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R_i-separation axioms via supra soft topological spaces

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Abstract

The aim of this study is to introduce and investigate two new classes of separation axioms called supra soft R_0 and supra soft R_1. They are defined in the spaces of supra soft topologies by using the notions of supra soft open sets and supra soft closure operator. We discuss the basic properties and characterizations of them. We also study the relationships between these classes and some other supra soft separation axioms with many results and explanative examples. Moreover, the connections between the properties of these classes and those in some generated soft topologies are presented. Finally, we show that these classes are preserved under subspaces, which means they are supra soft topological properties.

Keywords: Soft set, soft point, supra soft open set, supra soft topology, SS-kernel, supra soft R_0 and supra soft R_1 spaces.

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1. Introduction

In 1999, Molodtsov [38] initiated the notion of soft sets (S-sets) as a new approach to dealing with uncertainty data while modeling problems in the real world such as economics, engineering, data science, and health sciences. Many authors have applied the theory of S-sets as a mathematical framework for addressing practical problems such as [24, 43]. To found abstract structures inspired by S-sets, Shabir-Naz [44] introduced the idea of soft topology and studied various related concepts. Then, the intellectuals interested in abstract structures endeavored to navigate topological concepts to the spaces of topologies. For instance, it was introduced the concepts of soft compactness and Lindelöfness [6], soft separation axioms [30, 37, 42, 45], soft mappings [10, 31], soft menger [32], almost soft menger [17] and nearly soft menger spaces [18] and soft metric [25]. Furthermore, the notions of generalized open sets have been

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explored in soft topologies by some authors like soft semi-open sets [23], soft somewhat open sets [7], soft Q-set [3], weakly soft α-open sets [19] and weakly soft semi-open sets [20]. Some of these principles have been examined in the frame of supra fuzzy soft topology such as [40, 41].

In 1984, Mashhour et al. [35] introduced a new extension of topology, namely, supra topology. A supra topology on U is a collection μ of subsets of U, which is closed under an arbitrary union and $\emptyset$, $U \in \mu$. Then, a lot of topological researchers look at topological concepts via supra-topologies and scrutinize their properties [4, 11–13, 28]. As a natural extension, it was displayed the concept of supra soft topological spaces by Elsheikh et al. [29] as a hybridization of supra topological spaces and S-set theory. The structures of supra topology have attracted the attention of many authors. We tackle the main contributions in this framework, El-Shafei et al. [27] presented some types of operators and compactness in supra topologies. Abd El-latif examined the concepts of supra soft compact spaces [2] and supra strongly generalized soft closed sets [1]. Aras and Bayramov [21, 22] formulated supra soft separability axioms with respect to soft points. Whereas, Al-shami with his coauthors [5, 14] created four levels of supra soft separation axioms with respect to ordinary points and elucidated the interrelations among them. Also, Al-shami and El-Shafei [15] combined supra soft topology with a partial order relation. Then, they [16] established various types of supra soft separation axioms in ordered settings. We draw the reader's attention to that supra topological spaces have been employed to handle some real-life issues as debated in [9, 33], which confirm and enhance the importance of continued contributions to this scope of work.

The framework of this manuscript is the supra soft topology. We aim to introduce and study two new classes of separation properties called supra soft R_0 and R_1 in supra soft topological spaces by using the notions of supra S-open sets and supra S-closure operator. We investigate the basic properties and characterizations of them. The relationships between these new supra soft separation axioms and their relationships with some other properties are studied. Moreover, we describe the relationships between these properties in supra soft setting and those in some generated topologies and present some results and related theorems with some illustrate examples. We show that the supra soft separation axioms SSR_i, $i = 0, 1$ are supra soft topological properties.

2. Basic background

Through this paper, U refers to an initial universe set, T is the set of all parameters for U, and 2^U is the power set of U. In the following lines, we give some concepts and results about S-set theory.

Definition 2.1 ([38])**.** An S-set $G_T = (G, T)$ on U is defined as a set of ordered pairs

$$G_T = \{(t, G(t)) : t \in T,\ G(t) \in 2^U\},$$

where G is a mapping from T to 2^U. The class of all S-sets on U will be symbolized by $SS(U)$.

Definition 2.2 ([34])**.** For $H_T, G_T \in SS(U)$ and $x \in U$, we have following.

- If $H(t) = \emptyset$ (resp. $H(t) = U$) for all $t \in T$, then H_T is called a null (resp. an absolute) S-set and symbolized by $\widetilde{\emptyset}$ (resp. $\tilde{U}$).
- If $H(t) = \{x\}$ and $H(t') = \emptyset$ for every $t' \in T - \{t\}$, then H_T is called a soft point (or S-point) in $\tilde{U}$. It is symbolized by x_t. We write $x_t \tilde{\in} H_T$ if for the parameter $t \in T, x \in H(t)$. The class of all S-points in $\tilde{U}$ is denoted by $SP(U)$.
- The relative complement H^c_T of H_T is a mapping $H^c : T \longrightarrow 2^U$ given by $H^c(t) = U - H(t)$ for every $t \in T$. Clearly $(H^c_T)^c = H_T$.
- H_T is an S-subset of G_T, symbolized by $H_T \widetilde{\subseteq} G_T$, if $H(t) \subseteq G(t)$ for all $t \in T$.

- The S-union (resp. S-intersection) of H_T and G_T is an S-set K_T(resp. L_T) given by $K(t) = H(t) \cup G(t)$ (resp. $L(t) = H(t) \cap G(t)$) for all $t \in T$ and it is symbolized by $H_T \widetilde{\cup} G_T$ (resp. $H_T \widetilde{\cap} G_T$).

Definition 2.3 ([31])**.** Let $SS(U)$ and $SS(V)$ be two families of all S-sets on U with a set of parameters T and V with a set of parameters E, respectively. Let $u : U \longrightarrow V$ and $p : T \longrightarrow E$ be two maps, then the map $f_{up} : SS(U) \longrightarrow SS(V)$ is said to be a soft map (briefly, S-map) and we have:

(i) for $H_T \in SS(U)$, the image $f_{up}(H_T)$ of H_T is the S set on V given by $f_{up}(H_T)(e) = \cup\{u(H(t)) : t \in p^{-1}(e)\}$ if $p^{-1}(e) \neq \emptyset$ and $f_{up}(H_T)(e) = \widetilde{\emptyset}$ otherwise for any $e \in E$;
(ii) for $G_E \in SS(V)$, the preimage $f^{-1}_{up}(G_E)$ of V_E is the S-set on U given by $f^{-1}_{up}(G_E)(t) = u^{-1}(G(p(t)))$ for any $t \in T$.

The S-map f_{up} is called one-one (resp. onto, bijective), if u and p are one-one (resp. onto, bijective).

Definition 2.4 ([44])**.** A family $\tau \subseteq SS(U)$ under a fixed set of parameters T is called a soft topology on U if τ is closed under arbitrary S-union, finite S-intersections and $\widetilde{\emptyset}, \tilde{U} \in \tau$. The triple (U, τ, T) is called a soft topological space (or STS). Any element in τ is an S-open set, and the complement of any S-open set is an S-closed set.

For the S-set H_T in STS (U, τ, T), the S-closure $cl(H_T)$ of H_T is the S-intersection of all S-closed super sets of H_T. The S-interior $int(H_T)$ of H_T is the S-union of all S-open sets contained in H_T.

