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On some generalized numerical radius inequalities for Hilbert space operators

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Abstract

In this paper, it is shown, among other inequalities, that if $A, B \in \mathbb{B}(\mathbb{H})$, then, for $p \geqslant 1$, we have

$$2^{\frac{1}{p}-2}\left\| |A^*|^2+|B|^2 \right\|_p \leqslant 2^{\frac{1}{p}-3}\left(\|A^*+B\|_{2p}^2+\|A^*-B\|_{2p}^2 \right) \leqslant w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right)$$

and

$$w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leqslant 2^{\frac{1}{p}-1}\left(\left\| |A|^2 \right\|_p + \left\| |B^*|^2 \right\|_p \right) - (2^{\frac{1}{p}-1}-1)\left| \left\| |A|^2 \right\|_p - \left\| |B^*|^2 \right\|_p \right|.$$

Keywords: Numerical radius, norm inequality, Schatten p-norm.

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1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the algebra of all bounded linear operators on a complex separable Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $0 < p \leqslant \infty$ and $A \in \mathbb{B}(\mathbb{H})$, define $\|A\|_p$ by

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}.$$

For $1 \leqslant p \leqslant \infty$, this is the Schatten p-norm. When $p = 1$ the Schatten p-norm $\|A\|_1 = \operatorname{tr}|A|$ is the trace norm, when $p = 2$ the Schatten p-norm $\|A\|_2 = \left(\operatorname{tr}|A|^2 \right)^{1/2}$ is the Hilbert-Schmidt norm, and when $p = \infty$ the Schatten p-norm $\|A\|_\infty = \|A\|$ is the spectral norm.

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For $A \in \mathbb{B}(\mathbb{H})$, Yamazaki [13] showed that the numerical radius can be defined as

$$w(A) = \sup_{\theta \in \mathbb{R}} \left\| \mathrm{Re}(e^{i\theta} A) \right\|.$$

Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathbb{H})$, a generalization of the numerical radius has been introduced recently in [1] as the following

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N\left(\mathrm{Re}(e^{i\theta} A)\right)$$

for every $A \in \mathbb{B}(\mathbb{H})$. The norm $N(\cdot)$ is said to be self-adjoint if $N(A) = N(A^*)$ for every $A \in \mathbb{B}(\mathbb{H})$ and it is called unitarily invariant norm if it satisfies the invariant property $N(UAV^*) = N(A)$ for all $A \in \mathbb{B}(\mathbb{H})$ and for all unitary operators $U, V \in \mathbb{B}(\mathbb{H})$. Also, $N(\cdot)$ is called weakly unitarily invariant if $N(U^*AU) = N(A)$ for every $A \in \mathbb{B}(\mathbb{H})$ and every unitary $U \in \mathbb{B}(\mathbb{H})$. Obviously, $w(\cdot)$ is self-adjoint and weakly unitarily invariant norm. It is known that the numerical radius is equivalent to the spectral norm, that is

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\| \tag{1.1}$$

for every $A \in \mathbb{B}(\mathbb{H})$. Also, for every $A \in \mathbb{B}(\mathbb{H})$, the following estimate of the numerical radius

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \leqslant w^2(A) \leqslant \frac{1}{2}\left\||A|^2 + |A^*|^2\right\| \tag{1.2}$$

were given by Kittaneh (see [10, 11]). The inequality (1.2) refines the inequality (1.1). In [8], an improvement of the inequality (1.2) was given by Ghasvareh and Omidvar. They proved that

$$\frac{1}{8}\left(\|A + A^*\|^2 + \|A - A^*\|^2\right) \leqslant w^2(A) \leqslant \left\|\frac{|A|^2 + |A^*|^2}{2}\right\| - m\left(\left(\frac{|A| - |A^*|}{2}\right)^2\right) \tag{1.3}$$

for every $A \in \mathbb{B}(\mathbb{H})$, where $m(A) = \inf\{\langle Ax, x\rangle : x \in \mathbb{H}, \|x\| = 1\}$. In [1], Abu-Omar and Kittaneh proved the following inequality for the generalized numerical radius when $N(\cdot) = \|\cdot\|_2$:

$$\frac{1}{\sqrt{2}}\|A\|_2 \leqslant w_2(A) \leqslant \|A\|_2.$$

For inequalities in different settings that give several generalizations, refinements and applications of both $w(\cdot)$ and $w_N(\cdot)$, one can refer to [1–3, 5, 7, 9, 12–14], and references therein.

