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New oscillation and non-oscillation criteria for third order neutral differential equations with distributed deviating arguments

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Abstract

The oscillatory and non-oscillatory behavior of solutions of third-order neutral differential equations with distributed deviating arguments is discussed. New sufficient conditions that guarantee the oscillation of solutions are deduced. The obtained results improve and extend some recent criteria appeared in the literature. Two illustrative examples are given.

Keywords: Oscillation, non-oscillation, third order, neutral differential equation.

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1. Introduction

In this paper, we are concerned with the oscillation and asymptotic properties of solutions of a class of third-order neutral differential equations of the type

$$\left(\frac{1}{r_1(\varepsilon)}\left[\left(\frac{1}{r_2(\varepsilon)}w'(\varepsilon)\right)'\right]^{\alpha}\right)' + \int_c^d q(\varepsilon,\eta)\, f(x(\tau(\varepsilon,\eta)))\, d\eta = 0, \quad \varepsilon \geqslant \varepsilon_0, \tag{$E^{\pm}$}$$

where $w(\varepsilon) = x(\varepsilon) \pm b(\varepsilon)\, x(\delta(\varepsilon))$, and α is a quotient of odd positive integers. Moreover, we discuss the existence of non-oscillatory solutions tending to zero of (E^-) in the case of $\alpha = 1$.

Throughout this study, we assume that the following conditions are satisfied.

(A_1) $r_1(\varepsilon), r_2(\varepsilon), b(\varepsilon), \delta(\varepsilon) \in C([\varepsilon_0,\infty),(0,\infty))$, $\delta(\varepsilon) \leqslant \varepsilon$, and $\lim_{t\to\infty} \delta(\varepsilon) = \infty$;

(A_2) $\int_{\varepsilon_0}^{\infty} r_1^{\frac{1}{\alpha}}(\varepsilon)\, d\varepsilon = \int_{\varepsilon_0}^{\infty} r_2(\varepsilon)\, d\varepsilon = \infty$, $r_2'(\varepsilon) > 0$;

(A_3) $\tau(\varepsilon,\eta) \in C([\varepsilon_0,\infty)\times(c,d),\mathbb{R})$ is a non-decreasing function for η and $\liminf_{t\to\infty} \tau(\varepsilon,\eta) = \infty$, $q(\varepsilon,\eta) \in C([\varepsilon_0,\infty)\times(c,d),(0,\infty))$;

(A_4) $0 \leqslant b(\varepsilon) \leqslant b < 1$, $f \in C(\mathbb{R},\mathbb{R})$, $f'(x) > 0$, $f(x)/x^{\alpha} \geqslant \lambda$ for all $x \neq 0$ and for some $\lambda > 0$.

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Now, set $w^{[0]}(\varepsilon) = w(\varepsilon)$, $w^{[1]}(\varepsilon) = \frac{1}{r_2(\varepsilon)} w'(\varepsilon)$, $w^{[2]}(\varepsilon) = \frac{1}{r_1(\varepsilon)} \left[\left(w^{[1]}(\varepsilon)\right)'\right]^{\alpha}$.

By a solution of $(E^{\pm})$ we understand a nontrivial real valued function $x(\varepsilon)$, which satisfies that the property $\frac{1}{r_1(\varepsilon)} \left(\frac{1}{r_2(\varepsilon)} [x(\varepsilon) \pm b(\varepsilon) x(\delta(\varepsilon))]'\right)'$ is continuously differentiable and satisfies $(E^{\pm})$ for any $\varepsilon_1 \geqslant \varepsilon_0$. A solution of $(E^{\pm})$ is said to be oscillatory if it has an infinite set of zeros, otherwise it is termed non-oscillatory. Equation $(E^{\pm})$ is said to be oscillatory if all its solutions are oscillatory.

There are several applications to oscillation phenomena in real life, like as in the frame of continuous partial differential equations, and in particular in dynamical models, delay and oscillatory type effects are often modeled by external sources and/or nonlinear diffusion, perturbing the natural evolution of the related systems and those connected to mathematical biology see, e.g., ([6, 20, 25, 27]). In the last recent decades, the oscillatory behavior of solutions of various types of differential equations has received considerable interest see, e.g., [1–5, 7–19, 21–24, 26, 28–39]. Although the case of nonnegative neutral coefficients has recently received great interest by several authors (see [14, 30, 34, 35, 37]), however those which have negative neutral coefficients has not been received similar surge of interest (see [18, 19, 31]). To the best of our knowledge, we notice that the existence of nonoscillatory solutions has received less interest as well (see [11, 15, 16, 31]). For instance we mention some related papers that were concerned with some special cases of $(E^{\pm})$, and motivated this paper. Li [19] presented necessary and sufficient conditions for testing the existence of non-oscillatory solutions of the differential equation

$$\left(r(\varepsilon)[y(\varepsilon) - p(\varepsilon) y(\varepsilon - \sigma)]'\right)' + f(\varepsilon, y(\varepsilon - \delta)) = 0.$$

In [29], Mojsej and Tartalova were concerned with showing the existence of bounded non-oscillatory solutions of the differential equation

$$\left(\frac{1}{r_1(\varepsilon)} \left(\frac{1}{r_2(\varepsilon)} y'(\varepsilon)\right)'\right)' + q(\varepsilon) f(y(\varepsilon)) = 0.$$

Zhang et al. [38] studied the differential equation

$$\left(v(\varepsilon) \left[y(\varepsilon) + \int_a^b q(\varepsilon, s) y(\sigma(\varepsilon, s)) \, ds\right]''\right)' + \int_c^d p(\varepsilon, \zeta) f(y(\delta(\varepsilon, \zeta))) \, d\zeta = 0.$$

More recently Tian et al. [35] discussed the differential equation

$$\left(a(\varepsilon) \left(\left[y(\varepsilon) + \int_a^b r(\varepsilon, s) y(\tau(\varepsilon, s)) \, ds\right]''\right)^{\delta}\right)' + \int_c^d b(\varepsilon, \zeta) f(y(\sigma(\varepsilon, \zeta))) \, d\zeta = 0,$$

where $\delta \geqslant 0$ is a ratio of two odd positive integers. Meanwhile, Jiang et al. [18], were concerned with the differential equation with non-positive neutral coefficients of the form

$$\left(a(\varepsilon) \left(\left[y(\varepsilon) - \int_a^b r(\varepsilon, s) y(\tau(\varepsilon, s)) \, ds\right]''\right)^{\delta}\right)' + \int_c^d b(\varepsilon, \gamma) f(y(\eta(\varepsilon, \gamma))) \, d\gamma = 0,$$

where $\delta > 0$ is a ratio of two positive integers. In 2017, Wang et al. [37] studied the asymptotic behavior of solutions of the third order differential equation of the type

$$\left(p(\varepsilon) \left[\left(r(\varepsilon)[y(\varepsilon) + b(\varepsilon) y(\sigma(\varepsilon))]'\right)'\right]^{\alpha}\right)' + \int_c^d q(\varepsilon, s) f(y(\sigma(\varepsilon, s))) \, ds = 0,$$

where $\alpha > 0$ is a ratio of two odd positive integers. More recently, Zhao [39] discussed the oscillation of solutions of the differential equation

$$\left(\frac{1}{a(\varepsilon)}\left(\frac{1}{b(\varepsilon)}[y(\varepsilon)+r(\varepsilon)y(\sigma(\varepsilon))]'\right)'\right)'+q(\varepsilon)f(y(\delta(\varepsilon)))=0.$$

