}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\theta^{\beta}(\xi)\,d\xi. \tag{1.9}$$

Also, in the same paper [2], the authors obtained the above inequalities (1.7), (1.8), and (1.9), with ϕ is non-increasing as follows: let $\beta \geqslant 1$, $\gamma > 0$, and $\theta, \phi > 0$ on $[\upsilon_1, \upsilon_2] \subseteq (0, \infty)$, with ϕ be non-increasing. Then

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^\beta(\xi)}{\phi^\gamma(\xi)} d\xi \leqslant \frac{(\upsilon_2 - \upsilon_1)^\beta}{\phi^\gamma(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^\beta(\xi)\, d\xi - \frac{1}{\phi^\gamma(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^\beta \theta^\beta(\xi)\, d\xi. \tag{1.10}$$

If $0 < \beta < 1$, then

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^\beta(\xi)}{\phi^\gamma(\xi)} d\xi \geqslant (\upsilon_2 - \upsilon_1)^\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)} d\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^\beta \frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)} d\xi. \tag{1.11}$$

If $\beta < 0$, then

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^\beta(\xi)}{\phi^\gamma(\xi)} d\xi \leqslant (\upsilon_2 - \upsilon_1)^\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)} d\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^\beta \frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)} d\xi, \tag{1.12}$$

where

$$\Lambda(\xi) = \int_{\upsilon_1}^{\xi} \theta(\eta) d\eta, \quad \text{for all} \quad \xi > \upsilon_1.$$

The calculus of time scales was introduced by Hilger in his Ph.D. to devise a method that might unite continuous and discrete analysis [8]. More information about the theory of time scales, see [3, 5, 6].

In 2005, Řehák [10] obtained the time scale version of Hardy's inequality (1.1). In particular, he showed that if $\beta > 1$, θ is a nonnegative function and the delta integral $\int_{\upsilon_1}^{\infty} (\theta(\eta))^\beta \Delta\eta$ exists, then

$$\int_{\upsilon_1}^{\infty} \left(\frac{1}{\sigma(\xi) - \upsilon_1} \int_{\upsilon_1}^{\sigma(\xi)} \theta(\eta) \Delta\eta \right)^\beta \Delta\xi < \left(\frac{\beta}{\beta - 1} \right)^\beta \int_{\upsilon_1}^{\infty} \theta^\beta(\xi) \Delta\xi,$$

unless $\theta \equiv 0$. If, in addition, $\mu(\xi)/\xi \to 0$ as $\xi \to \infty$, then the constant is the best possible. Řehák considered as the first one generalizes Hardy's inequality on time scales. After that many authors established several generalizations of the dynamic Hardy's inequality. Here we mention papers [11–14].

The goal of this paper is to establish new dynamic inequalities on time scales. Our studies show the time scales version of inequalities (1.7)-(1.12). Some discrete and integral inequalities are obtained. This paper is arranged as follows. In Section 2, we cover the fundamental definitions and ideas of calculus on time scales. In Section 3, we state and prove our main results. In Section 4, we give some examples of our main results obtained.

2. Preliminaries

Before, we state our main results, let us review some time scales fundamentals. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\eta) := \inf\{s \in \mathbb{T} : s > \eta\}$, and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ for a time scale $\mathbb{T}$ is defined by $\mu(\eta) := \sigma(\eta) - \eta$. The interval $[\upsilon_1, \upsilon_2]$ in $\mathbb{T}$ is defined by $[\upsilon_1, \upsilon_2]_{\mathbb{T}} = \{\eta \in \mathbb{T} : \upsilon_1 \leqslant \eta \leqslant \upsilon_2\}$, i.e., $[\upsilon_1, \upsilon_2]_{\mathbb{T}} := [\upsilon_1, \upsilon_2] \cap \mathbb{T}$. If $\theta : \mathbb{T} \to \mathbb{R}$ is function, then the function $\theta^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $\theta^\sigma(\eta) = \theta(\sigma(\eta))$, for all $\eta \in \mathbb{T}$, i.e., $\theta^\sigma = \theta \circ \sigma$. For a function $\theta : \mathbb{T} \to \mathbb{R}$, we define $\theta^\Delta(\eta)$ as follows. Let $\eta \in \mathbb{T}$. If for all $\varepsilon > 0$ there exists a neighborhood U of η with

$$|(\theta(\sigma(\eta)) - \theta(s)) - \theta^\Delta(\eta)(\sigma(\eta) - s)| \leqslant \varepsilon |\sigma(\eta) - s)|, \quad \text{for all} \quad s \in U.$$

In this case, $\theta^\Delta(\eta)$ is said to be the delta derivative of θ at η and that θ is delta differentiable at η. In this research we use the delta integral which we may be defined as follows. For $\upsilon_1, \upsilon_2 \in \mathbb{T}$ and θ is a delta differentiable function, Cauchy integral is defined by $\int_{\upsilon_1}^{\upsilon_2} \theta^\Delta(\eta) = \theta(\upsilon_2) - \theta(\upsilon_1)$. An infinite integral is defined as $\int_{\upsilon_1}^{\infty} \theta(\eta)\Delta\eta = \lim_{\upsilon_2 \to \infty} \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)\Delta\eta$. The following key relations between $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ used as special cases of our results.

