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Hyers-Ulam-Gavruta stability of a Jensen's type quadratic-quadratic mapping in 2-Banach spaces

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Abstract

In this paper, our main objective is to study the Hyers-Ulam-Gavruta stability of a Jensen's type quadratic-quadratic mapping in 2-Banach Spaces, that is, we prove the Hyers-Ulam-Gavruta stability of the Jensen's type functional equation

$$g\left(\frac{x+y}{2}+z\right)+g\left(\frac{x+y}{2}-z\right)+g\left(\frac{x-y}{2}+z\right)+g\left(\frac{x-y}{2}-z\right)=g(x)+g(y)+4g(z),$$

in 2-Banach spaces by Hyers direct method.

Keywords: Hyers-Ulam-Gavruta stability, Jensen's type mapping, quadratic function, 2-Banach spaces, direct method.

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1. Introduction

We say that a functional equation is stable, if for every approximate solution, there exists an exact solution near to it. The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by Ulam [50] in 1940. A simulating and famous talk presented by Ulam [50] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those was the following question concerning the stability of homomorphisms.

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Theorem 1.1 ([50])**.** *Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$\rho(f(xy), f(x)f(y)) < \delta,$$

for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

Since then, this question has attracted the attention of many researchers. If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [19] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings, when G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

Theorem 1.2 (Hyers [19])**.** *Assume that G_1 and G_2 are Banach spaces. If a function $f : G_1 \to G_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

for some $\epsilon > 0$ and for all $x, y \in G_1$, then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in G_1$ and $A : G_1 \to G_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leqslant \epsilon \tag{1.2}$$

for all $x \in G_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G_1$, then A is linear.

Taking the above fact into account, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have Hyers-Ulam stability on (G_1, G_2). In the above Theorem, an additive function A satisfying the inequality (1.2) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the theorem formulated by Hyers was generalized by Aoki [5] and Bourgin [7] for additive mappings.

There is no reason for the Cauchy difference $f(x + y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [41] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam-stability for the Additive Cauchy Equation. This terminology is justified because the theorem of Rassias has strongly influenced mathematicians studying stability problems of functional equation. In fact, Rassias proved the following Theorem.

Theorem 1.3 ([41])**.** *Let X and Y be Banach spaces. Let $\theta \in (0, \infty)$ and let $p \in [0, 1)$. If a function $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

The findings of Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ualm-Rassias stability of functional equations. In 1982, Rassias [42] gave a further generalization of the result of Hyers and proved a theorem using weaker conditions controlled by a product of different powers of norms. His theorem is presented as follows.

Theorem 1.4 ([42])**.** *Let $f : X \to Y$ be a mapping from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \|x\|^p \|y\|^p \tag{1.3}$$

for all $x, y \in X$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leqslant p < \frac{1}{2}$. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

exists for all $x \in X$ and $A : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leqslant \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{1.4}$$

for all $x \in X$. If $p < 0$, then the inequality (1.3) holds for $x, y \neq 0$ and (1.4) for $x \neq 0$. If $p > 0$, then the inequality (1.3) holds for all $x, y \in X$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in X$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in R$ for each fixed $x \in X$, then A is R-linear mapping.

This type of stability involving a product of powers of norms is called Hyers-Ulam-Gavruta stability by Bouikhalence and Elquorachi [6], Nakmahachalasint [31, 32], Park and Nataji [36] and Sibaha et al. [46]. In 1991, Gajda [13] answered the question for $p > 1$, which was raised by Rassias [41]. This new concept is known as the Hyers-Ulam-Rassias stability of functional equations. The terminology, Hyers-Ulam-Rassias stability, is originated from these historical backgrounds. The terminology can also be applied to the case of other functional equations. In 1994, a further generalization of Rassias theorem was obtained by Gavruta [14] (see also [15]).

The stability concept introduced by Rassias [41] is significantly influenced by a number of Mathematicians to investigate the stability problem for various functional equations and there are many interesting results concerning the Ulam stability problems in ([1–3, 12, 20–24, 27–30, 33, 37, 43–45, 48, 49, 51]).

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function.

A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [47] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian Group. In [11], Czerwik proved the Cauchy-Rassias stability of the quadratic functional equation. Then, in 2006, Park et al. [37] are proved the general solution and Cauchy-Rassias stability of the Jensen's type quadratic-quadratic mapping in Banach spaces.

In 2010, by fixed point approach, Park [34] established the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$$

in non-Archimedean Banach spaces.