Very recently, Thandapani et al. [34] presented new oscillation results for the differential equation

$$\left(r_1(\varepsilon)\left(r_2(\varepsilon)\left[x(\varepsilon)+p(\varepsilon)x^{\beta}(c(\varepsilon))\right]'\right)'\right)'+f(\varepsilon)x^{\alpha}(\sigma(\varepsilon))=0,$$

where β, α are ratios of odd positive integers satisfying $0<\beta\leqslant 1, \alpha\geqslant 1$. In 2020, Qiu et al. [31] studied the conditions guarantee the existence of non-oscillatory solutions tending to zero of the dynamic equation on time scales of the type

$$\left(a(\varepsilon)\left(b(\varepsilon)[y(\varepsilon)+r(\varepsilon)y(f(\varepsilon))]^{\Delta}\right)^{\Delta}\right)^{\Delta}+g(\varepsilon, y(\delta(\varepsilon)))=0.$$

In this article, we aim to improve and extend some oscillation results of [39] for equation (E^{+}), and those given by [18, 39] for equation (E^{-}). Further, we study the case when there exist non-oscillatory solutions tending to zero of (E^{-}), using Knaster's theorem [17].

2. Preliminaries

Before giving our oscillation criteria, we outline the following auxiliary lemmas.

Lemma 2.1. *Let $x(\varepsilon)$ be a non-oscillatory solution of (E^{+}). Then there may exist $\varepsilon_1\geqslant\varepsilon_0$ such that $w(\varepsilon)$ for $\varepsilon\geqslant\varepsilon_1\geqslant\varepsilon_0$ has only one of the following two cases*

(C_1) $w(\varepsilon)w^{[1]}(\varepsilon)<0, w(\varepsilon)w^{[2]}(\varepsilon)>0$;
(C_2) $w(\varepsilon)w^{[1]}(\varepsilon)>0, w(\varepsilon)w^{[2]}(\varepsilon)>0$.

Proof. The proof can be deduced directly from the proof of Lemma 1 of [39]. □

Lemma 2.2 ([39]). *Assume that $x(\varepsilon)$ is a solution of (E^{+}), $w(\varepsilon)$ has the property (C_2). Then $(1-b)|w(\varepsilon)|\leqslant|x(\varepsilon)|\leqslant|w(\varepsilon)|$, for $\varepsilon\geqslant\varepsilon_1\geqslant\varepsilon_0$ and $\lim_{\varepsilon\to\infty}|w(\varepsilon)|=\lim_{\varepsilon\to\infty}|x(\varepsilon)|=\infty$.*

Lemma 2.3. *Assume that $x(\varepsilon)$ is an eventually positive solution of (E^{-}). Then there may exist $\varepsilon_1\geqslant\varepsilon_0$ such that for $\varepsilon\geqslant\varepsilon_1\geqslant\varepsilon_0$, $w(\varepsilon)$ has one of the following four cases*

(i) $w(\varepsilon)>0\ w^{[1]}(\varepsilon)>0, w^{[2]}(\varepsilon)>0$;
(ii) $w(\varepsilon)>0\ w^{[1]}(\varepsilon)<0, w^{[2]}(\varepsilon)>0$;
(iii) $w(\varepsilon)<0\ w^{[1]}(\varepsilon)<0, w^{[2]}(\varepsilon)>0$;
(iv) $w(\varepsilon)<0\ w^{[1]}(\varepsilon)<0, w^{[2]}(\varepsilon)<0$.

Proof. Let $x(\varepsilon)$ be an eventually positive solution of (E^{-}). Then one can find $\varepsilon_1\geqslant\varepsilon_0$ such that $x(\varepsilon)>0$, $x(\delta(\varepsilon))>0$, $x(\tau(\varepsilon,\eta))>0, \eta=[c,d]$ for $\varepsilon\geqslant\varepsilon_1$. Now since from (E^{-}), it follows that $\left(w^{[2]}(\varepsilon)\right)'<0$ eventually, then $w^{[2]}(\varepsilon)$ is decreasing and is of one sign for $\varepsilon\geqslant\varepsilon_1$. Now consider the case $w^{[2]}(\varepsilon)<0$, then there exists a constant $M>0$ such that $\frac{1}{r_1(\varepsilon)}\left[\left(w^{[1]}(\varepsilon)\right)'\right]^{\alpha}\leqslant -M<0$.

Then $\left[\left(w^{[1]}(\varepsilon)\right)'\right]\leqslant -r_1^{\frac{1}{\alpha}}(\varepsilon)M^{\frac{1}{\alpha}}$. Integrating from ε_1 to ε, we have $w^{[1]}(\varepsilon)\leqslant w^{[1]}(\varepsilon_1)-M^{\frac{1}{\alpha}}\int_{\varepsilon_1}^{\varepsilon} r_1^{\frac{1}{\alpha}}(s)\,ds$. Thus as $\varepsilon\to\infty$, by (A_2), we conclude that $w^{[1]}(\varepsilon)<0$.

Now since $r_2'(\varepsilon) > 0$ and

$$w^{[2]}(\varepsilon) = \frac{\left[r_2(\varepsilon) w''(\varepsilon) - r_2'(\varepsilon) w'(\varepsilon)\right]^{\alpha}}{(r_2(\varepsilon))^{2\alpha} r_1(\varepsilon)} < \mathbf{0},$$

we get $w''(\varepsilon) < 0$. But since $w^{[1]}(\varepsilon) = \frac{1}{r_2(\varepsilon)} w'(\varepsilon) < 0$, then $w'(\varepsilon) < 0$, which leads to $w(\varepsilon) < 0$, and so we get $w(\varepsilon) < 0, w^{[1]}(\varepsilon) < 0$, and $w^{[2]}(\varepsilon) < 0$. Thus we have the possibility that case (iv) holds.

Case 2: Let $w^{[2]}(\varepsilon) > 0$, Then either $w^{[1]}(\varepsilon) > 0$ or $w^{[1]}(\varepsilon) < 0$. If $w^{[1]}(\varepsilon) > 0$, then $w(\varepsilon) > 0$, while if $w^{[1]}(\varepsilon) < 0$, then $w(\varepsilon) > 0$ or $w(\varepsilon) < 0$. This means that when $w^{[2]}(\varepsilon) > 0$, and so we have the possibility of the three cases (i), (ii), (iii). This completes the proof. □

Lemma 2.4. *Suppose that* $x(\varepsilon)$ *is a solution of* (E^+) *which is eventually positive. Suppose further that* $w(\varepsilon)$ *satisfies the property* (C_1) *.If*

$$\int_{\varepsilon_0}^{\infty} r_2(\nu) \int_{\nu}^{\infty} \left[r_1(y) \int_{y}^{\infty} \int_{c}^{d} q(s,\eta)\, d\eta\, ds \right]^{\frac{1}{\alpha}} dy\, d\nu = \infty, \tag{2.1}$$

then $\lim_{\varepsilon \to \infty} x(\varepsilon) = 0$.