If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\eta) = \eta, \quad \theta^{\Delta}(\eta) = \theta'(\eta), \quad \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)\Delta\eta = \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)d\eta.$$

If $\mathbb{T} = h\mathbb{Z} = \{\eta = hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\sigma(\eta) = \eta + h, \quad \theta^{\Delta}(\eta) = \frac{\theta(\eta+h) - \theta(\eta)}{h}, \quad \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)\Delta\eta = \sum_{k=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} h\theta(hk).$$

If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\eta) = \eta + 1, \quad \theta^{\Delta}(\eta) = \theta(\eta+1) - \theta(\eta), \quad \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)\Delta\eta = \sum_{\eta=\upsilon_1}^{\upsilon_2-1} \theta(\eta).$$

If $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{\eta = q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$, then

$$\sigma(\eta) = q\eta, \quad \theta^{\Delta}(\eta) = \frac{\theta(qt) - \theta(\eta)}{(q-1)\eta}, \quad \int_{\upsilon_1}^{\upsilon_2} \theta(\eta)\Delta\eta = (q-1) \sum_{k=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^k \theta(q^k).$$

Lemma 2.1 (Chain Rule [5])**.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\phi : \mathbb{T} \to \mathbb{R}$ be a delta differentiable function on $\mathbb{T}^{\kappa}$, and $\theta : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then there is c in the real interval $[\eta, \sigma(\eta)]$ such that*

$$(\theta \circ \phi)^{\Delta}(\eta) = \theta'(\phi(c))\phi^{\Delta}(\eta). \tag{2.1}$$

Lemma 2.2 (Hölder's Inequality [1, Theorem 1.1.10])**.** *Assume $\upsilon_1, \upsilon_2 \in \mathbb{T}$ and $\theta, \phi \in C_{rd}([\upsilon_1, \upsilon_2]_{\mathbb{T}}, [0, \infty))$. If $\beta > 1$ with $1/\beta + 1/\gamma = 1$, then*

$$\int_{\upsilon_1}^{\upsilon_2} |\theta(\eta)\phi(\eta)|\Delta\eta \leqslant \left(\int_{\upsilon_1}^{\upsilon_2} |\theta(\eta)|^{\beta}\Delta\eta\right)^{\frac{1}{\beta}} \left(\int_{\upsilon_1}^{\upsilon_2} |\phi(\eta)|^{\gamma}\Delta\eta\right)^{\frac{1}{\gamma}}. \tag{2.2}$$

The inequality (2.2) is reversed if $0 < \beta < 1$ and if $\beta < 0$ or $\gamma < 0$.

Lemma 2.3 (Fubini's Theorem [4])**.** *Let θ be bounded and Δ-integrable over $R = [u_1, u_2) \times [\upsilon_1, \upsilon_2)$ and assume that the single integrals*

$$X(\xi) = \int_{u_1}^{u_2} \theta(\eta, \xi)\Delta\eta \quad \textit{and} \quad T(\eta) = \int_{\upsilon_1}^{\upsilon_2} \theta(\eta, \xi)\Delta\xi,$$

exist for each $\eta \in [u_1, u_2)$ and for each $\xi \in [\upsilon_1, \upsilon_2)$, respectively. Then the iterated integrals

$$\int_{u_1}^{u_2} \Delta\eta \int_{\upsilon_1}^{\upsilon_2} \theta(\eta, \xi)\Delta\xi \quad \textit{and} \quad \int_{\upsilon_1}^{\upsilon_2} \Delta\xi \int_{u_1}^{u_2} \theta(\eta, \xi)\Delta\eta,$$

exist and the equality

$$\int_{u_1}^{u_2} \Delta\eta \int_{\upsilon_1}^{\upsilon_2} \theta(\eta, \xi)\Delta\xi = \int_{\upsilon_1}^{\upsilon_2} \Delta\xi \int_{u_1}^{u_2} \theta(\eta, \xi)\Delta\eta,$$

holds.

3. Main results

Theorem 3.1. *Let* $\theta, \phi, \omega > 0$, *be* Δ*-integrable functions over* $[\upsilon_1, \upsilon_2]_{\mathbb{T}} \subseteq (0, \infty)$, $\gamma > 0$ *and define*

$$\Lambda(\xi) := \int_{\upsilon_1}^{\xi} \theta(\eta)\Delta\eta, \quad \xi \in [\upsilon_1, \upsilon_2]_{\mathbb{T}}. \tag{3.1}$$

If $\beta > 1$, *then we have following.*

i) *Assume* ϕ *is non-decreasing, then*

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi \leqslant \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)\,\omega(\eta)}{\phi^{\gamma}(\eta)} \left(\int_{\eta}^{\upsilon_2} \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \right)\Delta\eta. \tag{3.2}$$

ii) *Assume* ϕ *is non-increasing, then*

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi \leqslant \frac{1}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\eta)\,\omega(\eta) \left(\int_{\eta}^{\upsilon_2} \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \right)\Delta\eta. \tag{3.3}$$

Proof. Utilizing Hölder's inequality (2.2) and Fubini's Theorem 2.3 on time scales, we obtain that

i) If ϕ is non-decreasing, then

$$\begin{aligned}
\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi &= \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi) \left(\int_{\upsilon_1}^{\xi} \theta(\eta)\,\omega^{\frac{1}{\beta}}(\eta)\,\frac{1}{\omega^{\frac{1}{\beta}}(\eta)}\Delta\eta \right)^{\beta} \Delta\xi \\
&\leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi) \left[\left(\int_{\upsilon_1}^{\xi} \theta^{\beta}(\eta)\omega(\eta)\Delta\eta \right)^{\frac{1}{\beta}} \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\frac{\beta-1}{\beta}} \right]^{\beta} \Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi) \left(\int_{\upsilon_1}^{\xi} \theta^{\beta}(\eta)\omega(\eta)\Delta\eta \right) \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2} \int_{\upsilon_1}^{\xi} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta) \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\eta\Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2} \Delta\xi \int_{\upsilon_1}^{\xi} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta) \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\eta \\
&= \int_{\upsilon_1}^{\upsilon_2} \Delta\eta \int_{\eta}^{\upsilon_2} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta) \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \\
&\leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta)\theta^{\beta}(\eta)\omega(\eta) \left[\int_{\eta}^{\upsilon_2} \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \right] \Delta\eta.
\end{aligned}$$

ii) If ϕ is non-increasing, then, we proceed as in the proof (i) to obtain that

$$\begin{aligned}
\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi &\leqslant \int_{\upsilon_1}^{\upsilon_2} \Delta\eta \int_{\eta}^{\upsilon_2} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta) \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \\
&\leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta)\omega(\eta) \left[\int_{\eta}^{\upsilon_2} \left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta \right)^{\beta-1} \Delta\xi \right] \Delta\eta.
\end{aligned}$$