In 2014, Lee et al. [25] proved the generalized Hyers-Ulam stability of the mixed type additive-quadratic functional equation in Banach spaces. In the same year, Shen and Lan proved the general solution of a new quadratic functional equation of the form

$$f\left(x - \frac{y+z}{2}\right) + f\left(x + \frac{y-z}{2}\right) + f(x+z) = 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z). \tag{1.5}$$

Also studied the Ulam stability of this functional equation (1.5) in a real normed space and a non-Archimedean space. In 2018, Lee et al. [26] established the uniqueness theorems concerning the functional inequalities involving with an n-dimensional cubic-quadratic-additive equation of the form

$$\sum_{i=1}^{m} c_i f\left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\right) = 0$$

by using the Hyers direct method. In the next year, Park and Rassias [37] solved the additive functional equations

$$f(x+y+z) - f(x+y) - f(z) = s\left[f(x+y-z) + f(x-y+z) - 2f(x)\right] \tag{1.6}$$

and

$$f(x+y-z) + f(x-y+z) - 2f(x) = s\left[f(x+y+z) - f(x+y) - f(z)\right], \tag{1.7}$$

where s is a fixed nonzero complex number. Furthermore, they investigated the Hyers-Ulam stability of the additive functional equations (1.6) and (1.7) in complex Banach spaces. This is applied to prove the partial multipliers in Banach *-algebras, unital C*-algebras, Lie C*-algebras, JC*-algebras and C*-ternary algebras, associated with the additive functional equations (1.6) and (1.7). In 2012, Chung and Park [9] investigated the generalized Hyers-Ulam stability of the functional equations $f(x+y) = f(x) + f(y)$,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

and $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ in 2-Banach spaces. They also investigated the generalized Hyers-Ulam stability of the same equations in 2-Banach spaces with different assumptions (see [9]). In 2013, Patel and Patel [39] investigated the Hyers-Ulam stability of the functional equation

$$f(2x+y) - f(x+2y) = 3f(x) - 3f(y)$$

in 2-Banach spaces. In 2018, AL-Alia and Elkettani [4] introduced a new type of radical cubic functional equation related to Jensen mapping of the form

$$f\left(\sqrt[3]{x^3+y^3}\right) + f\left(\sqrt[3]{x^3-y^3}\right) = 2f(x)$$

and established general solution and some stability and hyper-stability results for the considered functional equation in 2-Banach spaces. Very recently, in 2021, Krzysztof [10] proved the Ulam stability of two general functional equations in several variables in 2-Banach spaces by applying fixed point method (see also [40]).

In this paper, by using Hyers direct method, we establish the Hyers-Ulam-Gavruta stability of the Jensen's type quadratic-quadratic mapping of the form

$$g\left(\frac{x+y}{2}+z\right) + g\left(\frac{x+y}{2}-z\right) + g\left(\frac{x-y}{2}+z\right) + g\left(\frac{x-y}{2}-z\right) = g(x) + g(y) + 4g(z), \tag{1.8}$$

for all $x, y, z \in A$, in 2-Banach spaces.

2. Preliminaries

In this section, we will provide some basic notations, definitions, and lemmas, which will be very useful to prove the main results.

Lemma 2.1 ([39]). *Let* A *and* B *be real vector spaces, and let* $g : A \to B$ *be a function that satisfies* (1.8) *if and only if* $g(x) = B(x, x) + C$, *for some symmetric bi-additive function* $B : A \times A \to B$, *for some* C *in* B . *Therefore every solution* g *of functional equation* (1.8) *with* $g(0) = 0$ *is also a quadratic function.*

In 1960, Gahler [16–18] introduced the concept of linear 2-normed spaces.

Definition 2.2 ([16]). Let A be a linear space over $\mathbb{R}$ with $\dim(A) > 1$ and let $\|.,.\| : A \times A \to \mathbb{R}$ be a function satisfying the following properties:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent;
2. $\|x, y\| = \|y, x\|$;
3. $\|\lambda x, y\| = |\lambda| \, \|x, y\|$;
4. $\|x, y + z\| \leqslant \|x, y\| + \|x, z\|$,

for each $x, y, z \in A$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called a 2-norm on A and the pair $(A, \|.,.\|)$ is a called a linear 2-normed space. Sometimes the condition (4) is called the triangle inequality.

In 2011, Park [35] introduced a basic property of linear 2-normed spaces as follows.

Lemma 2.3 ([38]). *Let* $(A, \|.,.\|)$ *be a 2-normed space. If* $\|x, y\| = 0$, *for all* $y \in A$, *then* $x = 0$.