Proof. The proof follows the same lines of the proof of the first part of [39, Theorem 1] and so it is omitted. □

Lemma 2.5. *Assume that* $x(\varepsilon)$ *is an eventually positive solution of* (E^-) *and assume that the corresponding* $w(\varepsilon)$ *has the property* (ii). *If* (2.1) *holds, then* $\lim_{\varepsilon \to \infty} x(\varepsilon) = 0$.

Proof. From the property (ii) there may exist $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) < 0, w^{[2]}(\varepsilon) > 0$. Going through as in the proof of [39, Theorem 1] , we get $\lim_{\varepsilon \to \infty} w(\varepsilon) = 0$. Now the proof of the fact that $x(\varepsilon)$ is bounded and $\lim_{\varepsilon \to \infty} x(\varepsilon) = 0$ follows from those of [18, Lemma 2.3] and so it is omitted. □

3. Oscillation theorems

Theorem 3.1. *Let the condition* (2.1) *be satisfied and* $\tau(\varepsilon, \eta) < \varepsilon$. *If there exists a function* $\varpi(\varepsilon)$ *such that* $\varpi(\varepsilon) \in C([\varepsilon_0, \infty), R)$, $\varpi(\varepsilon) > \varepsilon, \tau(\varpi(\varepsilon), \eta) \leqslant \varepsilon$, *and*

$$\lim_{\varepsilon \to \infty} \int_{\varepsilon}^{\varpi(\varepsilon)} \int_{c}^{d} q(s,\eta) \left[\int_{\varepsilon_0}^{\tau(s,\eta)} r_2(y) \int_{\varepsilon_0}^{y} r_1^{\frac{1}{\alpha}}(\nu)\, d\nu\, dy \right]^{\alpha} d\eta\, ds = \infty. \tag{3.1}$$

Then any proper solution x *of* (E^+) *is either oscillatory or vanishes eventually.*

Proof. Assume that x is an eventually positive solution of (E^+). Then by Lemma 2.1, $w(\varepsilon)$ has one of the two properties (C_1) or (C_2). Suppose first that $w(\varepsilon)$ has the property (C_1). Then by Lemma 2.4 we have $\lim_{\varepsilon \to \infty} x(\varepsilon) = 0$. Now if $w(\varepsilon)$ has the property (C_2), then there exists $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) > 0$ and $w^{[2]}(\varepsilon) > 0$ for $\varepsilon \geqslant \varepsilon_1$. Let ε_2 be such that $\tau(\varepsilon, \eta) \geqslant \varepsilon_2$, for $\varepsilon \geqslant \varepsilon_2 \geqslant \varepsilon_1$. Since $\left(w^{[2]}(\varepsilon)\right)' = -\int_c^d q(\varepsilon,\eta) f(x(\tau(\varepsilon,\eta)))\, d\eta < 0$ for $\varepsilon \geqslant \varepsilon_2$, then $w^{[2]}(\varepsilon)$ is a positive decreasing function. Now integrating both sides of (E^+) from ε to ∞, we get

$$w^{[2]}(\varepsilon) = w^{[2]}(\infty) + \int_{\varepsilon}^{\infty} \int_{c}^{d} q(s,\eta) f(x(\tau(s,\eta)))\, d\eta\, ds,$$

i.e.,

$$w^{[2]}(\varepsilon) \geqslant \int_{\varepsilon}^{\infty} \int_{c}^{d} q(s,\eta) f(x(\tau(s,\eta)))\, d\eta\, ds \geqslant \lambda \int_{\varepsilon}^{\infty} \int_{c}^{d} q(s,\eta) x^{\alpha}(\tau(s,\eta))\, d\eta\, ds.$$

Now by Lemma 2.2, we have

$$w^{[2]}(\varepsilon) \geqslant \lambda(1-b)^{\alpha}\int_{\varepsilon}^{\infty}\int_{c}^{d} q(s,\eta)\, w^{\alpha}(\tau(s,\eta))\, d\eta ds \geqslant \lambda(1-b)^{\alpha}\int_{\varepsilon}^{\varpi(\varepsilon)}\int_{c}^{d} q(s,\eta)\, w^{\alpha}(\tau(s,\eta))\, d\eta ds. \quad (3.2)$$

Integrating $w^{[2]}(\varepsilon) = w^{[2]}(\varepsilon)$ twice from ε_1 to ε, we get

$$w(\varepsilon) \geqslant \int_{\varepsilon_1}^{\varepsilon} r_2(s)\int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\left(w^{[2]}(y)\right)^{\frac{1}{\alpha}} dy ds.$$

Now since for $\varepsilon \geqslant \varepsilon_1$, we have

$$w(\tau(\varepsilon,\eta)) \geqslant \int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} r_2(s)\int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\left(w^{[2]}(y)\right)^{\frac{1}{\alpha}} dy ds,$$

then

$$w^{\alpha}(\tau(\varepsilon,\eta)) \geqslant \left[\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s)\int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\left(w^{[2]}(y)\right)^{\frac{1}{\alpha}} dy ds\right]^{\alpha}.$$

Substituting into (3.2), we get

$$w^{[2]}(\varepsilon) \geqslant \lambda(1-b)^{\alpha}\int_{\varepsilon}^{\varpi(\varepsilon)}\int_{c}^{d} q(s,\eta)\left[\int_{\varepsilon_1}^{\tau(s,\eta)} r_2(y)\int_{\varepsilon_1}^{y} r_1^{\frac{1}{\alpha}}(\nu)\left(w^{[2]}(\nu)\right)^{\frac{1}{\alpha}} d\nu dy\right]^{\alpha} d\eta ds.$$

Using the facts that $w^{[2]}(\varepsilon)$ is decreasing, $w^{[2]}(\tau(\varpi(\varepsilon),\eta))$ is nonincreasing, and $\frac{\partial\tau(\varpi(\varepsilon),\eta)}{\partial\eta} \geqslant 0$, we get

$$w^{[2]}(\varepsilon) \geqslant \lambda(1-b)^{\alpha} w^{[2]}(\tau(\varpi(\varepsilon),d))\int_{\varepsilon}^{\varpi(\varepsilon)}\int_{c}^{d} q(s,\eta)\left[\int_{\varepsilon_1}^{\tau(s,\eta)} \mathbf{r}_2(y)\int_{\varepsilon_1}^{y} r_1^{\frac{1}{\alpha}}(\nu)\, d\nu dy\right]^{\alpha} d\eta ds.$$

But since $w^{[2]}(\varepsilon)$ is decreasing and positive, we have

$$1 \geqslant \frac{w^{[2]}(\varepsilon)}{w^{[2]}(\tau(\varpi(\varepsilon),d))} \geqslant \lambda(1-b)^{\alpha}\int_{\varepsilon}^{\varpi(\varepsilon)}\int_{c}^{d} q(s,\eta)\left[\int_{\varepsilon_1}^{\tau(s,\eta)} \mathbf{r}_2(y)\int_{\varepsilon_1}^{y} r_1^{\frac{1}{\alpha}}(\nu)\, d\nu dy\right]^{\alpha} d\eta ds.$$