The proof is complete. □

Corollary 3.2. *In Theorem 3.1, if we put* $\omega(\eta) = 1$, *inequalities* (3.2) *and* (3.3), *respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \leqslant (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi, \tag{3.4}$$

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \leqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\xi) \Delta\xi - \frac{1}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \theta^{\beta}(\xi) \Delta\xi. \tag{3.5}$$

Proof. By putting $\omega(\eta) = 1$ in (3.2), we obtain that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta) \theta^{\beta}(\eta) \left(\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta - 1} \Delta\xi \right) \Delta\eta. \tag{3.6}$$

Using chain rule (2.1), there exists $c \in [\xi, \sigma(\xi)]$ such that

$$\left[(\xi - \upsilon_1)^{\beta} \right]^{\Delta} = \beta (c - \upsilon_1)^{\beta - 1} \geqslant \beta (\xi - \upsilon_1)^{\beta - 1}.$$

This implies

$$(\xi - \upsilon_1)^{\beta - 1} \leqslant \frac{1}{\beta} \left[(\xi - \upsilon_1)^{\beta} \right]^{\Delta}.$$

Thus, we have

$$\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta - 1} \Delta\xi \leqslant \frac{1}{\beta} \int_{\eta}^{\upsilon_2} \left[(\xi - \upsilon_1)^{\beta} \right]^{\Delta} \Delta\xi = \frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta}. \tag{3.7}$$

Combining (3.6) and (3.7), we get that

$$\begin{aligned} \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi &\leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta) \theta^{\beta}(\eta) \left(\frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta} \right) \Delta\eta \\ &= \frac{1}{\beta} (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\eta)} \Delta\eta - \frac{1}{\beta} \int_{\upsilon_1}^{\upsilon_2} (\eta - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\eta)} \Delta\eta, \end{aligned}$$

which is (3.4). Now, by putting $\omega(\eta) = 1$ in (3.3), we obtain that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2) \theta^{\beta}(\eta) \left(\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta - 1} \Delta\xi \right) \Delta\eta. \tag{3.8}$$

Combining (3.8) and (3.7), we get that

$$\begin{aligned} \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi &\leqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2) \theta^{\beta}(\eta) \left(\frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta} \right) \Delta\eta \\ &= \frac{1}{\beta} (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\upsilon_2)} \Delta\eta - \frac{1}{\beta} \int_{\upsilon_1}^{\upsilon_2} (\eta - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\upsilon_2)} \Delta\eta, \end{aligned}$$

which is (3.5). The proof is complete. □

Remark 3.3. In Corollary 3.2, if $\mathbb{T} = \mathbb{R}$, then, the inequalities (3.4) and (3.5) reduce to inequalities (1.7) and (1.10), respectively.

Remark 3.4. In Corollary 3.2, if $\mathbb{T} = \mathbb{R}$ and $\phi(\xi) = \xi$, then, the inequality (3.4) reduces to inequality (1.5).

Corollary 3.5. *In Corollary 3.2, setting* $\gamma = \beta$, *the inequalities* (3.4) *and* (3.5), *respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)} \right)^{\beta} \Delta\xi \leqslant (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\theta(\xi)}{\phi(\xi)} \right)^{\beta} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \left(\frac{\theta(\xi)}{\phi(\xi)} \right)^{\beta} \Delta\xi, \tag{3.9}$$

$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)} \right)^{\beta} \Delta\xi \leqslant \left(\frac{\upsilon_2 - \upsilon_1}{\phi(\upsilon_2)} \right)^{\beta} \int_{\upsilon_1}^{\upsilon_2} (\theta(\xi))^{\beta} \Delta\xi - \frac{1}{\phi^{\beta}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} (\theta(\xi))^{\beta} \Delta\xi. \tag{3.10}$$

Remark 3.6. In Corollary 3.5, if $\mathbb{T} = \mathbb{R}$, then, the inequality (3.10) reduces to inequality (16) in [2].

Corollary 3.7. *In Corollary 3.5, if* $\phi(\xi) = \xi$, *then, the inequality* (3.9) *becomes*

$$\begin{aligned}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{\Lambda(\xi)}{\xi}\right)^{\beta}\Delta\xi &\leqslant \int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\beta\xi^{\beta}} - \frac{(\xi-\upsilon_1)^{\beta}}{\beta\xi^{\beta}}\right)\theta^{\beta}(\xi)\Delta\xi \\ &\leqslant \int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\beta\xi^{\beta}}\right)\theta^{\beta}(\xi)\Delta\xi \\ &\leqslant \int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\beta}}{\beta\upsilon_1{}^{\beta}}\right)\theta^{\beta}(\xi)\Delta\xi \\ &\leqslant \int_{\upsilon_1}^{\upsilon_2}\left(\frac{\upsilon_2-\upsilon_1}{\upsilon_1}\right)^{\beta}\theta^{\beta}(\xi)\Delta\xi \\ &= \left(\frac{\upsilon_2-\upsilon_1}{\upsilon_1}\right)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\theta^{\beta}(\xi)\Delta\xi.\end{aligned}$$

For $\mathbb{T} = \mathbb{R}$, *we have*

$$\int_{\upsilon_1}^{\upsilon_2}\left(\frac{\Lambda(\xi)}{\xi}\right)^{\beta}d\xi \leqslant \left(\frac{\upsilon_2-\upsilon_1}{\upsilon_1}\right)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\theta^{\beta}(\xi)d\xi,$$

which is similar to Levinson inequality (1.2) *with the constant* $\left(\frac{\upsilon_2-\upsilon_1}{\upsilon_1}\right)^{\beta}$.

Remark 3.8. In Corollary 3.5, if $\mathbb{T} = \mathbb{R}$ and $\phi(\xi) = \xi$, then, the inequality (3.9) reduces to inequality (1.3).