Definition 2.5 ([29])**.** A family $\mu \subseteq SS(U)$ with a fixed set of parameters T is said to be a supra soft topology (briefly, SST) on U if $\widetilde{\emptyset}, \tilde{U} \in \mu$ and the S-union of any number of S-sets in μ belongs to μ. The triple (X, μ, T) is called a supra soft topological space (briefly, SSTS).

Definition 2.6 ([14])**.** Let (U, μ, T) be an SSTS, $H_T \in SS(U)$, and $x \in U$, then:

- every element of μ is called supra soft open set (briefly, SS-open set), and the set of all SS-open sets on U is denoted by $SSO(U)$;
- H_T is called a supra soft closed set (briefly, SS-closed set) in U if $H^c_T \in \mu$ and the set of all SS-closed sets is denoted by $SSC(U)$;
- if (U, τ, T) is an STS, we say that, μ is an SST associated with τ if $\tau \subseteq \mu$;
- an S-set H_T is called a supra soft neighborhood of an S-point x_t if there is an SS-open set G_T such that $x_t \widetilde{\in} G_T \widetilde{\subseteq} H_T$.

Definition 2.7 ([15, 26, 44])**.** for $H_T \in SS(U)$, $Y \subseteq U$, and $x \in U$, we have:

(1) $x \in H_T$ if $x \in H(t)$ for all $t \in T$, and $x \notin H_T$ if $x \notin H(t)$ for some $t \in T$;
(2) if $H(t) = \{x\}$ for all $t \in T$, then H_T is called an S-singleton point, denoted by x_T, we write $x_T \widetilde{\in} H_T \Longleftrightarrow x \in H_T \Longleftrightarrow x_t \widetilde{\in} H_T$ for all $t \in T$;
(3) for $x_t, y_r \in SP(U)$, we write $x_t \neq y_r$ if $x \neq y$ or $t \neq r$;
(4) $\tilde{Y} = (Y, T)$ refers to the S-set on U for which $Y(t) = Y$ for all $t \in T$, is called stable.

Definition 2.8 ([29])**.** Let (U, μ, T) be an SSTS and $F_T \in SS(U)$. The supra soft closure (briefly, SS-closure) of F_T, denoted by $cl_s(F_T)$ is the intersection of all SS-closed super sets of F_T, and the supra soft interior (briefly, SS-interior) of F_T, denoted by $int_s(F_T)$ is the union of all SS-open subsets of F_T.

Theorem 2.9 ([14])**.** *For two SS-sets G_T and F_T in SSTS (U, μ, T), we have:*

(i) $G_T \in SSC(U) \Longleftrightarrow cl_s(G_T) = G_T$;
(ii) $G_T \widetilde{\subseteq} F_T \Longrightarrow cl_s(G_T) \widetilde{\subseteq} cl_s(F_T)$;
(iii) $x_t \widetilde{\in} cl_s(G_T) \Longleftrightarrow F_T \widetilde{\cap} G_T \neq \widetilde{\emptyset}$ *for all* $F_T \in \mu$ *and* $x_t \widetilde{\in} F_T$.

Definition 2.10 ([14]). For SSTS (U, μ, T) and $Y \subseteq U$. The family $\mu_Y = \{\tilde{Y} \tilde{\cap} F_T : F_T \in \mu\}$ is an SST on Y, and (Y, μ_Y, T) is called an SS-subspace of (U, μ, T).

For the SST-subspace (Y, μ_Y, T) of (U, μ, T) and $H_T \in SS(Y)$ we have, H_T is an SS-open set in (Y, μ_Y, T) if and only if $H_T = \tilde{Y} \tilde{\cap} G_T$ for some $G_T \in \mu$.

Definition 2.11 ([14]). An S-map $f_{up} : (U, \mu, T) \longrightarrow (V, \theta, E)$ is called:

(i) SS-continuous if $f_{up}^{-1}(H_E) \in \mu$ for any SS-open set $H_E \in \theta$;
(ii) SS-open if $f_{up}(G_T) \in \theta$ for any SS-open set $G_T \in \mu$;
(iii) SS-homeomorphism if it is bijective, SS-continuous, and SS-open.

Definition 2.12 ([22]). An SSTS (U, μ, T) is said to be:

(i) supra soft T_0 (briefly, SST_0) iff for any $x_t \neq y_{t'} \in SP(U)$ there are SS-open sets H_T and F_T such that $x_t \tilde{\in} F_T$ and $y_{t'} \tilde{\notin} H_T$ or $y_{t'} \tilde{\in} H_T$ and $x_t \tilde{\notin} F_T$;
(ii) supra soft T_1 (briefly, SST_1) iff for any $x_t \neq y_{t'} \in SP(U)$ there are SS-open sets H_T and F_T such that $x_t \tilde{\in} H_T, y_{t'} \tilde{\notin} H_T$ and $y_{t'} \tilde{\in} F_T$, $x_t \tilde{\notin} F_T$;
(iii) supra soft T_2 (briefly, SST_2) iff for any $x_t \neq y_{t'} \in SP(U)$ there are SS-open sets H_T and F_T such that $x_t \tilde{\in} H_T$, $y_{t'} \tilde{\in} F_T$ and $H_T \tilde{\cap} F_T = \tilde{\emptyset}$.

Remark 2.13 ([22]). $SST_2 \Longrightarrow SST_1 \Longrightarrow SST_0$.

Definition 2.14 ([39]). An STS (U, τ, T) is called:

(i) soft R_0 (briefly, SR_0) if for any $x_e \neq y_e \in SP(U)$ with $cl(x_e) \neq cl(y_e)$ implies $cl(x_e) \tilde{\cap} cl(y_e) = \tilde{\emptyset}$;
(ii) soft R_1(briefly, SR_1) if for any $x_e \neq y_e \in SP(U)$ with $cl(x_e) \neq cl(y_e)$, there are disjoint S-open subsets H_T, G_T of U such that $x_e \in H_T$ and $y_e \in G_T$.

Remark 2.15 ([39]). Every SR_1 space is SR_0.

Definition 2.16 ([36]). A supra topological space (U, τ) is called:

(i) supra R_0 (or sup-R_0) if any $x \neq y \in U$ with $cl(x) \neq cl(y), cl(x) \cap cl(y) = \emptyset$;
(ii) supra R_1 (or sup-R_1) if any $x \neq y \in U$ with $cl(x) \neq cl(y)$, there are disjoint supra open subsets A, B of U such that $x \in A$ and $y \in B$.

3. On supra soft R_0 and R_1 spaces

In the following, we introduce and discuss a new class of supra soft separation properties called SSR_i, $i = 0, 1$ and investigate some characterizations for them.

First, we give some definitions and lemmas which are used in the sequel.

Definition 3.1. Let (U, μ, T) be an SSTS, $F_T \in SS(U)$, and $x_t \in SP(U)$. The supra soft kernel (briefly, SSK) of F_T is denoted by $SSK(F_T)$ and given as $SSK(F_T) = \tilde{\cap}\{G_T \in \mu : F_T \tilde{\subseteq} G_T\}$. In particular, the supra soft kernel of x_t is defined by $SSK(x_t) = \tilde{\cap}\{G_T \in \mu : x_t \tilde{\in} G_T\}$.