This is a contradiction with (3.1), and so the proof is completed. □

Theorem 3.2. *Assume that* (2.1) *holds. If*

$$\int_{\varepsilon_0}^{\infty}\int_{c}^{d} q(\varepsilon,\eta)\left[\int_{\varepsilon_0}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s)\, ds\right]^{\alpha} d\eta d\varepsilon = \infty, \quad (3.3)$$

then any proper solution of (E^{+}), *is either oscillatory or vanishes eventually.*

Proof. Assume that $x(\varepsilon)$ is an eventually positive solution. By Lemma 2.4, any solution $x(\varepsilon)$ of (E^{+}) tends to zero as $\varepsilon \to \infty$ in the case when $w(\varepsilon)$ has the property (C_1). Assume that $w(t)$ has the property (C_2). Then there may exist $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) > 0, w^{[2]}(\varepsilon) > 0$ for $\varepsilon \geqslant \varepsilon_1$. Thus $w^{[1]}(\varepsilon) > w^{[1]}(\varepsilon_1)$. By integrating from ε_1 to ε, we get

$$w(\varepsilon) \geqslant w^{[1]}(\varepsilon_1)\int_{\varepsilon_1}^{\varepsilon} r_2(s)\, ds = L\int_{\varepsilon_1}^{\varepsilon} r_2(s)\, ds, L > 0.$$

Moreover by Lemma 2.2, we have

$$x(\tau(\varepsilon,\eta)) \geqslant w(\tau(\varepsilon,\eta))(1-b) \geqslant (1-b) L \int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s)\, ds. \tag{3.4}$$

Let $\varepsilon_2 \geqslant \varepsilon_1$ be such that $\tau(\varepsilon,\eta) \geqslant \varepsilon_2$. Integrating Eq. (E^+) from ε_2 to ∞, we obtain

$$\mathbf{w}^{[2]}(\varepsilon_2) - \mathbf{w}^{[2]}(\infty) = \int_{\varepsilon_2}^{\infty}\int_c^d \mathbf{q}(s,\eta)\, \mathbf{f}(x(\tau(s,\eta)))\, d\eta ds.$$

Therefore

$$\int_{\varepsilon_2}^{\infty}\int_c^d \mathbf{q}(s,\eta)\, \mathbf{f}(x(\tau(s,\eta)))\, d\eta ds < \infty.$$

Using (A_4), we have

$$\lambda \int_{\varepsilon_2}^{\infty}\int_c^d \mathbf{q}(s,\eta)\, \mathbf{x}^{\alpha}(\tau(s,\eta))\, d\eta ds \leqslant \int_{\varepsilon_2}^{\infty}\int_c^d \mathbf{q}(s,\eta)\, \mathbf{f}(x(\tau(s,\eta)))\, d\eta ds,$$

i.e.,

$$\lambda \int_{\varepsilon_2}^{\infty}\int_c^d \mathbf{q}(s,\eta)\, \mathbf{x}^{\alpha}(\tau(s,\eta))\, d\eta ds < \infty.$$

Now using (3.4), we obtain

$$\lambda(1-b)^{\alpha} L^{\alpha} \int_{\varepsilon_2}^{\infty}\int_c^d q(s,\eta) \left(\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s)\, ds\right)^{\alpha} d\eta ds < \infty.$$

This is a contradiction with (3.3), and so the proof is completed. □

Theorem 3.3. *Suppose that* $\tau(\varepsilon,\eta) \leqslant \varepsilon, f(\nu_1\nu_2) \geqslant f(\nu_1) f(\nu_2)$ *for* $\nu_1,\nu_2 \in R$, *and* (2.1) *holds. If* $\int_0^1 \frac{1}{f\left(u^{\frac{1}{\alpha}}\right)} du < \infty$, *and*

$$\int_{\varepsilon_0}^{\infty}\int_c^d q(\varepsilon,\eta) \left(\int_{\varepsilon_0}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_0}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right)^{\alpha} d\eta d\varepsilon = \infty, \tag{3.5}$$

then any proper solution $x(\varepsilon)$ *of* (E^+) *is either oscillatory or vanishes eventually.*

Proof. Suppose that x is an eventually positive solution. By Lemma 2.4 any solution $x(\varepsilon)$ tends to zero as $\varepsilon \to \infty$ in the case when $w(\varepsilon)$ has the property (C_1). Suppose that $w(\varepsilon)$has the property (C_2). Then there may exist $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) > 0, w^{[2]}(\varepsilon) > 0$ for all $\varepsilon \geqslant \varepsilon_1$. But since $w^{[2]}(\varepsilon)$ is decreasing, we have

$$\mathbf{w}^{[1]}(\varepsilon) = \mathbf{w}^{[1]}(\varepsilon_1) + \int_{\varepsilon_1}^{\varepsilon} r_1^{\frac{1}{\alpha}}(s) \left(w^{[2]}(s)\right)^{\frac{1}{\alpha}} \mathbf{ds} \geqslant \left(w^{[2]}(\varepsilon)\right)^{\frac{1}{\alpha}} \int_{\varepsilon_1}^{\varepsilon} r_1^{\frac{1}{\alpha}}(s)\, \mathbf{ds}.$$

Therefore

$$\mathbf{w}'(\varepsilon) \geqslant \left[\mathbf{w}^{[2]}(\varepsilon)\right]^{\frac{1}{\alpha}} \mathbf{r}_2(\varepsilon) \int_{\varepsilon_1}^{\varepsilon} r_1^{\frac{1}{\alpha}}(s)\, \mathbf{ds}.$$

Thus

$$w(\varepsilon) \geqslant w(\varepsilon) - w(\varepsilon_1) \geqslant \int_{\varepsilon_1}^{\varepsilon} \left(w^{[2]}(s)\right)^{\frac{1}{\alpha}} r_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds.$$

But since $w^{[2]}(\varepsilon)$ is decreasing, then

$$w(\varepsilon) \geqslant \left(w^{[2]}(\varepsilon)\right)^{\frac{1}{\alpha}} \int_{\varepsilon_1}^{\varepsilon} r_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds. \tag{3.6}$$

Now by Lemma 2.2, it follows from (E^+) that

$$-\left(w^{[2]}(\varepsilon)\right)' = \int_c^d q(\varepsilon,\eta) f(x(\tau(\varepsilon,\eta)))\, d\eta \geqslant \int_c^d q(\varepsilon,\eta) f(1-b) f(w(\tau(\varepsilon,\eta)))\, d\eta.$$

So (3.6) leads to

$$-\left[w^{[2]}(\varepsilon)\right]' \geqslant \int_c^d q(\varepsilon,\eta) f(1-b) f\left(\left[w^{[2]}(\tau(\varepsilon,\eta))\right]^{\frac{1}{\alpha}}\right) f\left(\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right) d\eta.$$