Corollary 3.9. *In Corollary 3.2, if* $\mathbb{T} = h\mathbb{Z}$, $h > 0$, *then, the inequalities* (3.4) *and* (3.5), *respectively, reduce to*

- *if* ϕ *is non-decreasing, then*

$$\beta\sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}\frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \leqslant (\upsilon_2-\upsilon_1)^{\beta}\sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}\frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} - \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}(h\tau-\upsilon_1)^{\beta}\frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)};$$

- *if* ϕ *is non-increasing, then*

$$\beta\sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}\frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \leqslant \frac{(\upsilon_2-\upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)}\sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}\theta^{\beta}(h\tau) - \frac{1}{\phi^{\gamma}(\upsilon_2)}\sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1}(h\tau-\upsilon_1)^{\beta}\theta^{\beta}(h\tau),$$

where

$$\Lambda(h\tau) = \sum_{k=\frac{\upsilon_1}{h}}^{\tau-1}h\theta(hk), \quad \textit{for all} \quad \tau > \upsilon_1.$$

Corollary 3.10. *In Corollary 3.2, if* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, *then, the inequalities* (3.4) *and* (3.5), *respectively, reduce to*

- *if* ϕ *is non-decreasing, then*

$$\beta\sum_{\tau=\log_q\upsilon_1}^{\log_q\upsilon_2-1}q^{\tau}\frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \leqslant (\upsilon_2-\upsilon_1)^{\beta}\sum_{\tau=\log_q\upsilon_1}^{\log_q\upsilon_2-1}q^{\tau}\frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} - \sum_{\tau=\log_q\upsilon_1}^{\log_q\upsilon_2-1}q^{\tau}(q^{\tau}-\upsilon_1)^{\beta}\frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})};$$

- *if ϕ is non-increasing, then*

$$\beta \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \leqslant \frac{(\upsilon_2-\upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau}\theta^{\beta}(q^{\tau}) - \frac{1}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau}(q^{\tau}-\upsilon_1)^{\beta}\theta^{\beta}(q^{\tau}),$$

where

$$\Lambda(q^{\tau}) = (q-1)\sum_{k=\log_q \upsilon_1}^{\tau-1} q^{k}\theta(q^{k}), \quad \textit{for all} \quad \tau > \upsilon_1.$$

Theorem 3.11. *Let $\theta, \phi, \omega > 0$, be Δ-integrable functions over $[\upsilon_1, \upsilon_2]_{\mathbb{T}} \subseteq (0, \infty)$, $\gamma > 0$ and $\Lambda(\xi)$ be defined as in (3.1). If $\beta < 1$, then we have*

i) *assume ϕ is non-decreasing, then*

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi \geqslant \frac{1}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\eta)\,\omega(\eta)\left(\int_{\eta}^{\upsilon_2}\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi\right)\Delta\eta; \tag{3.11}$$

ii) *assume ϕ is non-increasing, then*

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi \geqslant \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)\,\omega(\eta)}{\phi^{\gamma}(\eta)}\left(\int_{\eta}^{\upsilon_2}\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi\right)\Delta\eta. \tag{3.12}$$

Proof. Utilizing reversed Hölder's inequality (2.2) and Fubini's Theorem 2.3 on time scales, we get that

i) if ϕ is non-decreasing, then

$$\begin{aligned}
\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi &= \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi)\left(\int_{\upsilon_1}^{\xi} \theta(\eta)\,\omega^{\frac{1}{\beta}}(\eta)\,\frac{1}{\omega^{\frac{1}{\beta}}(\eta)}\Delta\eta\right)^{\beta}\Delta\xi \\
&\geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi)\left[\left(\int_{\upsilon_1}^{\xi} \theta^{\beta}(\eta)\omega(\eta)\Delta\eta\right)^{\frac{1}{\beta}}\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\frac{\beta-1}{\beta}}\right]^{\beta}\Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\xi)\left(\int_{\upsilon_1}^{\xi} \theta^{\beta}(\eta)\omega(\eta)\Delta\eta\right)\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2}\int_{\upsilon_1}^{\xi} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta)\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\eta\Delta\xi \\
&= \int_{\upsilon_1}^{\upsilon_2}\Delta\xi\int_{\upsilon_1}^{\xi} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta)\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\eta \\
&= \int_{\upsilon_1}^{\upsilon_2}\Delta\eta\int_{\eta}^{\upsilon_2} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta)\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi \\
&\geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta)\omega(\eta)\left[\int_{\eta}^{\upsilon_2}\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi\right]\Delta\eta;
\end{aligned}$$

ii) if ϕ is non-increasing, then, we proceed as in the proof (i) to obtain that

$$\begin{aligned}
\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi &\geqslant \int_{\upsilon_1}^{\upsilon_2}\Delta\eta\int_{\eta}^{\upsilon_2} \phi^{-\gamma}(\xi)\theta^{\beta}(\eta)\omega(\eta)\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi \\
&\geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta)\theta^{\beta}(\eta)\omega(\eta)\left[\int_{\eta}^{\upsilon_2}\left(\int_{\upsilon_1}^{\xi} \frac{1}{\omega^{\frac{1}{\beta-1}}(\eta)}\Delta\eta\right)^{\beta-1}\Delta\xi\right]\Delta\eta.
\end{aligned}$$

The proof is complete. □

In the following, we have concluded some corollaries and remarks on the previous Theorem 3.11 with the condition $0 < \beta < 1$.