Definition 2.4 ([18, 52, 53]). A sequence $\{x_n\}$ in a linear 2-normed space A is called a Cauchy sequence if there are two points $y, z \in A$ such that y and z are linearly independent, and

$$\lim_{l,m\to\infty} \|x_l - x_m, y\| = 0 \quad \text{and} \quad \lim_{l,m\to\infty} \|x_l - x_m, z\| = 0.$$

Definition 2.5 ([18, 52, 53]). A sequence $\{x_n\}$ in a linear 2-normed space A is called a convergent sequence if there is an $x \in A$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in A$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of x_n. In this case, we also write $\lim\limits_{n\to\infty} x_n = x$.

Lemma 2.6 ([18, 52, 53]). *For a convergent sequence* x_n *in a linear 2-normed space* A,

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for all $y \in A$.

Definition 2.7 ([18, 52, 53]). A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

3. Stability of a Jensen's type quadratic-quadratic mapping for functions $g : (A, \|.\|) \to (A, \|.,.\|)$

Throughout this section, let us consider A be a real normed linear space and also consider that there is a 2-norm on A which makes $(A, \|.,.\|)$ be a 2-Banach space. Suppose for a function $g : (A, \|.\|) \to (A, \|.,.\|)$, we define a mapping $D_g : A \times A \to A$ by

$$D_g(x, y, z) = g\left(\frac{x+y}{2} + z\right) + g\left(\frac{x+y}{2} - z\right) + g\left(\frac{x-y}{2} + z\right) + g\left(\frac{x-y}{2} - z\right) - g(x) - g(y) - 4g(z),$$

for each $x, y, z \in A$.

In this section, we prove the Hyers-Ulam-Gavruta stability of the Jensen's type quadratic-quadratic mapping controlled by a product of powers of norms.

Theorem 3.1. *Let $\epsilon \geqslant 0$, $0 < s < 2$, $u > 0$. If $g : A \to A$ is a function satisfying*

$$\|D_g(x,y,z),w\| \leqslant \epsilon \|x\|^s \|y\|^s \|z\|^s \|w\|^u$$

for all $x,y,z \in A$, then there exists a unique quadratic function $Q : A \to A$ satisfying the functional equation (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon \|x\|^{3s} \|w\|^u}{4 - 2^{3s}},$$

for all $x,w \in A$.

Proof. Let us assume that the function $f : A \to A$ be defined by $f(x) = g(x) - g(0)$ for all $x \in A$. Then $f(0) = 0$. Consider

$$\begin{aligned}\|D_f(x,y,z),w\| = \Big\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) + f\left(\frac{x-y}{2}+z\right) + f\left(\frac{x-y}{2}-z\right) \\ - f(x) - f(y) - 4f(z), w\Big\| \leqslant \epsilon \|x\|^s \|y\|^s \|z\|^s \|w\|^u\end{aligned} \tag{3.1}$$

for each $x,w \in A$. Setting $x = y = z$ in (3.1), we have

$$\|f(2x) - 4f(x), w\| \leqslant \epsilon \|x\|^{3s} \|w\|^u \tag{3.2}$$

for each $x,w \in A$. Dividing the above equation (3.2) by 4, we get

$$\left\|f(x) - \frac{1}{4}f(2x), w\right\| \leqslant \frac{\epsilon}{4} \|x\|^{3s} \|w\|^u \tag{3.3}$$

for each $x,w \in A$. Now replacing x by $2x$ in the inequality (3.3), we have

$$\left\|f(2x) - \frac{1}{4}f(4x), w\right\| \leqslant \frac{\epsilon}{4} \|2x\|^{3s} \|w\|^u = \frac{\epsilon 2^{3s}}{4} \|x\|^{3s} \|w\|^u \tag{3.4}$$

for each $x,w \in A$. Again multiplying the above inequation (3.4) by $\frac{1}{4}$, then one can have

$$\left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x), w\right\| \leqslant \frac{\epsilon}{4}.\frac{2^{3s}}{4} \|x\|^{3s} \|w\|^u. \tag{3.5}$$