This with (A_4) leads to

$$\begin{aligned} -\left[w^{[2]}(\varepsilon)\right]' &\geqslant \lambda f(1-b) \int_c^d q(\varepsilon,\eta) f\left(\left[w^{[2]}(\tau(\varepsilon,\eta))\right]^{\frac{1}{\alpha}}\right) \left[\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right]^{\alpha} d\eta \\ &\geqslant \lambda f(1-b) f\left(\left[w^{[2]}(\varepsilon)\right]^{\frac{1}{\alpha}}\right) \int_c^d q(\varepsilon,\eta) \left[\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right]^{\alpha} d\eta. \end{aligned}$$

Thus

$$\frac{-\left[w^{[2]}(\varepsilon)\right]'}{f\left(\left[w^{[2]}(\varepsilon)\right]^{\frac{1}{\alpha}}\right)} \geqslant \lambda f((1-b)) \int_c^d q(\varepsilon,\eta) \left[\int_{\varepsilon_1}^{\tau(\varepsilon,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right]^{\alpha} d\eta.$$

Integrating from $\varepsilon_1$to ε,

$$\int_{\varepsilon_1}^{\varepsilon} \frac{-\left(w^{[2]}(s)\right)'}{f\left(\left[w^{[2]}(s)\right]^{\frac{1}{\alpha}}\right)} ds \geqslant \lambda f(1-b) \int_{\varepsilon_1}^{\varepsilon} \int_c^d q(\nu,\eta) \left[\int_{\varepsilon_1}^{\tau(\nu,\eta)} \mathbf{r}_2(s) \int_{\varepsilon_1}^{s} r_1^{\frac{1}{\alpha}}(y)\, dy ds\right]^{\alpha} d\eta d\nu.$$

Thus as $\varepsilon \to \infty$, we obtain

$$-\int_{\varepsilon_1}^{\infty} \frac{\left(w^{[2]}(s)\right)'}{f\left(\left[w^{[2]}(s)\right]^{\frac{1}{\alpha}}\right)} ds = \int_{w^{[2]}(\infty)}^{w^{[2]}(\varepsilon_1)} \frac{du}{f\left(u^{\frac{1}{\alpha}}\right)} < \infty.$$

This is a contradiction with condition (3.5) and so the proof is completed. □

Theorem 3.4. *Suppose that (2.1) and (3.1) hold. Suppose further that $\tau(\varepsilon,\eta) < \varepsilon$, and there may exist a function $\varpi(\varepsilon)$such that $\varpi(\varepsilon) \in C([\varepsilon_0,\infty), R)$,$\varpi(\varepsilon) > \varepsilon$, $\tau(\varpi(\varepsilon),\eta) \leqslant \varepsilon$. Then any proper solution x of (E^-) is either oscillatory or vanishes eventually.*

Proof. Suppose that x is an eventually positive solution of (E^-). By Lemma 2.3 we observe that for $\varepsilon \geqslant \varepsilon_1 \geqslant \varepsilon_0$, we may have one of four cases (i), (ii), (iii) or (iv). Assume first that the case (i) is satisfied. Then there exists $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0$, $w^{[1]}(\varepsilon) > 0$ and $w^{[2]}(\varepsilon) > 0$ for $\varepsilon \geqslant \varepsilon_1$. Then following the lines of the proof of Theorem 3.1, we get a contradiction with (3.1) . Suppose that case (ii) holds. Then it follows by Lemma 2.5 that $\lim_{\varepsilon\to\infty} x(\varepsilon) = 0$. If case (iii) or (iv) holds, then as the proof of the last part of [18, Theorem 3.1], it follows that $x(\varepsilon)$ vanishes eventually and thus the proof is completed. □

Theorem 3.5. *Let (2.1) and (3.3) be satisfied. Then any proper solution $x(\varepsilon)$ of (E^-) is either oscillatory or vanishes eventually.*

Proof. Suppose that x is an eventually positive solution, and $w(\varepsilon)$ has the property (i). Then there may exist $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) > 0, w^{[2]}(\varepsilon) > 0$ for $\varepsilon \geqslant \varepsilon_1$. Following the lines of the proof of Theorem 3.2, we get a contradiction with (3.3). The proof for the cases (ii), (iii), (iv) will be as in Theorem 3.4, and so it is omitted. □

Theorem 3.6. *Suppose that* $\tau(\varepsilon,\eta) \leqslant t, f(\nu_1\nu_2) \geqslant f(\nu_1) f(\nu_2)$ *for* $\nu_1, \nu_2 \in \mathbb{R}$, *and* $\int_0^1 \frac{1}{f\left(s^{\frac{1}{\alpha}}\right)} ds < \infty$. *If* (2.1) *and* (3.5) *hold, then any proper solution* $x(\varepsilon)$ *of* (E^-) *is either oscillatory or satisfies* $\lim_{\varepsilon\to\infty} x(\varepsilon) = 0$.

Proof. Suppose that x is an eventually positive solution of (E^-), and $w(\varepsilon)$ has the property (i). Then there may exist $\varepsilon_1 \geqslant \varepsilon_0$ such that $w(\varepsilon) > 0, w^{[1]}(\varepsilon) > 0, w^{[2]}(\varepsilon) > 0$ for all $\varepsilon \geqslant \varepsilon_1$. Following the lines of the proof of Theorem 3.3, we get a contradiction with condition (3.5) and the proof of the cases (ii), (iii), (iv) will go through as in Theorem 3.4. □

Remark 3.7. Although our technique of the proof of our oscillation results depends on that given in [39], however beside our results are dealing with negative neutral coefficients which was not considered by [39], the author there was dealing with the special case $\alpha = 1$ and $\int_c^d q(\varepsilon,\eta) f(x(\tau(\varepsilon,\eta))) d\eta = q(\varepsilon) f(x(\delta(\varepsilon)))$.

4. Non-oscillation theorems

Now, we study the existence of nonoscillatory solutions of (E^-)in the case $\alpha = 1$, we assume that (A_1), (A_3), and (A_4) hold. We also assume that $\lim_{\varepsilon\to\infty} b(\varepsilon) = b_0$, where $0 \leqslant b_0 < 1$.