Corollary 3.12. *In Theorem 3.11, in the case $0 < \beta < 1$, if we put $\omega(\eta) = 1$, inequalities (3.11) and (3.12), respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \geqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\xi)\Delta\xi - \frac{1}{\phi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta}\theta^{\beta}(\xi)\Delta\xi, \tag{3.13}$$

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \geqslant (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi. \tag{3.14}$$

Proof. By putting $\omega(\eta) = 1$ in (3.11), we get that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta) \left(\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta-1} \Delta\xi \right) \Delta\eta. \tag{3.15}$$

Using chain rule (2.1), there exists $c \in [\xi, \sigma(\xi)]$ such that

$$\left[(\xi - \upsilon_1)^{\beta}\right]^{\Delta} = \beta(c - \upsilon_1)^{\beta-1} \leqslant \beta(\xi - \upsilon_1)^{\beta-1}.$$

This implies

$$(\xi - \upsilon_1)^{\beta-1} \geqslant \frac{1}{\beta}\left[(\xi - \upsilon_1)^{\beta}\right]^{\Delta}.$$

Thus

$$\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta-1} \Delta\xi \geqslant \frac{1}{\beta} \int_{\eta}^{\upsilon_2} \left[(\xi - \upsilon_1)^{\beta}\right]^{\Delta} \Delta\xi = \frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta}. \tag{3.16}$$

Combining (3.15) and (3.16), we get that

$$\begin{aligned} \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi &\geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta) \left(\frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta} \right) \Delta\eta \\ &= \frac{1}{\beta}(\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\upsilon_2)} \Delta\eta - \frac{1}{\beta} \int_{\upsilon_1}^{\upsilon_2} (\eta - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\upsilon_2)} \Delta\eta, \end{aligned}$$

which is (3.13). Now, by putting $\omega(\eta) = 1$ in (3.12), we get that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi \geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta)\theta^{\beta}(\eta) \left(\int_{\eta}^{\upsilon_2} (\xi - \upsilon_1)^{\beta-1} \Delta\xi \right) \Delta\eta. \tag{3.17}$$

Combining (3.17) and (3.16), we get that

$$\begin{aligned} \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi &\geqslant \int_{\upsilon_1}^{\upsilon_2} \phi^{-\gamma}(\eta)\theta^{\beta}(\eta) \left(\frac{(\upsilon_2 - \upsilon_1)^{\beta} - (\eta - \upsilon_1)^{\beta}}{\beta} \right) \Delta\eta \\ &= \frac{1}{\beta}(\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\eta)} \Delta\eta - \frac{1}{\beta} \int_{\upsilon_1}^{\upsilon_2} (\eta - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\eta)}{\phi^{\gamma}(\eta)} \Delta\eta, \end{aligned}$$

which is (3.14). The proof is complete. □

Remark 3.13. In Corollary 3.12, if $\mathbb{T} = \mathbb{R}$, then, the inequalities (3.13) and (3.14) reduce to inequalities (1.8) and (1.11), respectively.

Remark 3.14. In Corollary 3.12, if $\mathbb{T} = \mathbb{R}$ and $\phi(\xi) = \xi$, then, the inequality (3.13) reduces to inequality (1.6).

Corollary 3.15. *In Corollary 3.12, setting $\gamma = \beta$, the inequalities (3.13) and (3.14), respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi \geqslant \left(\frac{\upsilon_2 - \upsilon_1}{\phi(\upsilon_2)}\right)^{\beta} \int_{\upsilon_1}^{\upsilon_2} (\theta(\xi))^{\beta} \Delta\xi - \frac{1}{\phi^{\beta}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} (\theta(\xi))^{\beta} \Delta\xi, \tag{3.18}$$

$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi \geqslant (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\theta(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \left(\frac{\theta(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi. \tag{3.19}$$

Remark 3.16. In Corollary 3.15, if $\mathbb{T} = \mathbb{R}$, then, the inequality (3.19) reduces to inequality (17) in [2].

Remark 3.17. In Corollary 3.15, if $\mathbb{T} = \mathbb{R}$ and $\phi(\xi) = \xi$, then, the inequality (3.18) reduces to inequality (1.4).

Corollary 3.18. *In Corollary 3.12, if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then, the inequalities (3.13) and (3.14), respectively, reduce to*

- *if ϕ is non-decreasing, then*

$$\beta \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \geqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \theta^{\beta}(h\tau) - \frac{1}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} (h\tau - \upsilon_1)^{\beta} \theta^{\beta}(h\tau);$$

- *if ϕ is non-increasing, then*

$$\beta \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \geqslant (\upsilon_2 - \upsilon_1)^{\beta} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} - \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} (h\tau - \upsilon_1)^{\beta} \frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)},$$

where

$$\Lambda(h\tau) = \sum_{k=\frac{\upsilon_1}{h}}^{\tau-1} h\theta(hk), \quad \textit{for all} \quad \tau > \upsilon_1.$$

Corollary 3.19. *In Corollary 3.12, if $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, then, the inequalities (3.13) and (3.14), respectively, reduce to*

- *if ϕ is non-decreasing, then*

$$\beta \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \geqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \theta^{\beta}(q^{\tau}) - \frac{1}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} (q^{\tau} - \upsilon_1)^{\beta} \theta^{\beta}(q^{\tau});$$

- *if ϕ is non-increasing, then*

$$\beta \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \geqslant (\upsilon_2 - \upsilon_1)^{\beta} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} - \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} (q^{\tau} - \upsilon_1)^{\beta} \frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})},$$

where

$$\Lambda(q^{\tau}) = (q-1) \sum_{k=\log_q \upsilon_1}^{\tau-1} q^{k} \theta(q^{k}), \quad \textit{for all} \quad \tau > \upsilon_1.$$

Also, in the following, we have concluded some corollaries and remarks on the Theorem 3.11, but with the condition $\beta < 0$.