Lemma 3.2. *Let (U, μ, T) be an SSTS and $F_T \in SS(U)$, then $SSK(F_T) = \tilde{\cup}\{x_t \in SP(U) : cl_s(x_t) \tilde{\cap} F_T \neq \tilde{\emptyset}\}$.*

Proof. Let $x_t \tilde{\in} SSK(F_T)$. Assume that $cl_s(x_t) \tilde{\cap} F_T = \tilde{\emptyset}$, we get $F_T \tilde{\subseteq} (cl_s(x_t))^c$ and so, $x_t \tilde{\notin} (cl_s(x_t))^c$ which is an SS-open set contains F_T. This contradicts with $x_t \tilde{\in} SSK(F_T)$. So $cl_s(x_t) \tilde{\cap} F_T \neq \tilde{\emptyset}$ and $SSK(F_T) \tilde{\subseteq} \tilde{\cup}\{x_t \in SP(U) : cl_s(x_t) \tilde{\cap} F_T \neq \tilde{\emptyset}\}$.
Conversely, let $cl_s(x_t) \tilde{\cap} F_T \neq \tilde{\emptyset}$. Suppose $x_t \tilde{\notin} SSK(F_T)$, there is $G_T \in \mu$ such that $F_T \tilde{\subseteq} G_T$ and $x_t \tilde{\notin} G_T$. Let $y_r \tilde{\in} cl_s(x_t) \tilde{\cap} F_T$, we have $y_r \tilde{\in} cl_s(x_t)$ and G_T is an SS-open set containing y_r imply $x_t \tilde{\in} G_T$, a contradiction. So that $x_t \tilde{\in} SSK(F_T)$. This completes the proof. □

Lemma 3.3. *Let* $(\mathcal{U}, \mu, T)$ *be an* SSTS *and* $x_t \in SP(\mathcal{U})$, *then* $x_t \widetilde{\in} cl_s(y_r)$ *if and only if* $y_r \widetilde{\in} SSK(x_t)$.

Proof. It is clear. □

Definition 3.4. An SSTS $(\mathcal{U}, \mu, T)$ is said to be:

(i) supra soft R_0 (briefly, SSR_0) iff $x_t \neq y_r \in SP(\mathcal{U})$ with $x_t \widetilde{\in} cl_s(y_r)$ implies $y_r \widetilde{\in} cl_s(x_t)$;

(ii) supra soft R_1 (briefly, SSR_1) iff $x_t \neq y_r \in SP(\mathcal{U})$ with $cl_s(x_t) \neq cl_s(y_r)$, there are SS-open sets F_T, G_T such that $x_t \widetilde{\in} F_T$ and $y_r \widetilde{\in} G_T$ with $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$.

Theorem 3.5. *Every* SSR_1 *space is* SSR_0.

Proof. Let $x_t \neq y_r \in SP(\mathcal{U})$ with $x_t \widetilde{\notin} cl_s(y_r)$, then $cl_s(x_t) \neq cl_s(y_r)$. Since $(\mathcal{U}, \mu, T)$ is SSR_1, there is $G_T \in \mu$ such that $y_r \widetilde{\in} G_T$ and $x_t \widetilde{\notin} G_T$. Thus $y_r \widetilde{\notin} cl_s(x_t)$, and this completes the proof. □

The converse of the above theorem is not necessary true, the next example shows it.

Example 3.6. Let $\mathcal{U}$ be an infinite set. The class $\mu = \{\widetilde{\emptyset}\} \cup \{H_T : (H(t))^c$ is a finite subset of $\mathcal{U}$ for all $t \in T\}$ is SST on $\mathcal{U}$ and $(\mathcal{U}, \mu, T)$ is called an SS-cofinite space. Now one can verify μ is SSR_0. But it is not SSR_1. Indeed, suppose that $(\mathcal{U}, \mu, T)$ is SSR_1 and $x_t \neq y_r \in SP(\mathcal{U})$ with $cl_s(x_t) \neq cl_s(y_r)$, there are $F_T, G_T \in \mu$ such that $x_t \widetilde{\in} F_T$ and $y_r \widetilde{\in} G_T$ with $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$ implies $(F(t))^c \cup (G(r))^c = \mathcal{U}$. Since $(F(t))^c$ and $(G(r))^c$ are finite subsets of $\mathcal{U}$, this means that $\mathcal{U}$ is finite. This is a contradiction. Thus $(\mathcal{U}, \mu, T)$ is not SSR_1.

Theorem 3.7. *Let* $(\mathcal{U}, \tau, T)$ *be an* STS *and* μ *be an* SST *associated with* τ, *then* $(\mathcal{U}, \mu, T)$ *is* SSR_0 *if and only if* $cl_s(x_t) \widetilde{\subseteq} F_T$ *for any* $F_T \in \mu$ *and* $x_t \widetilde{\in} F_T$.

Proof. Let $(\mathcal{U}, \mu, T)$ be SSR_0. Suppose $cl_s(x_t) \widetilde{\nsubseteq} F_T$ for some $F_T \in \mu$ and $x_t \widetilde{\in} F_T$, there exists an S-point y_r such that $y_r \widetilde{\in} cl_s(x_t)$, $y_r \widetilde{\notin} F_T$. So that $y_r \widetilde{\cap} F_T = \widetilde{\emptyset}$ for some $F_T \in \mu, x_t \widetilde{\in} F_T$ where x_t, y_r are different S-points in $\mathcal{U}$. Thus $x_t \widetilde{\notin} cl_s(y_r)$. This is a contradiction. Hence, the necessary part holds. Conversely, let $x_t \widetilde{\notin} cl_s(y_r)$, there is an SS-open set G_T contains x_t such that $y_r \widetilde{\cap} G_T = \widetilde{\emptyset}$ imply $y_r \widetilde{\notin} G_T$. Clearly $cl_s(x_t) \widetilde{\subseteq} G_T$, we have $y_r \widetilde{\notin} cl_s(x_t)$. We obtain the desired result. □

Theorem 3.8. *Let* $(\mathcal{U}, \tau, T)$ *be an* STS *and* μ *be an* SST *associated with* τ, *then* $(\mathcal{U}, \mu, T)$ *is* SSR_0 *if and only if for any* $F_T \in SSC(\mathcal{U})$ *with* $x_t \widetilde{\notin} F_T$ *implies* $F_T \widetilde{\subseteq} G_T$ *and* $x_t \widetilde{\notin} G_T$ *for some* $G_T \in \mu$.

Proof.

"$\Longrightarrow$" Let $F_T \in SSC(\mathcal{U})$ and $x_t \widetilde{\notin} F_T$, then $x_t \widetilde{\in} F_T^c$ which is an SS-open set contains x_t. Since $(\mathcal{U}, \mu, T)$ is SSR_0, we have $cl_s(x_t) \widetilde{\subseteq} F_T^c$. Take $G_T = (cl_s(x_t))^c$, then $G_T \in \mu$ and $x_t \widetilde{\notin} G_T$.

"$\Longleftarrow$" Let $x_t \neq y_r \in SP(\mathcal{U})$. Suppose that $x_t \widetilde{\notin} cl_s(y_r) \in SSC(\mathcal{U})$. By hypothesis, there is an SS-open set G_T such that $cl_s(y_r) \widetilde{\subseteq} G_T$ with $x_t \widetilde{\notin} G_T$, this implies $x_t \widetilde{\in} G_T^c$. So, $cl_s(x_t) \widetilde{\subseteq} G_T^c$ but $y_r \widetilde{\notin} G_T^c$. Thus $y_r \widetilde{\notin} cl_s(x_t)$ and the result holds. □

Theorem 3.9. *For* SSTS $(\mathcal{U}, \mu, T)$, *the next statements are equivalent:*

(1) $(\mathcal{U}, \mu, T)$ *is* SSR_0;
(2) $F_T \in SSC(\mathcal{U}) \Longrightarrow F_T = SSK(F_T)$;
(3) $H_T \in SSC(\mathcal{U})$ *and* $x_t \widetilde{\in} H_T \Longrightarrow SSK(x_t) \widetilde{\subseteq} H_T$;
(4) $x_t \in SP(\mathcal{U}) \Longrightarrow SSK(x_t) \widetilde{\subseteq} cl_s(x_t)$.