Theorem 4.1. *Let* $R_1(\varepsilon_0) < \infty$, *and*

$$\int_{\varepsilon_0}^{\infty}\int_{\varepsilon_0}^{s}\int_c^d r_1(s) q(u,\eta) f(kR_1(\tau(u,\eta))) d\eta du ds < \infty, \tag{4.1}$$

for some $k \neq 0$, *and*

$$\lim_{\varepsilon\to\infty} R_1(\delta(\varepsilon))/R_1(\varepsilon) = 1,$$

where $R_1(\varepsilon) = \int_\varepsilon^\infty r_2(s) ds$. *Then there exists an eventually positive solution* $x(\varepsilon)$ *of* (E^-), *with* $\lim_{\varepsilon\to\infty} x(\varepsilon) = 0$, *where* $w^{[1]}(\varepsilon)$ *and* $w^{[2]}(\varepsilon)$ *are both eventually negative.*

Proof. Assume that (4.1) holds for $k > 0$. The case $k < 0$ has similar arguments. Now since

$$\lim_{\varepsilon\to\infty} R_1(\delta(\varepsilon))/R_1(\varepsilon) = 1,$$

we may choose $b < B < 1$ such that $b(\varepsilon) R_1(\delta(\varepsilon))/R_1(\varepsilon) \leqslant B$ and $\varepsilon_1 \geqslant \varepsilon_0$ so large such that

$$\int_{\varepsilon_1}^{\infty}\int_{\varepsilon_1}^{s}\int_c^d r_1(s) q(u,\eta) f(kR_1(\tau(u,\eta))) d\eta du ds < (1-B) a, \tag{4.2}$$

where $a = \frac{k}{2}$. Consider the equation

$$x(\varepsilon) = b(\varepsilon) x(\delta(\varepsilon)) + (1-B) aR_1(\varepsilon) + \int_\varepsilon^\infty\int_{\varepsilon_1}^{y}\int_{\varepsilon_1}^{s}\int_c^d r_1(s) r_2(y) q(u,\eta) f(x(\tau(u,\eta))) d\eta du ds dy. \tag{4.3}$$

It is easily proved that any solution of (4.3) also satisfies (E^-). Now consider the Banach space Φ of all bounded real functions $x(\varepsilon)$ with norm $\sup_{\varepsilon\geqslant\varepsilon_1} |x(\varepsilon)|$, endowed with the usual pointwise ordering $\leqslant$: for $x_1, x_2 \in \Phi$, $x_1 \leqslant x_2$ is equivalent to $x_1(\varepsilon) \leqslant x_2(\varepsilon)$ for all $\varepsilon \geqslant \varepsilon_1$. Then Φ is partially ordered. Define a subset Ω_1 of Φ as

$$\Omega_1 = \{x \in \Phi : (1-B) aR_1(\varepsilon) \leqslant x(\varepsilon) \leqslant 2aR_1(\varepsilon), \varepsilon \geqslant \varepsilon_1\}.$$

For any subset $Y \subset \Omega_1$, it is clear that $\inf Y \in \Omega_1$ and $\sup Y \in \Omega_1$. Moreover, define an operator F: $\Omega_1 \to \Phi$ as

$$(Fx)(\varepsilon) = b(\varepsilon) x(\delta(\varepsilon)) + (1-B) aR_1(\varepsilon) + \int_\varepsilon^\infty\int_{\varepsilon_1}^{y}\int_{\varepsilon_1}^{s}\int_c^d r_1(s) r_2(y) q(u,\eta) f(x(\tau(u,\eta))) d\eta du ds dy.$$

Then the mapping F satisfies the assumptions of Knaster's fixed point theorem [17], and satisfies following.

(I) F maps Ω_1 into itself. Indeed, if $x \in \Omega_1$, then by (4.2), we have

$$(1-B)\,aR_1(\varepsilon) \leqslant (Fx)(\varepsilon) \leqslant 2b(\varepsilon)\,aR_1(\delta(\varepsilon)) + (1-B)\,aR_1(\varepsilon) + (1-B)\,aR_1(\varepsilon).$$

Then

$$(1-B)\,aR_1(\varepsilon) \leqslant (Fx)(\varepsilon) \leqslant 2BaR_1(\varepsilon) + (1-B)\,aR_1(\varepsilon) + (1-B)\,aR_1(\varepsilon) = 2aR_1(\varepsilon).$$

(II) Since f is increasing, F is nondecreasing. That is, for any $x_1, x_2 \in \Omega_1, x_1 \leqslant x_2$ implies that $Fx_1 \leqslant Fx_2$. By Knaster's fixed point theorem [17], there may exist an $x \in \Omega_1$ such that $Fx = x$, that is, $x(\varepsilon)$ is a non-oscillatory solution of (E^-), with

$$\begin{aligned}\int_{\varepsilon}^{\infty}\int_{\varepsilon_1}^{y}\int_{\varepsilon_1}^{s}\int_{c}^{d} & r_1(s)\,r_2(y)\,q(u,\eta)\,f(x(\tau(u,\eta)))\,d\eta du ds dy \\ &\leqslant R_1(\varepsilon)\int_{\varepsilon_1}^{\infty}\int_{\varepsilon_1}^{s}\int_{c}^{d} r_1(s)\,q(u,\eta)\,f(kR_1(\tau(u,\eta)))\,d\eta du ds,\end{aligned}$$

and

$$\lim_{\varepsilon\to\infty} R_1(t)\int_{\varepsilon_1}^{\infty}\int_{\varepsilon_1}^{s}\int_{c}^{d} r_1(s)\,q(u,\eta)\,f(kR_1(\tau(u,\eta)))\,d\eta du ds = 0.$$

Thus we arrive at $w(\varepsilon)$ vanishes eventually and by [31, Lemma 2.2], $x(\varepsilon)$ vanishes eventually. Now for $\varepsilon \in [\varepsilon_1, \infty)$, we obtain

$$w^{[1]}(\varepsilon) = -(1-B)\,aR_1(\varepsilon) - \int_{\varepsilon_1}^{\varepsilon}\int_{\varepsilon_1}^{s}\int_{c}^{d} r_1(s)\,q(u,\eta)\,f(x(\tau(u,\eta)))\,d\eta du ds < 0,$$

and

$$w^{[2]}(\varepsilon) = -\int_{\varepsilon_1}^{\varepsilon}\int_{c}^{d} q(u,\eta)\,f(x(\tau(u,\eta)))\,d\eta du < 0.$$

This completes the proof. □

Theorem 4.2. *Suppose that* $R_2(\varepsilon_0) < \infty$ *and*

$$\int_{\varepsilon_0}^{\infty}\int_{c}^{d} q(\varepsilon,\eta)\,f(kR_2(\tau(\varepsilon,\eta)))\,d\eta d\varepsilon < \infty. \tag{4.4}$$

If $\lim_{\varepsilon\to\infty} R_2(\delta(\varepsilon))/R_2(\varepsilon) = 1$, *where* $R_2(\varepsilon) = \int_{\varepsilon}^{\infty}\int_{y}^{\infty} r_1(s)\,r_2(y)\,ds dy$, *then* (E^-) *has an eventually positive solution* $x(\varepsilon)$, *with* $\lim_{\varepsilon\to\infty} x(\varepsilon) = 0$, *where* $w^{[1]}(\varepsilon)$ *is eventually negative while* $w^{[2]}(\varepsilon)$ *is eventually positive.*

Proof. Assume that (4.4) holds for $k > 0$. The proof of the case $k < 0$ is similar. Now since

$$\lim_{\varepsilon\to\infty} R_2(\delta(\varepsilon))/R_2(\varepsilon) = 1,$$

we may choose $b < M < 1$ such that $b(\varepsilon)\,R_2(\delta(\varepsilon))/R_2(\varepsilon) \leqslant M$ and $\varepsilon_1 \geqslant \varepsilon_0$ so large such that

$$\int_{\varepsilon_1}^{\infty}\int_{c}^{d} q(\varepsilon,\eta)\,f(kR_2(\tau(\varepsilon,\eta)))\,d\varepsilon d\eta < (1-M)\,a,$$

where $a = \frac{k}{2}$. Consider the equation

$$x(\varepsilon) = b(\varepsilon)\,x(\delta(\varepsilon)) + (1-M)\,aR_2(\varepsilon) + \int_{\varepsilon}^{\infty}\int_{y}^{\infty}\int_{s}^{\infty}\int_{c}^{d} r_1(s)\,r_2(y)\,q(u,\eta)\,f(x(\tau(u,\eta)))\,d\eta du ds dy. \tag{4.5}$$