Corollary 3.20. *In Theorem 3.11, in the case $\beta < 0$, if we put $\omega(\eta) = 1$, inequalities (3.11) and (3.12), respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \leqslant \frac{(\upsilon_2-\upsilon_1)^{\beta}}{\varphi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} \theta^{\beta}(\xi)\Delta\xi - \frac{1}{\varphi^{\gamma}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi-\upsilon_1)^{\beta}\theta^{\beta}(\xi)\Delta\xi, \tag{3.20}$$

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \leqslant (\upsilon_2-\upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\xi)}{\varphi^{\gamma}(\xi)}\Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\xi)}{\varphi^{\gamma}(\xi)}\Delta\xi. \tag{3.21}$$

Proof. By putting $\omega(\eta) = 1$ in (3.11), we clarify that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \geqslant \int_{\upsilon_1}^{\upsilon_2} \varphi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta)\left(\int_{\eta}^{\upsilon_2}(\xi-\upsilon_1)^{\beta-1}\Delta\xi\right)\Delta\eta. \tag{3.22}$$

Using chain rule (2.1), there exists $c \in [\xi, \sigma(\xi)]$ such that

$$\left[(\xi-\upsilon_1)^{\beta}\right]^{\Delta} = \beta(c-\upsilon_1)^{\beta-1} \geqslant \beta(\xi-\upsilon_1)^{\beta-1}.$$

This implies

$$(\xi-\upsilon_1)^{\beta-1} \geqslant \frac{1}{\beta}\left[(\xi-\upsilon_1)^{\beta}\right]^{\Delta}.$$

Thus

$$\int_{\eta}^{\upsilon_2}(\xi-\upsilon_1)^{\beta-1}\Delta\xi \geqslant \frac{1}{\beta}\int_{\eta}^{\upsilon_2}\left[(\xi-\upsilon_1)^{\beta}\right]^{\Delta}\Delta\xi = \frac{(\upsilon_2-\upsilon_1)^{\beta}-(\eta-\upsilon_1)^{\beta}}{\beta}. \tag{3.23}$$

Combining (3.22) and (3.23), we get that

$$\begin{aligned}\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi &\geqslant \int_{\upsilon_1}^{\upsilon_2} \varphi^{-\gamma}(\upsilon_2)\theta^{\beta}(\eta)\left(\frac{(\upsilon_2-\upsilon_1)^{\beta}-(\eta-\upsilon_1)^{\beta}}{\beta}\right)\Delta\eta \\ &= \frac{1}{\beta}(\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\upsilon_2)}\Delta\eta - \frac{1}{\beta}\int_{\upsilon_1}^{\upsilon_2}(\eta-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\upsilon_2)}\Delta\eta.\end{aligned}$$

This leads to

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \leqslant (\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\upsilon_2)}\Delta\eta - \int_{\upsilon_1}^{\upsilon_2}(\eta-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\upsilon_2)}\Delta\eta,$$

which is (3.20). Now, by putting $\omega(\eta) = 1$ in (3.12), we clarify that

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \geqslant \int_{\upsilon_1}^{\upsilon_2} \varphi^{-\gamma}(\eta)\theta^{\beta}(\eta)\left(\int_{\eta}^{\upsilon_2}(\xi-\upsilon_1)^{\beta-1}\Delta\xi\right)\Delta\eta. \tag{3.24}$$

Combining (3.24) and (3.23), we get that

$$\begin{aligned}\int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi &\geqslant \int_{\upsilon_1}^{\upsilon_2} \varphi^{-\gamma}(\eta)\theta^{\beta}(\eta)\left(\frac{(\upsilon_2-\upsilon_1)^{\beta}-(\eta-\upsilon_1)^{\beta}}{\beta}\right)\Delta\eta \\ &= \frac{1}{\beta}(\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\eta)}\Delta\eta - \frac{1}{\beta}\int_{\upsilon_1}^{\upsilon_2}(\eta-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\eta)}\Delta\eta.\end{aligned}$$

This leads to

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{\Lambda^{\beta}(\xi)}{\varphi^{\gamma}(\xi)} \Delta\xi \leqslant (\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\eta)}\Delta\eta - \int_{\upsilon_1}^{\upsilon_2}(\eta-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\eta)}{\varphi^{\gamma}(\eta)}\Delta\eta,$$

which is the required inequality (3.21). The proof is complete. □

Remark 3.21. In Corollary 3.20, if $\mathbb{T} = \mathbb{R}$, then, the inequalities (3.20) and (3.21) reduce to inequalities (1.9) and (1.12), respectively.

Corollary 3.22. *In Corollary 3.20, setting $\gamma = \beta$, the inequalities (3.20) and (3.21), respectively, reduce to*

$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi \leqslant \left(\frac{\upsilon_2 - \upsilon_1}{\phi(\upsilon_2)}\right)^{\beta} \int_{\upsilon_1}^{\upsilon_2} (\theta(\xi))^{\beta} \Delta\xi - \frac{1}{\phi^{\beta}(\upsilon_2)} \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} (\theta(\xi))^{\beta} \Delta\xi,$$
$$\beta \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\Lambda(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi \leqslant (\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \left(\frac{\theta(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \left(\frac{\theta(\xi)}{\phi(\xi)}\right)^{\beta} \Delta\xi.$$

Corollary 3.23. *In Corollary 3.20, if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then, the inequalities (3.20) and (3.21), respectively, reduce to*

- *if ϕ is non-decreasing, then*

$$\beta \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \leqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \theta^{\beta}(h\tau) - \frac{1}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} (h\tau - \upsilon_1)^{\beta} \theta^{\beta}(h\tau);$$

- *if ϕ is non-increasing, then*

$$\beta \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\Lambda^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} \leqslant (\upsilon_2 - \upsilon_1)^{\beta} \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} \frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)} - \sum_{\tau=\frac{\upsilon_1}{h}}^{\frac{\upsilon_2}{h}-1} (h\tau - \upsilon_1)^{\beta} \frac{\theta^{\beta}(h\tau)}{\phi^{\gamma}(h\tau)},$$

where

$$\Lambda(h\tau) = \sum_{k=\frac{\upsilon_1}{h}}^{\tau-1} h\theta(hk), \quad \text{for all} \quad \tau > \upsilon_1.$$

Corollary 3.24. *In Corollary 3.20, if $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, then, the inequalities (3.20) and (3.21), respectively, reduce to*