Proof.

(1) $\Longrightarrow$ (2) Let $F_T \in SSC(\mathcal{U})$. Suppose that $x_t \widetilde{\notin} F_T$, we have $x_t \widetilde{\in} F_T^c$ which is an SS-open set containing x_t. Since $(\mathcal{U}, \mu, T)$ is SSR_0, we get $cl_s(x_t) \widetilde{\subseteq} F_T^c$ this implies that $cl_s(x_t) \widetilde{\cap} F_T = \widetilde{\emptyset}$. From Lemma 3.2, we get $x_t \widetilde{\notin} SSK(F_T)$. So that $F_T = SSK(F_T)$.

(2) $\Longrightarrow$ (3) It follows from that $K_T\widetilde{\subseteq}G_T$ implies $SSK(K_T)\widetilde{\subseteq}SSK(G_T)$.

(3) $\Longrightarrow$ (4) It is clear.

(4) $\Longrightarrow$ (1) Let $x_t \neq y_r \in SP(U)$ with $x_t\widetilde{\in}cl_s(y_r)$. From Lemma 3.3, we have $y_r\widetilde{\in}SSK(x_t)$. Since $x_t\widetilde{\in}cl_s(x_t)$ which is an SS-closed set and from (4), we get $y_r\widetilde{\in}SSK(x_t)\widetilde{\subseteq}cl_s(x_t)$, that is $y_r\widetilde{\in}cl_s(x_t)$ and this completes the proof. □

Proposition 3.10. *An* SSTS (U, μ, T) *is* SSR_0 *if and only if* $cl_s(x_t)\widetilde{\subseteq}SSK(x_t)$ *for all* $x_t \in SP(U)$.

Proof. It follows from Lemma 3.3 and Theorem 3.7. □

Theorem 3.11. *For* SSTS (U, μ, T), *the next statements are equivalent:*

(i) (U, μ, T) *is* SSR_0;

(ii) *for any* $F_T \in SSC(U)$ *with* $x_t\widetilde{\notin}F_T \Longrightarrow cl_s(x_t)\widetilde{\cap}F_T=\widetilde{\emptyset}$;

(iii) *for any* $x_t \neq y_r \in SP(U)$, *either* $cl_s(x_t)=cl_s(y_r)$ *or* $cl_s(x_t)\widetilde{\cap}cl_s(y_r)=\widetilde{\emptyset}$.

Proof.

(i) $\Longrightarrow$ (ii) Let $F_T \in SSC(U)$ and $x_t\widetilde{\notin}F_T$, then $x_t\widetilde{\in}F_T^c \in \mu$. From Theorem 3.7, we get $cl_s(x_t)\widetilde{\subseteq}F_T^c$. So $cl_s(x_t)\widetilde{\cap}F_T=\widetilde{\emptyset}$.

(ii) $\Longrightarrow$ (iii) Let $x_t \neq y_r \in SP(U)$ with $cl_s(x_t)\neq cl_s(y_r)$, there is $z_e\widetilde{\in}cl_s(x_t)$ and $z_e\widetilde{\notin}cl_s(y_r)$ or $z_e\widetilde{\in}cl_s(y_r)$ and $z_e\widetilde{\notin}cl_s(x_t)$. So there is $F_T \in \mu$ such that $y_r\widetilde{\notin}F_T$, $z_t\widetilde{\in}F_T$ and so, $x_t\widetilde{\in}F_T$. Therefore $x_t\widetilde{\notin}cl_s(y_r)$. From (ii) we obtain $cl_s(x_t)\widetilde{\cap}cl_s(y_r)=\widetilde{\emptyset}$. The proof of the remaining case is similar.

(iii) $\Longrightarrow$ (i) Let $x_t \neq y_r \in SP(U)$ with $x_r\widetilde{\notin}cl_s(y_r)$, then $cl_s(x_t)\neq cl_s(y_r)$. From (iii) we get, $cl_s(x_t)\widetilde{\cap}cl_s(y_r)$ $=\widetilde{\emptyset} \Longrightarrow y_r\widetilde{\in}cl_s(y_r)\widetilde{\subset}(cl_s(x_t))^c$. Therefore $y_r\widetilde{\notin}cl_s(x_t)$. The result holds. □

Corollary 3.12. *Let* (U, τ, T) *be an STS and* μ *be an SSTassociated with* τ, *then* (U, μ, T) *is* SSR_0 *if and only if for any two different S-points* x_t, y_r *in* U *with* $cl_s(x_t)\neq cl_s(y_r) \Longrightarrow cl_s(x_t)\widetilde{\cap}cl_s(y_r) = \widetilde{\emptyset}$.

Proof. It follows from above theorem. □

Lemma 3.13. *Let* (U, μ, T) *be an SSTS,* $x_t, y_r \in SP(U)$, *then* $SSK(x_t) \neq SSK(y_r)$ *if and only if* $cl_s(x_t)\neq cl_s(y_r)$.

Proof. Suppose that $SSK(x_t) \neq SSK(y_r)$, there is $z_e \in SP(U)$ with $z_e\widetilde{\in}SSK(x_t)$ and $z_e\widetilde{\notin}SSK(y_r)$. Since $z_e\widetilde{\in}SSK(x_t)$. From Lemma 3.2, we get $x_t\widetilde{\cap}cl_s(z_e) \neq \widetilde{\emptyset}$ implies $x_t\widetilde{\in}cl_s(z_e)$, that is $cl_s(x_t)\widetilde{\subseteq}cl_s(z_e)$. Similarly, since $z_e\widetilde{\notin}SSK(y_r)$ we get $y_r\widetilde{\notin}cl_s(z_e)$. Since $cl_s(x_t)\widetilde{\subset}cl_s(z_e)$ and $y_r\widetilde{\notin}cl_s(z_e)$, we have $y_r\widetilde{\notin}cl_s(x_t)$. Conversely, let $cl_s(x_t) \neq cl_s(y_r)$, there is $z_e \in SP(U)$ with $z_e\widetilde{\in}cl_s(x_t)$ and $z_e\widetilde{\notin}cl_s(y_r)$. Thus, there is an SS-open set containing z_e and so x_t but not y_r. Hence $y_r\widetilde{\notin}SSK(x_t)$. The proof is complete. □

Theorem 3.14. *An* SSTS (U, μ, T) *is* SSR_0 *if and only if* $x_t \neq y_r \in SP(U)$ *with* $SSK(x_t) \neq SSK(y_r)$ *implies* $SSK(x_t)\widetilde{\cap}SSK(y_r) = \widetilde{\emptyset}$.

Proof.