In this paper, we give some inequalities that give upper and lower bounds of the generalized numerical radius when $N(\cdot)$ is the Schatten p-norm.

2. Preliminary results

In this section, we want to give an upper and lower bound for the generalized numerical radius when $N(\cdot)$ is the Schatten p-norm. First, we start with the following lemma. Part (a) can be found in [6], while part (b) can be obtained by applying the fact that $w(\cdot)$ is weakly unitarily invariant to the operator $\tilde{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ and the unitary operator $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, and using part (a). From the basic properties of unitarily invariant norms, we can obtain parts (c) and (d). Also, we can obtain part (e) from the definition of $\|\cdot\|_p$.

Lemma 2.1. *Let* $A, B \in \mathbb{B}(\mathbb{H}), p \geqslant 1,$ *and* $r > 0$.

(a) $w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B)\}$.

(b) $w\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = w(A)$.

(c) $\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_p = \left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{\frac{1}{p}}$.

(d) $\|A^*\| = \|A\|$ *and* $\|A\|_p = \|A^*\|_p$.

(d) $\||A|^r\|_p = \|A\|_{rp}^r$.

We want the following lemma from [2].

Lemma 2.2. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \sqrt{2}w_2(A).$$

Also, we need with the following lemma (see [3]).

Lemma 2.3. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$w_N\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geqslant \frac{1}{2}N\left(\begin{bmatrix} 0 & A+B^* \\ B+A^* & 0 \end{bmatrix}\right)$$

and

$$w_N\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left(\begin{bmatrix} 0 & e^{i\theta}A + e^{-i\theta}B^* \\ e^{i\theta}B + e^{-i\theta}A^* & 0 \end{bmatrix}\right).$$

Depending on Lemma 2.3, we have the following corollary.

Corollary 2.4. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then, for* $p \geqslant 1$, *we have*

$$w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geqslant 2^{\frac{1}{p}-1}\|A \pm B^*\|_p.$$

Proof. By taking $N(\cdot) = \|\cdot\|_p$ in Lemma 2.3, we have

$$\begin{aligned} w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geqslant \frac{1}{2}\left\|\begin{bmatrix} 0 & A+B^* \\ B+A^* & 0 \end{bmatrix}\right\|_p &= \frac{1}{2}\left(\|A+B^*\|_p^p + \|B+A^*\|_p^p\right)^{1/p} \quad \text{(by Lemma 2.1 (c))} \\ &= 2^{\frac{1}{p}-1}\|A+B^*\|_p \quad \text{(by Lemma 2.1 (d))}. \end{aligned}$$

To prove that $w_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geqslant 2^{\frac{1}{p}-1}\|A - B^*\|_p$, we just replace A and B by iA and iB, respectively. □

Now, we are ready to give our first result in this paper.

Theorem 2.5. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then, for* $p \geqslant 1$, *we have*

$$2^{\frac{1}{p}-2}\left\||A^*|^2 + |B|^2\right\|_p \leqslant 2^{\frac{1}{p}-3}\left(\|A^*+B\|_{2p}^2 + \|A^*-B\|_{2p}^2\right) \leqslant w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

Proof. The parallelogram law asserts that for $A, B \in \mathbb{B}(\mathbb{H})$, we have

$$\frac{|A^*|^2 + |B|^2}{2} = \left|\frac{A^*+B}{2}\right|^2 + \left|\frac{A^*-B}{2}\right|^2. \tag{2.1}$$