- *if ϕ is non-decreasing, then*

$$\beta \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \leqslant \frac{(\upsilon_2 - \upsilon_1)^{\beta}}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \theta^{\beta}(q^{\tau}) - \frac{1}{\phi^{\gamma}(\upsilon_2)} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} (q^{\tau} - \upsilon_1)^{\beta} \theta^{\beta}(q^{\tau});$$

- *if ϕ is non-increasing, then*

$$\beta \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\Lambda^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} \leqslant (\upsilon_2 - \upsilon_1)^{\beta} \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} \frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})} - \sum_{\tau=\log_q \upsilon_1}^{\log_q \upsilon_2 - 1} q^{\tau} (q^{\tau} - \upsilon_1)^{\beta} \frac{\theta^{\beta}(q^{\tau})}{\phi^{\gamma}(q^{\tau})},$$

where

$$\Lambda(q^{\tau}) = (q - 1) \sum_{k=\log_q \upsilon_1}^{\tau-1} q^{k} \theta(q^{k}), \quad \text{for all} \quad \tau > \upsilon_1.$$

4. Examples

In this section, we give some examples to illustrate our results obtained.

Example 4.1. Define functions $\theta, \phi : [\upsilon_1, \upsilon_2]_{\mathbb{T}} \to \mathbb{R}$ by $\theta(\eta) = 1$ and $\phi(\xi) = 1$. Then $\theta, \phi > 0$ and ϕ is non-decreasing. Applying Corollary 3.2 with $\beta = 2$ and $\gamma = 2$, the left side of (3.4) becomes

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{1}{\phi^{\gamma}(\xi)} \left(\int_{\upsilon_1}^{\xi} \theta(\eta) \Delta\eta \right)^{\beta} \Delta\xi = 2 \int_{\upsilon_1}^{\upsilon_2} \left(\int_{\upsilon_1}^{\xi} \Delta\eta \right)^{2} \Delta\xi = 2 \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi. \tag{4.1}$$

Using chain rule (2.1), there exists $c \in [\xi, \sigma(\xi)]$ such that

$$\left[(\xi - \upsilon_1)^3\right]^{\Delta} = 3(c - \upsilon_1)^2 \geqslant 3(\xi - \upsilon_1)^2.$$

This implies

$$2 \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi \leqslant \frac{2}{3} \int_{\upsilon_1}^{\upsilon_2} \left[(\xi - \upsilon_1)^3\right]^{\Delta} \Delta\xi = \frac{2}{3}(\upsilon_2 - \upsilon_1)^3. \tag{4.2}$$

Combining (4.1) and (4.2), we find that the left side of (3.4) is as follows

$$\beta \int_{\upsilon_1}^{\upsilon_2} \frac{1}{\phi^{\gamma}(\xi)} \left(\int_{\upsilon_1}^{\xi} \theta(\eta) \Delta\eta \right)^{\beta} \Delta\xi \leqslant \frac{2}{3}(\upsilon_2 - \upsilon_1)^3. \tag{4.3}$$

For the right side of (3.4), we have

$$\begin{aligned}
(\upsilon_2 - \upsilon_1)^{\beta} \int_{\upsilon_1}^{\upsilon_2} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^{\beta} \frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)} \Delta\xi &= (\upsilon_2 - \upsilon_1)^2 \int_{\upsilon_1}^{\upsilon_2} \Delta\xi - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi \\
&= (\upsilon_2 - \upsilon_1)^2 (\upsilon_2 - \upsilon_1) - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi \\
&= (\upsilon_2 - \upsilon_1)^3 - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi.
\end{aligned} \tag{4.4}$$

Using chain rule (2.1), we have

$$\int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi \leqslant \frac{1}{3} \int_{\upsilon_1}^{\upsilon_2} \left[(\xi - \upsilon_1)^3\right]^{\Delta} \Delta\xi = \frac{1}{3}(\upsilon_2 - \upsilon_1)^3. \tag{4.5}$$

This implies

$$\begin{aligned}
-\int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi &\geqslant -\frac{1}{3}(\upsilon_2 - \upsilon_1)^3, \\
(\upsilon_2 - \upsilon_1)^3 - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi &\geqslant (\upsilon_2 - \upsilon_1)^3 - \frac{1}{3}(\upsilon_2 - \upsilon_1)^3, \\
(\upsilon_2 - \upsilon_1)^3 - \int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi &\geqslant \frac{2}{3}(\upsilon_2 - \upsilon_1)^3.
\end{aligned} \tag{4.6}$$

Also, from (4.5), since $\frac{1}{3}(\upsilon_2 - \upsilon_1)^3 \leqslant (\upsilon_2 - \upsilon_1)^3$, we have

$$\int_{\upsilon_1}^{\upsilon_2} (\xi - \upsilon_1)^2 \Delta\xi \leqslant (\upsilon_2 - \upsilon_1)^3.$$

Since $\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi$ is nonnegative, then

$$(\upsilon_2-\upsilon_1)^3-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi\leqslant(\upsilon_2-\upsilon_1)^3. \tag{4.7}$$

Combining (4.4), (4.6), and (4.7), we find that the right side of (3.4) is as follows

$$\frac{2}{3}(\upsilon_2-\upsilon_1)^3\leqslant(\upsilon_2-\upsilon_1)^\beta\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)}\Delta\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^\beta\frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)}\Delta\xi\leqslant(\upsilon_2-\upsilon_1)^3. \tag{4.8}$$

So, it is clear from (4.3) and (4.8) that they confirm the result described by (3.4) in Corollary 3.2.