"$\Longrightarrow$" Let (U, μ, T) is SSR_0 and $x_t \neq y_r \in SP(U)$ with $SSK(x_t) \neq SSK(y_r)$. By Lemma 3.13, we get $cl_s(x_t) \neq cl_s(y_r)$. Suppose $SSK(x_t)\widetilde{\cap}SSK(y_r) \neq \widetilde{\emptyset}$, there is $z_e\widetilde{\in}SSK(x_t)\widetilde{\cap}SSK(y_r)$. Since $z_e\widetilde{\in}SSK(x_t)$, from Lemma 3.3, we have $x_t\widetilde{\in}cl_s(z_e)$ implies $cl_s(x_t)\widetilde{\subset}cl_s(z_e)$. Since $x_t\widetilde{\in}cl_s(x_t)$ and from Corollary 3.12, we get $cl_s(x_t) = cl_s(z_t)$. Similarly, since $z_e\widetilde{\in}SSK(y_r)$, we have $cl_s(y_r) =cl_s(z_e)=cl_s(x_t)$. This is contradiction. Therefore, $SSK(x_e)\widetilde{\cap}SSK(y_r) = \widetilde{\emptyset}$.

"$\Longleftarrow$" Let $x_t \neq y_r \in SP(U)$ with $cl_s(x_t) \tilde{\neq} cl_s(y_r)$. From Lemma 3.13, we have $SSK(x_t) \neq SSK(y_r)$. By hypothesis, we get $SSK(x_t) \widetilde{\cap} SSK(y_r) = \widetilde{\emptyset}$. Suppose that $cl_s(x_t) \widetilde{\cap} cl_s(y_r) \tilde{\neq} \widetilde{\emptyset}$, there is z_e in $\tilde{U}$ such that $z_e \widetilde{\in} cl_s(x_t)$ and $z_e \widetilde{\in} cl_s(y_r)$. Form Lemma 3.3, we have $x_t \widetilde{\in} SSK(z_e)$ and $y_r \widetilde{\in} SSK(z_e)$ and by Lemma 3.2, we obtain, $SSK(x_t) \widetilde{\cap} SSK(z_e) \neq \widetilde{\emptyset}$ and $SSK(y_r) \widetilde{\cap} SSK(z_e) \neq \widetilde{\emptyset}$. By hypothesis we get, $SSK(x_t) = SSK(z_e)$ and $SSK(y_r) = SSK(z_e)$. Hence $SSK(x_t) = SSK(y_r)$. So, $SSK(x_t) \widetilde{\cap} SSK(y_r) \neq \widetilde{\emptyset}$. This is a contradiction. Hence $cl_s(x_t) \widetilde{\cap} cl_s(y_r) = \widetilde{\emptyset}$. From Corollary 3.12, we obtain the result. □

Proposition 3.15. *An SSTS (U, μ, T) is SSR_0 if and only if $cl_s(x_t) \widetilde{\subseteq} SSK(x_t)$ for all $x_t \in SP(U)$.*

Proof. It follows from Definition 3.1 and Theorem 3.7. □

From Lemma 3.3 and the above proposition one can verify the next corollary.

Corollary 3.16. *An SSTS (U, μ, T) is SSR_0 for any $x_t \in SP(U)$, $SSK(x_t) = cl_s(x_t)$.*

Theorem 3.17. *An SSTS (U, μ, T) is SSR_1 if and only if for any $x_t \neq y_r \in SP(U)$ with $SSK(x_t) \neq SSK(y_r)$, there are $F_T, G_T \in \mu$ such that $cl_s(x_t) \widetilde{\subseteq} F_T, cl_s(y_r) \widetilde{\subseteq} G_T$ and $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$.*

Proof. It follows from Lemma 3.13. □

Proposition 3.18. *For an SSTS (U, μ, T), the next statements are equivalence:*

(1) *(U, μ, T) is SSR_1;*
(2) *for any $x_t \neq y_r \in SP(U)$ with $x_t \widetilde{\notin} cl_s(y_r)$, there are $H_T, G_T \in \mu$ such that $x_t \widetilde{\in} H_T, y_r \widetilde{\in} G_T$ and $H_T \widetilde{\cap} G_T = \widetilde{\emptyset}$;*
(3) *for any $x_t \neq y_r \in SP(U)$ with $cl_s(x_t) \tilde{\neq} cl_s(y_r)$, there are F_T, $G_T \in \mu$ such that $cl_s(x_t) \widetilde{\subseteq} F_T$ and $cl_s(y_r) \widetilde{\subseteq} G_T$ with $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$.*

Proof. It follows from the above theorem and Lemma 3.13. □

Theorem 3.19. *Every SS-subspace (Y, μ_Y, T) of an SSR_i space (U, μ, T) is SSR_i, $i = 0, 1$.*

Proof. We will show the case $i = 1$. The proof of the rest case is similar. To prove that (Y, μ_Y, T) is SSR_1, let $x_t \neq y_r \in SP(Y)$ with $cl_s(x_t) \tilde{\neq} cl_s(y_r)$, then also x_t, y_r are different S-points in $\tilde{U}$ with $cl_s(x_t) \tilde{\neq} cl_s(y_r)$. Since (U, μ, T) is SSR_1, there are $F_T, G_T \in \mu$ such that $x_t \widetilde{\in} F_T, y_r \widetilde{\in} G_T$, and $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$. So there are SS-open sets in Y say, $H^Y_T = \tilde{Y} \widetilde{\cap} F_T \in \mu_Y$ and $V^Y_T = \tilde{Y} \widetilde{\cap} G_T \in \mu_Y$, which are containing x_t, y_r, respectively, with $H^Y_T \widetilde{\cap} V^Y_T = \widetilde{\emptyset}$. The result holds. □

Corollary 3.20. *An open SS-subspace of SSR_i space need not be SSR_i space, $i = 0, 1$.*

Proof. Clearly, it follows from that, the S-intersection of an S-open set and an SS-open set need not to be SS-open soft. □

Remark 3.21. Clearly, every SR_i space is SSR_i, for $i = 0, 1$.

4. More characterizations and relations

This section is devoted to investigating how the concepts of supra soft R_0 and R_1 spaces navigate between supra soft topological spaces and their parametric supra topological spaces. To demonstrate the obtained relationships we provide some elucidative examples.

Definition 4.1. Let (U, σ) be a supra topological space and T be a fixed set of parameters. The family $\mu_\sigma = \{H_T : H(t) = A \text{ for all } t \in T \text{ and } A \in \sigma\}$ defines an SST called stable SST on U derived from σ. In general, an SSTS (U, μ, T) is called stable if any SS-open set in (U, μ, T) is stable.

Definition 4.2. An SSTS (U, μ, T) is called a strong stable if $\mu = \{H_T : H(t) = B$ for all $t \in T$ and $B \subset U\}$. In this case any SS-singleton point x_T is SS-open set.

Definition 4.3. Let (U, μ, T) be an SSTS, then the class $\mu_t = \{F(t) : F_T \in \mu\}$ for each $t \in T$ defines a supra topology on U which is called a parametric supra topology [14].

Remark 4.4.

1. If (U, μ, T) is a strong stable SSTS, we have:
   (i) any element in (U, μ, T) is both SS-open and SS-closed set;
   (ii) (U, μ_t) is a discrete space for all $t \in T$;
   (iii) every stable SSTS (U, μ_σ, T) is a subspace of a strong stable SSTS (U, μ, T);
   (iv) every strong stable SSTS (U, μ, T) is a subspace of soft discrete space $\tau = SS(U)$;
2. If (U, σ) is a discrete topological space, then the SST μ_σ, which is defined in Definition 3.1, is a strong stable SST on U.

Definition 4.5. For an SST (U, μ, T), if $F_T^c \in \mu$ for every $F_T \in \mu$, then (U, μ, T) is called a complemental SST.