Therefore, we have

$$2^{\frac{1}{p}-2}\left\||A^*|^2 + |B|^2\right\|_p = 2^{\frac{1}{p}-1}\left\|\frac{|A^*|^2 + |B|^2}{2}\right\|_p$$

$$
\begin{aligned}
&= 2^{\frac{1}{p}-1}\left\|\left|\frac{A^*+B}{2}\right|^2+\left|\frac{A^*-B}{2}\right|^2\right\|_p \quad \text{(by the inequality (2.1))}\\
&\leqslant 2^{\frac{1}{p}-1}\left(\left\|\left|\frac{A^*+B}{2}\right|^2\right\|_p+\left\|\left|\frac{A^*-B}{2}\right|^2\right\|_p\right)\\
&= 2^{\frac{1}{p}-1}\left(\left\|\frac{A^*+B}{2}\right\|_{2p}^2+\left\|\frac{A^*-B}{2}\right\|_{2p}^2\right) \quad \text{(by Lemma 2.1 (e))}\\
&= 2^{\frac{1}{p}-3}\left(\|A^*+B\|_{2p}^2+\|A^*-B\|_{2p}^2\right)\\
&\leqslant w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \quad \text{(by Corollary 2.4).}
\end{aligned}
$$

□

Using Theorem 2.5, we have the following corollaries.

Corollary 2.6. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then, for* $p \geqslant 1$, *we have*

$$
2^{\frac{1}{p}-2}\left\||A^*|^2+|A|^2\right\|_p \leqslant 2^{\frac{1}{p}-3}\left(\|A^*+A\|_{2p}^2+\|A^*-A\|_{2p}^2\right) \leqslant w_{2p}^2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right). \tag{2.2}
$$

Proof. The inequality (2.2) follows from Theorem 2.5 by taking $A = B$. □

Corollary 2.7. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$
\frac{1}{2}\operatorname{tr}\left(|A^*|^2+|B|^2\right) \leqslant \frac{1}{2}\left(\|A^*\|_2^2+\|B\|_2^2\right) \leqslant w_2^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right). \tag{2.3}
$$

In particular,

$$
\frac{1}{\sqrt{2}}\|A\|_2 \leqslant w_2(A). \tag{2.4}
$$

Proof. The inequality (2.3) follows from Theorem 2.5 by taking $p = 1$. The inequality (2.4) follows from the second inequality in the inequality (2.3) by taking $A = B$ and applying Lemma 2.2. □

Corollary 2.8. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$
\frac{1}{4}\left\||A^*|^2+|B|^2\right\| \leqslant \frac{1}{8}\left(\|A^*+B\|^2+\|A^*-B\|^2\right) \leqslant w^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right). \tag{2.5}
$$

In particular,

$$
\frac{1}{4}\left\||A^*|^2+|A|^2\right\| \leqslant \frac{1}{8}\left(\|A^*+A\|^2+\|A^*-A\|^2\right) \leqslant w^2(A). \tag{2.6}
$$

Proof. The inequality (2.5) follows from Theorem 2.5 by taking $p = \infty$. Also, we obtain the inequality (2.6) from the inequality (2.5) by taking $A = B$. □

The inequality (2.6) is similar to the first inequality in the inequality (1.3) given by Ghasvareh and Omidvar in [8], which means that Theorem 2.5 gives a generalization to the lower bound of $w^2(A)$ given by Ghasvareh and Omidvar in [8].

We need the following lemma (see [4]) to complete our work.

Lemma 2.9. *Let* $a, b \in [0, \infty)$.

(a) *If* $1 \leqslant r < \infty$, *then*

$$a^r + b^r \leqslant (a+b)^r - (2^r - 2)\min(a^r, b^r).$$

(b) *If* $0 < r \leqslant 1$, *then*

$$a^r + b^r \geqslant (a+b)^r - (2^r - 2)\min(a^r, b^r).$$

After replacing $\min(a^r, b^r)$ by $\frac{a^r+b^r-|a^r-b^r|}{2}$, the following corollary can be obtained from Lemma 2.9 by direct computations.

Corollary 2.10. *Let* $a, b \in [0, \infty)$.