Example 4.2. Define functions $\theta,\phi:[\upsilon_1,\upsilon_2]_{\mathbb{T}}\to\mathbb{R}$ by $\theta(\eta)=\eta-\upsilon_1$ and $\phi(\xi)=\xi-\upsilon_1$. Then $\theta,\phi>0$ and ϕ is non-decreasing. Applying Corollary 3.2 with $\beta=2$ and $\gamma=2$, the left side of (3.4) becomes

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{1}{\phi^\gamma(\xi)}\left(\int_{\upsilon_1}^{\xi}\theta(\eta)\Delta\eta\right)^\beta\Delta\xi=2\int_{\upsilon_1}^{\upsilon_2}\frac{1}{(\xi-\upsilon_1)^2}\left(\int_{\upsilon_1}^{\xi}(\eta-\upsilon_1)\Delta\eta\right)^2\Delta\xi. \tag{4.9}$$

Using chain rule (2.1), there exists $c\in[\eta,\sigma(\eta)]$ such that

$$\left[(\eta-\upsilon_1)^2\right]^\Delta=2(c-\upsilon_1)\geqslant2(\eta-\upsilon_1).$$

This implies

$$\int_{\upsilon_1}^{\xi}(\eta-\upsilon_1)\Delta\eta\leqslant\frac{1}{2}\int_{\upsilon_1}^{\xi}\left[(\eta-\upsilon_1)^2\right]^\Delta\Delta\eta=\frac{1}{2}(\xi-\upsilon_1)^2. \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{1}{\phi^\gamma(\xi)}\left(\int_{\upsilon_1}^{\xi}\theta(\eta)\Delta\eta\right)^\beta\Delta\xi\leqslant2\int_{\upsilon_1}^{\upsilon_2}\frac{1}{(\xi-\upsilon_1)^2}\left(\frac{(\xi-\upsilon_1)^2}{2}\right)^2\Delta\xi=\frac{1}{2}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi. \tag{4.11}$$

Using chain rule (2.1), there exists $c\in[\xi,\sigma(\xi)]$ such that

$$\left[(\xi-\upsilon_1)^3\right]^\Delta=3(c-\upsilon_1)^2\geqslant3(\xi-\upsilon_1)^2.$$

This implies

$$\frac{1}{2}\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi\leqslant\frac{1}{6}\int_{\upsilon_1}^{\upsilon_2}\left[(\xi-\upsilon_1)^3\right]^\Delta\Delta\xi=\frac{1}{6}(\upsilon_2-\upsilon_1)^3. \tag{4.12}$$

Combining (4.11) and (4.12), we find that the left side of (3.4) is as follows

$$\beta\int_{\upsilon_1}^{\upsilon_2}\frac{1}{\phi^\gamma(\xi)}\left(\int_{\upsilon_1}^{\xi}\theta(\eta)\Delta\eta\right)^\beta\Delta\xi\leqslant\frac{1}{6}(\upsilon_2-\upsilon_1)^3. \tag{4.13}$$

For the right side of (3.4), we have

$$\begin{aligned}(\upsilon_2-\upsilon_1)^\beta&\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)}\Delta\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^\beta\frac{\theta^\beta(\xi)}{\phi^\gamma(\xi)}\Delta\xi\\&=(\upsilon_2-\upsilon_1)^2\int_{\upsilon_1}^{\upsilon_2}\frac{(\xi-\upsilon_1)^2}{(\xi-\upsilon_1)^2}\Delta\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\frac{(\xi-\upsilon_1)^2}{(\xi-\upsilon_1)^2}\Delta\xi\\&=(\upsilon_2-\upsilon_1)^2\int_{\upsilon_1}^{\upsilon_2}\Delta\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi\\&=(\upsilon_2-\upsilon_1)^2(\upsilon_2-\upsilon_1)-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi=(\upsilon_2-\upsilon_1)^3-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi.\end{aligned} \tag{4.14}$$

Using chain rule (2.1), we have

$$\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi \leqslant \frac{1}{3}\int_{\upsilon_1}^{\upsilon_2}\left[(\xi-\upsilon_1)^3\right]^{\Delta}\Delta\xi = \frac{1}{3}(\upsilon_2-\upsilon_1)^3. \tag{4.15}$$

This implies

$$\begin{aligned} -\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi &\geqslant -\frac{1}{3}(\upsilon_2-\upsilon_1)^3, \\ (\upsilon_2-\upsilon_1)^3-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi &\geqslant (\upsilon_2-\upsilon_1)^3-\frac{1}{3}(\upsilon_2-\upsilon_1)^3, \\ (\upsilon_2-\upsilon_1)^3-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi &\geqslant \frac{2}{3}(\upsilon_2-\upsilon_1)^3. \end{aligned} \tag{4.16}$$

Also, from (4.15), since $\frac{1}{3}(\upsilon_2-\upsilon_1)^3 \leqslant (\upsilon_2-\upsilon_1)^3$, we have

$$\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi \leqslant (\upsilon_2-\upsilon_1)^3.$$

Since $\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi$ is nonnegative, then

$$(\upsilon_2-\upsilon_1)^3-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^2\Delta\xi \leqslant (\upsilon_2-\upsilon_1)^3. \tag{4.17}$$

Combining (4.14), (4.16), and (4.17), we find that the right side of (3.4) is as follows

$$\frac{2}{3}(\upsilon_2-\upsilon_1)^3 \leqslant (\upsilon_2-\upsilon_1)^{\beta}\int_{\upsilon_1}^{\upsilon_2}\frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi-\int_{\upsilon_1}^{\upsilon_2}(\xi-\upsilon_1)^{\beta}\frac{\theta^{\beta}(\xi)}{\phi^{\gamma}(\xi)}\Delta\xi \leqslant (\upsilon_2-\upsilon_1)^3. \tag{4.18}$$

So, it is clear from (4.13) and (4.18) that they confirm (3.4) in Corollary 3.2.

5. Conclusion

In the present article, using of the time scales version of the chain rule, Fubini's theorem, and Hölder's inequality, we have successfully obtained some new dynamic inequalities. The proven dynamic inequalities show the time scales version of inequalities similar to Hardy's inequality. Future work might involve studying different generalizations of inequalities similar to Hardy's inequality utilizing the results demonstrated in this article.

Authors' contributions

W.M.H. and A.A.-D. wrote the main manuscript text and conceptualization and H.M.-O. and H.M.R formal analysis and investigation. All authors read and reviewed the manuscript.

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