It is clear that any solution of (4.5) must satisfy (E^-). Consider the Banach space Φ as in Theorem 4.1. Now define a subset Ω_2 of Φ as

$$\Omega_2 = \{x \in \Phi : (1-M)\, aR_2(\varepsilon) \leqslant x(\varepsilon) \leqslant 2aR_2(\varepsilon), \varepsilon \geqslant \varepsilon_1\},$$

for any subset $Y \subset \Omega_2$. It is clear that $\inf Y \in \Omega_2$ and $\sup Y \in \Omega_2$. Moreover define an operator F: $\Omega_2 \to \Phi$ as

$$(Fx)(\varepsilon) = b(\varepsilon)\, x(\delta(\varepsilon)) + (1-M)\, aR_2(\varepsilon) + \int_\varepsilon^\infty \int_y^\infty \int_s^\infty \int_c^d r_1(s)\, r_2(y)\, q(u,\eta)\, f(x(\tau(u,\eta)))\, d\eta du ds dy,$$

$\varepsilon \in [\varepsilon_1, \infty)$. As in the proof of Theorem 4.1, F satisfies the conditions of the fixed point theorem of Knaster [17] and so there may exist an $x \in \Omega_2$ such that $Fx = x$, i.e., $x(\varepsilon)$ is a non-oscillatory solution of (E^-), but since

$$\int_\varepsilon^\infty \int_y^\infty \int_s^\infty \int_c^d r_1(s)\, r_2(y)\, q(u,\eta)\, f(x(\tau(u,\eta)))\, d\eta du ds dy \leqslant R_2(\varepsilon) \int_{\varepsilon_1}^\infty \int_c^d q(u,\eta)\, f(kR_2(\tau(u,\eta)))\, d\eta du$$

and

$$\lim_{\varepsilon\to\infty} R_2(\varepsilon) \int_{\varepsilon_1}^\infty \int_c^d q(u,\eta)\, f(kR_2(\tau(u,\eta)))\, d\eta du = 0,$$

we conclude that $w(\varepsilon)$ vanishes eventually and consequently due to [31, Lemma 2.2], $x(\varepsilon)$ vanishes eventually. Now we obtain for $\varepsilon \in [\varepsilon_1, \infty)$,

$$w^{[1]}(\varepsilon) = -(1-M) \int_\varepsilon^\infty r_1(s)\, ds - \int_\varepsilon^\infty \int_s^\infty \int_c^d r_1(s)\, q(u,\eta)\, f(x(\tau(u,\eta)))\, d\eta du ds < 0,$$

and

$$w^{[2]}(\varepsilon) = (1-M) + \int_\varepsilon^\infty \int_c^d q(u,\eta)\, f(x(\tau(u,\eta)))\, d\eta du > 0.$$

This completes the proof. □

Remark 4.3. Although our nonoscillatory results depend on the technique of [31], the authors there considered a neutral dynamic third order equation in the special case $\int_c^d q(\varepsilon,\eta)\, f(x(\tau(\varepsilon,\eta)))\, d\eta = f(\varepsilon, x(g(\varepsilon)))$.

5. Examples

Example 5.1. Consider the third-order differential equation

$$\left(e^{-\varepsilon}\left(x(\varepsilon) + e^{-\pi} x(\varepsilon-\pi)\right)'\right)'' + \left(1 - e^{-2\pi}\right) \int_{-\frac{5}{2}\pi}^{-\pi} e^{-\varepsilon-\eta} x(\varepsilon+\eta)\, d\eta = 0. \tag{5.1}$$

Here, $r_1(\varepsilon) = 1$, $r_2(\varepsilon) = e^\varepsilon, \alpha = 1, b(\varepsilon) = e^{-\pi}, c = -\frac{5}{2}\pi, d = -\pi, q(\varepsilon,\eta) = e^{-\varepsilon-\eta}, \tau(\varepsilon,\eta) = \varepsilon+\eta$, and $f(x(\tau(\varepsilon,\eta))) = \left(1-e^{-2\pi}\right) x(\varepsilon+\eta)$. Thus clearly, the assumptions of Theorem 3.2 hold. Therefore, every solution $x(\varepsilon)$ of (5.1) is either oscillatory or vanishes eventually. In fact $x(\varepsilon) = e^\varepsilon \sin\varepsilon$ is an oscillatory solution of (5.1).

Example 5.2. Consider the differential equation

$$\left(\varepsilon^4\left(\varepsilon^2\left[x(\varepsilon)-\frac{\varepsilon-1}{2\varepsilon}x(\varepsilon-3)\right]'\right)'\right)'+\frac{2}{3\pi^2}\int_\pi^{2\pi}\frac{\eta}{\varepsilon^2}x\left(\frac{\varepsilon}{2}\right)d\eta=0,\quad \varepsilon\geqslant 2. \tag{5.2}$$

Here $r_1(\varepsilon)=\frac{1}{\varepsilon^4}, r_2(\varepsilon)=\frac{1}{\varepsilon^2}, b(\varepsilon)=\frac{\varepsilon-1}{2\varepsilon}, \delta(\varepsilon)=\varepsilon-3, \tau(\varepsilon,\eta)=\frac{\varepsilon}{2}, f(x)=x, q(\varepsilon,\eta)=\frac{2\eta}{3\varepsilon^2\pi^2}, c=\pi, d=2\pi$. It is obvious that the coefficients of (5.2) satisfy (A_1), (A_3), and (A_4). Moreover it is clear that $\lim_{\varepsilon\to\infty}b(\varepsilon)=\frac{1}{2}$. Now since $\int_{\varepsilon_0}^\infty r_2(u)\,du=\int_2^\infty\frac{d\varepsilon}{\varepsilon^2}<1$, and $R_1(\varepsilon)=\int_\varepsilon^\infty r_2(u)\,du=\frac{1}{\varepsilon}<1$, thus $\lim_{\varepsilon\to\infty}R_1(\delta(\varepsilon))/R_1(\varepsilon)=\lim_{\varepsilon\to\infty}\frac{\varepsilon}{\varepsilon-3}=1, f(kR_1(\tau(\varepsilon,\eta)))=kR_1(\tau(\varepsilon,\eta))<k<\infty, k>0$, and

$$\int_{\varepsilon_0}^\infty\int_{\varepsilon_0}^s\int_c^d r_1(u)\,q(u,\eta)\,f(kR_1(\tau(u,\eta)))\,d\eta du ds<k\int_2^\infty\int_2^s\int_\pi^{2\pi}\frac{2\eta}{3s^4u^2\pi^2}d\eta du ds<\infty.$$

Thus by Theorem 4.1, it follows that (5.2) has an eventually positive solution $x(\varepsilon)$ that vanishes eventually, where $w^{[1]}(\varepsilon)<0$, $w^{[2]}(\varepsilon)<0$ eventually.