(a) If $1 \leqslant r < \infty$, *then*

$$2^{r-1}(a^r + b^r) - (2^{r-1} - 1)\,|a^r - b^r| \leqslant (a+b)^r \leqslant 2^{r-1}(a^r + b^r).$$

(b) If $0 < r \leqslant 1$, *then*

$$2^{r-1}(a^r + b^r) \leqslant (a+b)^r \leqslant 2^{r-1}(a^r + b^r) - (2^{r-1} - 1)\,|a^r - b^r|.$$

Now, we are ready to give our second result in this paper.

Theorem 2.11. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then, for* $p \geqslant 1$, *we have*

$$w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant 2^{\frac{1}{p}-1}\left(\left\||A|^2\right\|_p + \left\||B^*|^2\right\|_p\right) - (2^{\frac{1}{p}-1} - 1)\left|\left\||A|^2\right\|_p - \left\||B^*|^2\right\|_p\right|.$$

Proof. By taking $N(\cdot) = \|\cdot\|_p$ in Lemma 2.3, we have

$$\begin{aligned}
w_{2p}^{2p}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leqslant \left(\frac{1}{2}\right)^{2p} \sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & e^{i\theta}A + e^{-i\theta}B^* \\ e^{i\theta}B + e^{-i\theta}A^* & 0 \end{bmatrix}\right\|_{2p}^{2p} \\
&= \left(\frac{1}{2}\right)^{2p} \sup_{\theta\in\mathbb{R}} \left(\|e^{i\theta}A + e^{-i\theta}B^*\|_{2p}^{2p} + \|e^{i\theta}B + e^{-i\theta}A^*\|_{2p}^{2p}\right) \quad \text{(by Lemma 2.1 (c))} \\
&= \left(\frac{1}{2}\right)^{2p} \sup_{\theta\in\mathbb{R}} \left(2\,\|e^{i\theta}A + e^{-i\theta}B^*\|_{2p}^{2p}\right) \quad \text{(by Lemma 2.1 (d))} \\
&\leqslant 2^{1-2p} \sup_{\theta\in\mathbb{R}} \left(\|e^{i\theta}A\|_{2p} + \|e^{-i\theta}B^*\|_{2p}\right)^{2p} \\
&= 2^{1-2p} \left(\sup_{\theta\in\mathbb{R}} \left(\|e^{i\theta}A\|_{2p} + \|e^{-i\theta}B^*\|_{2p}\right)\right)^{2p} \\
&\leqslant 2^{1-2p} \left(\sup_{\theta\in\mathbb{R}} \|e^{i\theta}A\|_{2p} + \sup_{\theta\in\mathbb{R}} \|e^{-i\theta}B^*\|_{2p}\right)^{2p} \\
&= 2^{1-2p} \left(\sup_{\theta\in\mathbb{R}} |e^{i\theta}|\,\|A\|_{2p} + \sup_{\theta\in\mathbb{R}} |e^{i\theta}|\,\|B^*\|_{2p}\right)^{2p} \\
&= 2^{1-2p} \left(\|A\|_{2p} + \|B^*\|_{2p}\right)^{2p} \\
&\leqslant \|A\|_{2p}^{2p} + \|B^*\|_{2p}^{2p} = \left\||A|^2\right\|_p^p + \left\||B^*|^2\right\|_p^p \quad \text{(by Lemma 2.1 (e))}.
\end{aligned}$$

So, we have

$$w_{2p}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant \left(\left\||A|^2\right\|_p^p + \left\||B^*|^2\right\|_p^p\right)^{1/p}. \tag{2.7}$$

Now, applying part (b) of Corollary 2.10 to the inequality (2.7) completes the proof. □

Using Theorem 2.11, we have the following corollary.

Corollary 2.12. *Let* $A, B \in \mathbb{B}(\mathbb{H})$*, then, for* $p \geqslant 1$*, we have*

$$w_{2p}^2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \leqslant 2^{\frac{1}{p}} \left\| |A|^2 \right\|_p . \tag{2.8}$$

In particular,

$$w_2(A) \leqslant \|A\|_2 . \tag{2.9}$$

Proof. The inequality (2.8) follows from Theorem 2.11 by taking $A = B$. The inequality (2.9) can be obtained from the inequality (2.8) by taking $p = 1$, then applying Lemma 2.2. □

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