By using the equations (3.3) and (3.5), we will have

$$\begin{aligned}\left\|f(x) - \frac{1}{4^2}f(2^2x), w\right\| = \left\|f(x) - \frac{1}{16}f(4x), w\right\| = \left\|f(x) - \frac{1}{4}f(2x), w\right\| + \left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x), w\right\| \\ \leqslant \frac{\epsilon}{4} \|x\|^{3s} \|w\|^u + \frac{\epsilon}{4}.\frac{2^{3s}}{4} \|x\|^{3s} \|w\|^u\end{aligned}$$

for all $x,w \in A$. Then the above inequality can be written as

$$\left\|f(x) - \frac{1}{4^2}f(2^2x), w\right\| \leqslant \left[1 + \frac{2^{3s}}{4}\right] \frac{\epsilon}{4} \|x\|^{3s} \|w\|^u, \tag{3.6}$$

for each $x,w \in A$. By applying induction on n and using (3.6), we arrive at

$$\begin{aligned}\left\|f(x) - \frac{1}{4^n}f(2^nx), w\right\| \leqslant \frac{\epsilon}{4} \|x\|^{3s} \|w\|^u \sum_{k=0}^{n-1} \frac{2^{3sk}}{4^k} \\ \leqslant \frac{\epsilon}{4} \|x\|^{3s} \|w\|^u \left[\frac{1 - 2^{(3s-2)n}}{1 - 2^{3s-2}}\right] \leqslant \frac{\epsilon}{4 - 2^{3s}} \|x\|^{3s} \|w\|^u\end{aligned} \tag{3.7}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\begin{aligned}\left\|\frac{1}{4^m}f(2^m x) - \frac{1}{4^n}f(2^n x), w\right\| &= \left\|\frac{1}{4^{m-n+n}}f(2^{m-n+n}x) - \frac{1}{4^n}f(2^n x), w\right\| \\ &= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}}f(2^{m-n}.2^n x) - f(2^n x), w\right\| \\ &\leqslant \frac{1}{4^n}\left[\frac{\epsilon}{4}\|2^n x\|^{3s}\|w\|^u\right]\sum_{k=0}^{m-n-1}\frac{2^{3sk}}{4^k} \\ &\leqslant \left[\frac{\epsilon}{4}\|x\|^{3s}\|w\|^u\right]\sum_{k=0}^{m-n-1}2^{(3s-2)(n+k)},\end{aligned}$$

for all $x, w \in A$. Therefore, we have

$$\left\|\frac{1}{4^m}f(2^m x) - \frac{1}{4^n}f(2^n x), w\right\| \leqslant \frac{\epsilon}{4}\|x\|^{3s}\|w\|^u\left[\frac{2^{(3s-2)n}\left(1 - 2^{(3s-2)(m-n)}\right)}{1 - 2^{3s-2}}\right] \tag{3.8}$$

for all $x, w \in A$. Then the above inequality (3.8) vanishes as m, n approach to infinity, for all $x, w \in A$. Therefore, $\left\{\frac{1}{4^n}f(2^n x)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Now, we will define a function $Q : A \to A$ by

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^n x),$$

for all $x \in A$. Then by using equation (3.7), we get

$$\lim_{n\to\infty}\left\|f(x) - \frac{1}{4^n}f(2^n x), w\right\| \leqslant \frac{\epsilon}{4}\|x\|^{3s}\|w\|^u\left(\frac{1}{1 - 2^{3s-2}}\right) \tag{3.9}$$

for all $x, w \in A$. Therefore, from the equations (3.8) and (3.9), we get

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon\|x\|^{3s}\|w\|^u}{4 - 2^{3s}} \tag{3.10}$$

for all $x, w \in A$. Next, we have to show that the quadratic function Q satisfies the functional equation (1.8). Now, for all $x, w \in A$, we will have

$$\begin{aligned}\|D_Q(x, y, z), w\| &= \lim_{n\to\infty}\frac{1}{4^n}\|D_f(2^n x, 2^n y, 2^n z), w\| \\ &= \lim_{n\to\infty}\frac{\epsilon}{4^n}(\|2^n x\|^s\|2^n y\|^s\|2^n z\|^s)\|w\|^u \\ &= \lim_{n\to\infty}2^{(s-2)n}\epsilon(\|x\|^s\|y\|^s\|z\|^s)\|w\|^u = 0\end{aligned}$$

for all $w \in A$. Thus $\|D_Q(x, y, z), w\| = 0$, for all $w \in A$. Also we have $D_Q(x, y, z) = 0$. Finally, to complete the proof, we have to prove that the function Q is unique. Suppose that Q' is an another quadratic function satisfying (3.1) and (3.10). Since Q and Q' are quadratic, then for all $X \in A$,