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References

[1] R. P. Agarwal, M. Bohner, T. Li, C. Zhang, *Oscillation of Third-Order Nonlinear Delay Differential Equations*, Taiwanese J. Math., **17** (2013), 545–558. 1

[2] R. P. Agarwal, C. Zhang, T. Li, *Some remarks on oscillation of second order neutral differential equations*, Appl. Math. Comput., **274** (2016), 178–181.

[3] M. Bohner, T. Li, *Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient*, Appl. Math. Lett., **37** (2014), 72–76.

[4] T. Candan, R. S. Dahiya, *Oscillation of third order functional differential equations with delay*, Electron. J. Differ. Equ. Conf., **10** (2003), 79–88.

[5] K.-S. Chiu, T. Li, *Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments*, Math. Nachr., **292** (2019), 2153–2164. 1

[6] A. Columbu, S. Frassu, G. Viglialoro, *Refined criteria toward boundedness in an attraction–repulsion chemotaxis system with nonlinear productions*, Appl. Anal., (2023), 1–17. 1

[7] I. Dassios , O. Bazighifan, *Oscillation Conditions for Certain Fourth-Order Non-Linear Neutral Differential Equation*, Symmetry, **12** (2020), 1–9. 1

[8] J. Džurina, S. R. Grace, I. Jadlovská, T. Li, *Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term*, Math. Nachr., **293** (2020), 910–922.

[9] E. M. Elabbasy, T. S. Hassan, B. M. Elmatary, *Oscillation criteria for third order delay nonlinear differential equations*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), 1–11.

[10] A. A. El-Gaber, M. M. A. El-Sheikh, E. I. El-Saedy, *Oscillation of super-linear fourth-order differential equations with sub-linear neutral terms*, Bound. Value Probl., **2022** (2022), 1–14.

[11] M. M. A. El-Sheikh, *Oscillation and nonoscillation criteria for second order nonlinear differential equations. I*, J. Math. Anal. Appl., **179** (1993), 14–27. 1

[12] M. M. A. El-Sheikh, R. A. Sallam, D. I. Elimy, *Oscillation criteria for second order nonlinear equations with damping*, Adv. Differ. Equ. Control Process., **8** (2011), 127–142.

[13] M. M. A. El-Sheikh, R. A. Sallam, E. I. El-Saedy, *On the oscillatory behavior of solutions of second order nonlinear neutral delay differential equations*, Wseas Trans. Math., **17** (2018), 51–57.

[14] M. M. A. El-Sheikh, R. A. Sallam, S. Salem, *Oscillation of nonlinear third- order differential equations with several sublinear neutral terms*, Math. Slovaca, **71** (2021) 1411–1426. 1

[15] J. R. Graef, *Some nonoscillation criteria for higher order nonlinear differential equations*, Pacific J. Math., **66** (1976), 125–129. 1

[16] J. R. Graef, *Oscillation, Nonoscillation, and growth of solutions of nonlinear functional differential equations of arbitrary order*, J. Math. Anal. Appl., **60** (1977), 398-409. 1

[17] I. Gyori, G. Ladas, *Oscillation theory of delay differential equations with applications*, Oxford University Press, New York, 1991. 1, 4, II, 4

[18] C. Jiang, Y. Jiang, T. Li, *Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments*, Adv. Difference Equ., **2016** (2016), 14 pages. 1, 1, 2, 3

[19] W. Li, *Classifications and Existence of Nonoscillatory Solutions of Second Order Nonlinear Neutral Differential Equations*, Ann. Polon. Math., **65** (1997), 283–302. 1

[20] T. Li, S. Frassu, G. Viglialoro, *Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption*, Z. Angew. Math. Phys., **74** (2023), 21 pages. 1

[21] T. Li, Y. V. Rogovchenko, *Oscillation of second-order neutral differential equations*, Math. Nachr., **288** (2015), 1150–1162. 1

[22] T. Li, Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett., **61** (2016), 35–41.

[23] T. Li, Y. V. Rogovchenko, *On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations*, Appl. Math. Lett., **67** (2017), 53–59.

[24] T. Li, Y. V. Rogovchenko, *Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations*, Monatsh. Math., **184** (2017), 489–500. 1

[25] T. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., **70** (2019), 18 pages. 1

[26] T. Li, Y. V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett., **105** (2020), 7 pages. 1

[27] T. Li, G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime*, Differential Integral Equations, **34** (2021), 315–336. 1

[28] O. Moaaz, E. M. Elabbasy, E. Shaaban, *Oscillation Criteria for A Class of Third Order Damped Differential*, Arab J. Math. Sci., **24** (2018), 16–30. 1

[29] I. Mojsej, A. Tartal'ová, *On bounded nonoscillatory solutions of third-order nonlinear differential equations*, Cent. Eur. J. Math., **7** (2009), 717–724. 1

[30] G. Qin, C. Huang, Y. Xie, F. Wen, *Asymptotic behavior for third-order quasi-linear differential equations*, Adv. Difference Equ., **2013** (2013), 8 pages. 1

[31] Y.-C. Qiu, I. Jadlovská, K.-S. Chiu, T. Li, *Existence of nonoscillatory solutions tending to zero of third-order neutral dynamic equations on time scales*, Adv. Difference Equ., **2020** (2020), 9 pages. 1, 1, 4, 4, 4.3

[32] R. A. Sallam, M. M. A. El-Sheikh, E. I. El-Saedy, *On the oscillation of second order nonlinear neutral delay differential equations*, Math. Solovaca, **71** (2021), 859–870.

[33] R. A. Sallam, S. Salem, M. M. A. El-Sheikh, *Oscillation of solutions of third order nonlinear neutral differential equations*, Adv. Difference Equ., **2020** (2020), 25 pages.

[34] E. Thandapani, M. M. A. El-Sheikh, R. Sallam, S. Salem, *On the oscillatory behavior of third order differential equations with a sublinear neutral term*, Math. Solovaca, **70** (2020), 95–106. 1, 1

[35] Y. Tian, Y. Cail, Y. Fu, T. Li, *Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, Adv. Difference Equ., **2015** (2015), 14 pages. 1, 1

[36] A. Tiryaki, M. F. Aktaş, *Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping*, J. Math. Anal. Appl., **325** (2007), 54–68.

[37] H. Wang, G. Chen, Y. Jiang, C. Jiang, T. Li, *Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, J. Math. Comput. Sci., **17** (2017), 194–199. 1, 1

[38] Q. Zhang, L. Gao, Y. Yu, *Oscillation criteria for third-order neutral differential equations with continuously distributed delay*, Appl. Math. Lett., **25** (2012), 1514–1519. 1

[39] Y.-P. Zhao, *Oscillation criteria for a class of third-order differential equations with neutral term*, J. Adv. Math. Comput. Sci., **33** (2019), 1–7. 1, 1, 2, 2.2, 2, 2, 3.7