Example 4.6.

(1) Let $U = \{a, b, c\}$, $T = \{t_1, t_2\}$, and the family $\theta = \{\emptyset, U, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Then θ is an supra topology on U and from Definition 4.1, we have the family $\mu_\theta = \{\widetilde{\emptyset}, \tilde{U}, \widetilde{\{c\}}, \widetilde{\{a, b\}}, \widetilde{\{b, c\}}, \widetilde{\{a, c\}}\}$ is a stable SST on U.

(2) Let $U = \{x, y, z\}$, $T = \{t_1, t_2\}$ and $\tau = SS(U)$ be a soft discrete topology on U. The family $\mu = \{\widetilde{\emptyset}, \tilde{U}, x_T, y_T, z_T, \widetilde{\{x, y\}}, \widetilde{\{x, z\}}, \widetilde{\{y, z\}}\}$ is a strong stable SST on U and any element in μ is both SS-open and SS-closed set. Moreover, μ is a complemental SST and it is a subspace of τ. On other hand, $\mu_{t_1} = \mu_{t_2} = \{\emptyset, U, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ is a discrete topology on U.

Theorem 4.7. *Every strong stable* SSTS (U, μ, T) *is* $SSR_i, i = 0, 1$.

Proof. We will prove only the case $i = 1$. The proof of the other cases is similar. Let (U, μ, T) be a strong stable SSTS and $x_t \neq y_r \in SP(U)$ such that $cl_s(x_t) \neq cl_s(y_r)$. From Definition 4.2, there are SS-open sets x_T, y_T such that $x_t \tilde{\in} x_T$ and $y_t \tilde{\in} y_T$ with $x_T \tilde{\cap} y_T = \widetilde{\emptyset}$. Hence (U, μ, T) is SSR_1. □

Corollary 4.8. *Every stable* SSTS (U, μ, T) *is* $SSR_i, i = 0, 1$.

Theorem 4.9. *Every complemental* SSTS (U, μ, T) *is* SSR_i, $i = 0, 1$.

Proof. We will prove only the case $i = 1$. The proof of other case is similar. Let $x_t \neq y_r \in SP(U)$ and $x_t \tilde{\notin} cl_s(y_r)$, then $x_t \tilde{\in} (cl_s(y_r))^c = F_T \in \mu$. Since (U, μ, T) is a complemental SSTS, we have $y_r \tilde{\in} cl_s(y_r) = G_T \in \mu$. Clearly, $F_T \tilde{\cap} G_T = \widetilde{\emptyset}$ and so, from Proposition 3.18 (2), we obtain (U, μ, T) is SSR_1. □

Theorem 4.10. *If* (U, μ, T) *is* SSR_i, *then* (U, μ_t) *is sup-*R_i *for all* $t \in T$ *and* $i = 0, 1$.

Proof. We will prove the case $i = 1$. The proof of the case $i = 0$ is similar. To prove that (U, μ_t) is sup-R_1, let $x \neq y \in U$ with $cl(x) \neq cl(y)$, then either $x \notin cl(y)$ or $y \notin cl(x)$. Thus $x_t \tilde{\notin} cl_s(y_r)$ or $y_r \tilde{\notin} cl_s(x_t)$ imply $cl_s(x_t) \neq cl_s(y_r)$. Since (U, μ, T) is SSR_1, there are $F_T, G_T \tilde{\in} \mu$ such that $x_t \tilde{\in} F_T$ and $y_r \tilde{\in} G_T$ with $F_T \tilde{\cap} G_T = \widetilde{\emptyset}$ and so, there are $F(t)$ and $G(t) \in \mu_t$ such that $x \in F(t)$ and $y \in G(t)$ with $F(t) \cap G(t) = \emptyset$ for all $t \in T$. Therefore (U, μ_t) is sup-R_1 for all $t \in T$. □

The next example shows that the converse of the above theorem may not be true.

Example 4.11. Let $U=\{u_1,u_2\}$ and $T=\{t_1,t_2\}$. Consider the family $\mu=\{\widetilde{\emptyset},\tilde{U},F_{1T},F_{2T},F_{3T},F_{4T}\}$, where $F_{1T}=\{(t_1,\{u_1\})\}$, $F_{2T}=\{(t_1,\{u_1\}),(t_2,\{u_1\})\}$, $F_{3T}=\{(t_1,\{u_1\}),(t_2,\{u_2\})\}$, and $F_{4T}=\{(t_1,\{u_1\}),(t_2,U)\}$. Then μ is an SST on U and the class $\mu_{t_2}=\{\emptyset,U,\{a\},\{b\}\}$ is a supra topology on U. It is clear that (U,μ_{t_2}) is sup-R_1 and sup-R_0. But (U,μ,T) is not SSR_0. Indeed, for $(u_1)_{t_1}\neq(u_2)_{t_1}\in SP(U)$ we have, $\tilde{U}=cl_s((u_1)_{t_1})\tilde{\neq}cl_s((u_2)_{t_1})=(u_2)_{t_1}$ but $cl_s((u_1)_{t_1})\tilde{\cap}cl_s((u_2)_{t_1})\tilde{\neq}\widetilde{\emptyset}$. Hence (U,μ,T) is not SSR_1.

Proposition 4.12. *If (U,μ,T) is a strong stable SSTS, then (U,μ,T) is SSR_i if and only if (U,μ_t) is sup-R_i for all $t\in T$ and $i=0,1$.*

Proof. We will give the proof for $i=1$. The proof for the case $i=0$ is similar.

"$\Longrightarrow$" The proof follows from that of Theorem 4.10.

"$\Longleftarrow$" Let $x_t\neq y_r\in SP(U)$ with $cl_s(x_t)\tilde{\neq}cl_s(y_r)$, then $x\neq y$ with $cl(x)\neq cl(y)$. Since (U,μ_t) is sup-R_1, there are supra open subsets F,K of U such that $x\in F$ and $y\in K$ with $F\cap K=\emptyset$ imply there are SS-open sets $H_T,V_T\tilde{\in}\mu$ such that $F=H(t)$ and $K=V(t)$ for all $t\in T$ with $x_t\tilde{\in}H_T$ and $y_r\tilde{\in}V_T$ with $H_T\tilde{\cap}V_T=\widetilde{\emptyset}$. This completes the proof. □

Theorem 4.13. *A supra topological space (U,θ) is sup-R_i if and only if (U,μ_θ,T) is SSR_i, $i=0,1$.*

Proof. We will give the proof for $i=1$. The proof for the case $i=0$ is similar.

"$\Longrightarrow$" It is similar to that of the converse part in the above proposition.

"$\Longleftarrow$" Let $x_t\neq y_r\in SP(U)$ with $cl_s(x_t)\tilde{\neq}cl_s(y_r)$, either $x\notin cl_s(y_r)$ or $y\notin cl_s(x_t)$ implies $x_t\tilde{\notin}cl_s(y_r)$ or $y_r\tilde{\notin}cl_s(x_t)$, then $cl_s(x_t)\tilde{\neq}cl_s(y_r)$. Since (U,μ_θ,T) is SSR_1, there are F_T, $G_T\tilde{\in}\mu_\theta$ such that $x_t\tilde{\in}F_T$, $y_r\tilde{\in}G_T$ and $F_T\tilde{\cap}G_T=\widetilde{\emptyset}$. Thus, there are disjoint supra open sets A, $B\in\theta$ such that $x\in F(t)=A$ and $y\in G(t)=B$ for all $t\in T$. Hence (U,θ) is sup-R_1. □

Theorem 4.14. *If (U,μ,T) is SST_i, then it is SSR_{i-1}, $i=1,2$.*

Proof. We will show the case $i=1$. The proof for the case $i=2$ is obvious.