$$Q(2^n x) = 4^n Q(x), \qquad Q'(2^n x) = 4^n Q'(x).$$

Hence, for all $x \in A$, we have

$$\|Q(x) - Q'(x), w\| = \frac{1}{4^n}\|Q(2^n x) - Q'(2^n x), w\|$$

$$= \frac{1}{4^n}\left[\|Q(2^n x) - f(2^n x), w\| + \left\|f(2^n x) - Q'(2^n x), w\right\|\right]$$
$$= \frac{1}{4^n}\left[\frac{\epsilon\,\|2^n x\|^{3s}\,\|w\|^u}{4-2^{3s}} + \frac{\epsilon\,\|2^n x\|^{3s}\,\|w\|^u}{4-2^{3s}}\right]$$
$$= \frac{1}{4^n}\left[\frac{2\epsilon\,\|2^n x\|^{3s}\,\|w\|^u}{4-2^{3s}}\right]$$
$$= \frac{1}{4^n}\left[\frac{2^{3ns}.2\epsilon\,\|x\|^{3s}\,\|w\|^u}{4-2^{3s}}\right] = \frac{2^{n(3s-2)}.2\epsilon\,\|x\|^{3s}\,\|w\|^u}{4-2^{3s}} = 0 \quad \text{as } n \to \infty$$

for all $w \in A$. Therefore, $\|Q(x) - Q'(x), w\| = 0$. Then $Q(x) = Q'(x)$ for all $x \in A$. This completes the theorem. □

Corollary 3.2. *Suppose that $\theta \in [0, \infty)$, $0 < s < 2$, $u > 0$. If $g : A \to A$ is a function satisfying the inequality*

$$\|D_g(x, y, z), w\| \leqslant \theta\,\|x\|^s\,\|y\|^s\,\|z\|^s\,\|w\|^u$$

for all $x, y, z \in A$, then there exists a unique quadratic function $Q : A \to A$ satisfying the functional equation (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\theta\,\|x\|^{3s}\,\|w\|^u}{4-2^{3s}},$$

for all $x, w \in A$.

Theorem 3.3. *Let $\epsilon \geqslant 0$, $s > 2$, $u > 0$. If $g : A \to A$ is a function that satisfies the inequality*

$$\|D_g(x, y, z), w\| \leqslant \epsilon\,\|x\|^s\,\|y\|^s\,\|z\|^s\,\|w\|^u$$

for all $x, y, z \in A$. Then there exists a unique quadratic mapping $Q : A \to A$ satisfying the functional equation (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon\,\|x\|^{3s}\,\|w\|^u}{2^{3s}-4},$$

for all $x, w \in A$.

Proof. By equation (3.2) of Theorem 3.1, we have

$$\|f(2x) - 4f(x), w\| \leqslant \epsilon\,\|x\|^{3s}\,\|w\|^u \tag{3.11}$$

for each $x, w \in A$. Now replacing x by $\left(\frac{x}{2}\right)$ in (3.11), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| \leqslant \frac{\epsilon}{2^{3s}}\,\|x\|^{3s}\,\|w\|^u = \epsilon 2^{-3s}\,\|x\|^{3s}\,\|w\|^u \tag{3.12}$$

for all $x, w \in A$. Again replacing x by $\left(\frac{x}{2}\right)$ in (3.12), we have

$$\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \leqslant \epsilon 2^{-2(3s)}\,\|x\|^{3s}\,\|w\|^u \tag{3.13}$$

for each $x, w \in A$. Combining (3.12) and (3.13), we will have

$$\left\|f(x) - 4^2 f\left(\frac{x}{2^2}\right), w\right\| = \left\|f(x) - 16f\left(\frac{x}{4}\right), w\right\| = \left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| + 4\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\|$$
$$\leqslant \epsilon 2^{-s}\,\|x\|^{3s}\,\|w\|^u + 4.\epsilon 2^{-2(3s)}\,\|x\|^{3s}\,\|w\|^u$$
$$= \epsilon\,\|x\|^{3s}\,\|w\|^u\left(2^{-3s} + 4.2^{-2(3s)}\right)$$