Let (U,μ,T) be SST_1 and F_T be an SS-open set contains x_t. We want to show that $cl_s(x_t)\tilde{\subseteq}F_T$. So let $y_r\tilde{\notin}F_T$, then $x_t\tilde{\notin}cl_s(y_r)$ and x_t,y_r are different S-points. Since (U,μ,T) is SST_1, there is $H_T\tilde{\in}\mu$ such that $y_r\tilde{\in}H_T$ and $x_t\tilde{\notin}H_T$, then $y_t\tilde{\notin}cl_s(x_t)$. Therefore $cl_s(x_t)\tilde{\subseteq}F_T$. This completes the proof. □

The converse of the above theorem may not be true. The next example shows it.

Example 4.15. Let $U=\{x,y\}$ and $T=\{t_1,t_2\}$. The family $\mu=\{\widetilde{\emptyset},\tilde{U},G_{1T},G_{2T}\}$, where, $G_{1T}=\{(t_1,U)\}$ and $G_{2T}=\{(t_2,U)\}$ is an SST on U. One can verify (U,μ,T) is SSR_0 and SSR_1 but not SST_1. Indeed, for two S-points x_{t_1},y_{t_1}, the SS-open sets which are containing x_{t_1} are $\tilde{U}$ and G_{1T} but also, they are containing y_{t_1}. Thus (U,μ,T) is not SST_1. Moreover, one can check that (U,μ,T) is not SST_2.

Theorem 4.16. *For SSTS (U,μ,T), the next items hold:*

(1) *(U,μ,T) is $SST_1\Longleftrightarrow$ it is both SSR_0 and SST_0;*
(2) *(U,μ,T) is $SST_2\Longleftrightarrow$ it is both SSR_1 and SST_0.*

Proof. Necessity for two cases follows from Theorem 4.14 and Remark 2.13. The proof of converse part is obvious, for two cases. □

Corollary 4.17. *(U,μ,T) is $SST_2\Longleftrightarrow$ it is both SSR_1 and SST_1.*

Definition 4.18. A property is called an SS-topological property if the property is preserved by SS-homeomorphism.

Theorem 4.19. *For a bijective SS-continuous map $f_{up}:(U,\mu,T)\longrightarrow(V,\theta,E)$, if (V,θ,E) is SSR_i, then (U,μ,T) is also SSR_i, $i=0.1$.*

Proof. We will prove only the case $i = 1$. The proof of the rest case is similar. To show that (U, μ, T) is SSR_1, let $x_t \neq y_{t'} \in SP(U)$. Since $f_u p$ is one-one, there are two different S-points $a_e, b_{e'}$ in V such that $f_u p\,(x_t) = a_e$ and $f_{up}\left(y_{t'}\right) = b_{e'}$. Since (V, θ, E) is SSR_1, then there are two SS-open sets $G_{1E}, G_{2E} \in \theta$ such that $a_e \widetilde{\in} G_{1E}$ and $b_{e'} \widetilde{\in} G_{2E}$and so, $x_t \widetilde{\in} f^{-1}_{up}(G_{1E})$ and $y_{t'} \widetilde{\in} f^{-1}_{up}(G_{2E})$. Since f_{up} is SS-continuous, we have $f^{-1}_{up}\,(G_{1E}), f^{-1}_{up}(G_{2E})$ are SS-open sets in (U, μ, T) with $f^{-1}_{up}\,(G_{1E}\,) \widetilde{\cap} f^{-1}_{up}\,(G_{2E}) = \widetilde{\emptyset}$. This complete the proof. □

Theorem 4.20. *For a bijective SS-open map $f_{up} : (U, \mu, T) \longrightarrow (V, \theta, E)$, if (U, μ, T) is SSR_i, then (V, θ, E) is also SSR_i , $i = 0, 1$.*

Proof. We will prove only the case $i = 1$. The proof of the rest case is similar. To show that (V, θ, E) is SSR_1. Let $a_e \neq b_{e'} \in SP(V)$. Since f_{up} is onto, there are two different S-points $x_t, y_{t'}$ in U such that $f_{up}\,(x_t) = a_e$ and $f_{up}(y_{t'}) = b_{e'}$. By hypothesis, there are two disjoint SS-open sets $F_{1T}, F_{2T} \in \mu$ such that $x_t \widetilde{\in} F_{1E}, y_{t'} \widetilde{\in} F_{2E}$ and so, $a_e \widetilde{\in} f_{up}(F_{1T})$ and $b'_e \widetilde{\in} f_{up}(F_{2T})$. Since f_{up} is SS-open, we have $f_{up}(F_{1T})$, $f_{up}(F_{2T})$ are SS-open sets in (V, θ, E) with $f_{up}\,(F_{1E}) \widetilde{\cap} f_{up}\,(F_{2T}) = \widetilde{\emptyset}$. The result holds. □

From the two above theorems, we have the next theorem.

Theorem 4.21. *Let $f_{up} : (U, \mu, T) \longrightarrow (V, \theta, E)$ be an SS-homeomorphism map, then (U, μ, T) is SSR_iif and only if (V, θ, E) is SSR_i , $i = 0.1$.*

Corollary 4.22. *The supra soft properties SSR_i are SS-topological property, for $i = 0, 1$.*

Corollary 4.23. *From Remarks 2.13, 2.15, and Corollary 4.17, the following implications hold and describe the relationships between SSR_i and other soft separation properties.*

$$\begin{array}{ccccc} SST_2 & \Longrightarrow & SST_1 & \Longrightarrow & SST_0 \\ \Downarrow & & \Downarrow & & \\ SSR_1 & \Longrightarrow & SSR_0 & & \\ \Uparrow & & \Uparrow & & \\ SR_1 & \Longrightarrow & SR_0 & & \end{array}$$

5. Conclusion

The theory of soft sets proves its efficiency as a fruitful mathematical approach to tackling the incompleteness and vagueness of knowledge, which is the utmost importance to cognitive analysis and artificial intelligence. This theory has been exploited to initiate the concept of soft topologies and some of their generalizations such as supra soft topologies and infra soft topologies [8].

In this work, we have defined and studied a new class of supra soft properties called supra soft R_0 and supra soft R_1 axioms in supra soft topological spaces, and have obtained basic properties of these axioms. We have investigated more characterizations and relationships between these classes and other soft separability properties. Several results and supported examples have been presented. Also, we have showed that the properties of begin SSR_i and $i = 0, 1$ are supra soft topological properties.

In future works, we will continue study some generalizations of some topological notions via supra soft topological spaces such as supra soft δ-topology. It is stated that the results obtained in this paper may be useful for further research on supra soft topology and its applications.

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References

[1] A. M. Abd El-latif, *Soft supra strongly generalized closed sets*, J. Intell. Fuzzy Syst., **31** (2016), 1311–1317. 1
[2] A. M. Abd El-latif, *On soft supra compactness in supra soft topological spaces*, Tbilisi Math. J., **11** (2018), 169–178. 1
[3] S. Al-Ghour, *Boolean algebra of soft Q-Sets in soft topological spaces*, Appl. Comput. Intell. Soft Comput., **2022** (2022), 9 pages. 1
[4] T. M. Al-shami, *Complete Hausdorffness and complete regularity on supra topological spaces*, J. Appl. Math., **2021** (2021), 7 pages. 1
[5] T. M. Al-shami, J. C. R. Alcantud, A. A. Azzam, *Two new families of supra-soft topological spaces defined by separation axioms*, Mathematics, **10** (2022), 18 pages. 1
[6] T. M. Al-shami, *Compactness on soft topological ordered spaces and its application on the information system*, J. Math., **2021** (2021), 12 pages. 1
[7] T. M. Al-shami, *Soft somewhat open sets: Soft separation axioms and medical application to nutrition*, Comput. Appl. Math., **41** (2022), 22 pages. 1
[8] T. M. Al-shami, *New soft structure: Infra soft topological spaces*, Math. Probl. Eng., **2021** (2021), 12 pages. 5
[9] T. M. Al-shami, I. Alshammari, *Rough sets models inspired by supra-topology structures*, Artif. Intell. Rev., **56** (2023), 6855–6883. 1
[10] T. M. Al-shami, I. Alshammari, B. A. Asaad, *Soft maps via soft somewhere dense sets*, Filomat, **34** (2020), 3429–3440. 1
[11] T. M. Al-shami, E. A. Abo-Tabl, B. A. Asaad, *Investigation of limit points and separation axioms using supra β-open sets*, Missouri J. Math. Sci., **32** (2020), 171–187. 1
[12] T. M. Al-shami, E. A. Abo-Tabl, B. A. Asaad, M. A. Arahet, *Limit points and separation axioms with respect to supra semi-open sets*, Eur. J. Pure Appl. Math., **13** (2020), 427–443.
[13] T. M. Al-shami, B. A. Asaad, M. A. El-Gayar, *Various types of supra pre-compact and supra pre-Lindelöf spaces*, Missouri J. Math. Sci., **32** (2020), 1–20. 1
[14] T. M. Al-shami, M. E. El-Shafei, *Two types of separation axioms on supra soft separation spaces*, Demonstr. Math., **52** (2019), 147–165. 1, 2.6, 2.9, 2.10, 2.11, 4.3
[15] T. M. Al-shami, M. E. El-Shafei, *On supra soft topological ordered spaces*, Arab J. Basic Appl. Sci., **26** (2019), 433–445. 1, 2.7
[16] T. M. Al-shami, M. E. El-Shafei, *Two new forms of ordered soft separation axioms*, Demonstr. Math., **53** (2020), 8–26. 1
[17] T. M. Al-shami, L. D. R. Kočinac, *Almost soft Menger and weakly soft Menger spaces*, Appl. Comput. Math., **21** (2022), 35–51. 1
[18] T. M. Al-shami, L. D. R. Kočinac, *Nearly soft Menger spaces*, J. Math., **2020** (2020), 9 pages. 1
[19] T. M. Al-shami, A. Mhemdi, *A weak form of soft α-open sets and its applications via soft topologies*, AIMS Math., **8** (2023) 11373–11396. 1
[20] T. M. Al-shami, A. Mhemdi, R. Abu-Gdairi, *A Novel framework for generalizations of soft open sets and its applications via soft topologies*, Mathematics, **11** (2023), 1–16. 1
[21] C. G. Aras, S. Bayramov, *Separation axioms in supra soft topological spaces*, Filomat, **32** (2018), 3479–3486. 1
[22] C. G. Aras, S. Bayramov, *Results of some separation axioms in supra soft topological spaces*, TWMS J. Appl. Eng. Math., **9** (2019), 58–63. 1, 2.12, 2.13
[23] B. Chen, *Soft semi-open sets and related properties in soft topological spaces*, Appl. Math. Inf. Sci., **7** (2013), 287–294. 1
[24] O. Dalkılıç, N. Demirtaş, *Algorithms for covid-19 outbreak using soft set theory: estimation and application*, Soft Comput., **27** (2023), 3203–3211. 1
[25] S. Das, S. K. Samanta, *Soft Matric*, Ann. Fuzzy Math. Inform., **6** (2013), 77–94. 1
[26] M. E. El-Shafei, M. Abo-Elhamayel, T. M. Al-shami, *Partial soft separation axioms and soft compact spaces*, Filomat, **32** (2018), 4755–4771. 2.7
[27] M. E. El-Shafei, M. Abo-Elhamayel, T. M. Al-shami, *Further notions related to new operators and compactness via supra soft topological spaces*, Int. J. Adv. Math., **1** (2019), 44–60. 1
[28] M. E. El-Shafei, A. H. Zakari, T. M. Al-shami, *Some applications of supra preopen sets*, J. Math., **2020** (2020), 11 pages. 1
[29] S. A. El-Sheikh, A. M. Abd El-Latif, *Decompositions of some types of supra soft sets and soft continuity*, Int. J. Math. Trends Technol., **9** (2014), 37–56. 1, 2.5, 2.8
[30] D. N. Georgiou, A. C. Mergaritis, *Soft set theory and topology*, Appl. Gen. Topol., **15** (2014), 93–109. 1
[31] A. Kharal, B. Ahmad, *Mapping on soft classes*, New Math. Nat. Comput., **7** (2011), 471–481. 1, 2.3
[32] L. D. R. Kočinac, T. M. Al-shami, V. Çetkin, *Selection principles in the context of soft sets: Menger spaces*, Soft Comput., **25** (2021), 12693–12702. 1
[33] A. M. Kozae, M. Shokry, M. Zidan, *Supra topologies for digital plane*, AASCIT Commun., **3** (2016), 1–10. 1
[34] P. K. Maji, R. Biswas, R. Roy, *Soft set theory*, Comput. Math. Appl., **45** (2003), 555–562. 2.2
[35] A. S. Mashhour *On supra topological spaces*, Indian J. Pure Appl. Math., **14** (1983), 502–510. 1
[36] A. S. Mashhour, F. H. Khedr, S, A. Abd El-Bakkey, *On supra-R_0 and supra-R_1 spaces*, Indian J. Pure Appl. Math., **16** (1985), 1300–1306. 2.16
[37] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl., **62** (2011), 3524–3528. 1

[38] D. Molodtsov, *Soft set theory—first results*, Comput. Math. Appl., **37** (1999), 19–31. 1, 2.1

[39] S. Saleh, *On some new properties in soft topological spaces*, Ann. Fuzzy Math. Inform., **17** (2019), 303–312. 2.14, 2.15

[40] S. Saleh, R. Abu-Gdairi, T. M. Al-shami, M. S. Abdo, *On categorical property of fuzzy soft topological spaces*, Appl. Math. Inf. Sci., **16** (2022), 635–641. 1

[41] S. Saleh, T. M. Al-shami, A. Mhemdi, *On some new types of fuzzy soft compact spaces*, J. Math., **2023** (2023), 8 pages. 1

[42] S. Saleh, K. Hur, *On some lower soft separation axioms*, Ann. Fuzzy Math. Inform., **19** (2020), 61–72. 1

[43] G. Santos-García, J. C. R. Alcantud, *Ranked soft sets*, Expert Systems, **40** (2023), 1–24. 1

[44] M. Shabir, M. Naz, *On soft topological spaces*, Comput. Math. Appl., **61** (2011), 1786–1799. 1, 2.4, 2.7

[45] O. Tantawy, S. A. El-Sheikh, S. Hamde, *Separation axioms on soft topological spaces*, Ann. Fuzzy Math. Inform., **11** (2016), 511–525. 1