for each $x, w \in A$. By using induction on n, we arrive at

$$\begin{aligned}\left\|f(x)-4^n f\left(\frac{x}{2^n}\right), w\right\| &\leqslant \epsilon \|x\|^s \|w\|^u \sum_{k=0}^{n-1} 4^k.2^{-3s(k+1)} \\ &= \epsilon \|x\|^{3s} \|w\|^u \sum_{k=0}^{n-1} 2^{k(2-3s)-3s} \leqslant \epsilon \|x\|^{3s} \|w\|^u 2^{-3s}\left[\frac{1-2^{(2-3s)n}}{1-2^{2-3s}}\right]\end{aligned} \tag{3.14}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\begin{aligned}\left\|4^m f\left(\frac{x}{2^m}\right)-4^n f\left(\frac{x}{2^n}\right), w\right\| &= \left\|4^{m-n+n} f\left(\frac{x}{2^{m-n+n}}\right)-4^n f\left(\frac{x}{2^n}\right), w\right\| \\ &= 4^n \left\|4^{m-n} f\left(\frac{x}{2^{m-n}.2^n x}\right)-f\left(\frac{x}{2^n}\right), w\right\| \\ &\leqslant 4^n \epsilon \left\|\frac{x}{2^n}\right\|^{3s} \|w\|^u \sum_{k=0}^{m-n-1} 2^{k(2-3s)-3s} \\ &= \epsilon \|x\|^{3s} \|w\|^u 2^{(2-3s)n} \sum_{k=0}^{m-n-1} 2^{k(2-3s)-3s} \\ &= \epsilon \|x\|^{3s} \|w\|^u \sum_{k=0}^{m-n-1} 2^{(n+k)(2-3s)-3s} \\ &= \epsilon \|x\|^{3s} \|w\|^u 2^{n(2-3s)-3s}\left[\frac{1-2^{(2-3s)(m-n)}}{1-2^{2-3s}}\right] \longrightarrow 0 \ \text{ as } n \to \infty\end{aligned}$$

for all $x \in A$ and $w \in A$. Therefore, $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Since A is a 2-Banach space, the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is 2-converges for all $x \in A$. We define a mapping $Q : A \to A$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right),$$

for all $x \in A$. Now, from the equation (3.14), we get

$$\lim_{n\to\infty} \left\|f(x)-4^n f\left(\frac{x}{2^n}\right), w\right\| \leqslant \epsilon \|x\|^{3s} \|w\|^u \left(\frac{2^{-3s}}{1-2^{2-3s}}\right) = \frac{\epsilon \|x\|^{3s} \|w\|^u}{2^{3s}-4}$$

for all $x, w \in A$. Hence,

$$\|g(x)-Q(x)-g(0), w\| \leqslant \frac{\epsilon \|x\|^{3s} \|w\|^u}{2^{3s}-4},$$

for all $x, w \in A$. The remaining part of the proof is similar to the above Theorem 3.1. □

Corollary 3.4. *If* $\theta \in [0, \infty)$, $s > 2$, $u > 0$ *and if* $g : A \to A$ *is a function which satisfies the inequality*

$$\|D_g(x, y, z), w\| \leqslant \theta \|x\|^s \|y\|^s \|z\|^s \|w\|^u$$

for all $x, y, z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the functional equation* (1.8) *such that*

$$\|g(x)-Q(x)-g(0), w\| \leqslant \frac{\theta \|x\|^{3s} \|w\|^u}{2^{3s}-4},$$

for all $x, w \in A$.

Next, we will establish the Hyers-Ulam-Gavruta stability of the functional equation (1.8), which is controlled by the product of different powers of norms.

Theorem 3.5. *Let $\epsilon \geqslant 0$, $0 < r, s, t < 2$, $u > 0$. If $g : A \to A$ is a function satisfying the inequality*

$$\|D_g(x,y,z), w\| \leqslant \epsilon \|x\|^r \|y\|^s \|z\|^t \|w\|^u$$

for all $x, y, z \in A$, then there exists a unique quadratic mapping $Q : A \to A$ satisfying the Jensen's type quadratic-quadratic mapping (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\|x\|^{r+s+t}}{4 - 2^{r+s+t}} \|w\|^u \epsilon,$$

for all $x, w \in A$.

Proof. Suppose we define a function $f : A \to A$ by $f(x) = g(x) - g(0)$ for all $x \in A$. Then $f(0) = 0$. Assume that

$$\begin{aligned}\|D_f(x,y,z), w\| = \Bigg\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) + f\left(\frac{x-y}{2} + z\right) + f\left(\frac{x-y}{2} - z\right) \\ - f(x) - f(y) - 4f(z), w\Bigg\| \leqslant \epsilon \|x\|^r \|y\|^s \|z\|^t \|w\|^u\end{aligned} \tag{3.15}$$

for every $x, w \in A$. Put $x = y = z$ in (3.15), we have

$$\|f(2x) - 4f(x), w\| \leqslant \epsilon \|x\|^{r+s+t} \|w\|^u \tag{3.16}$$

for each $x, w \in A$. Dividing the above equation (3.16) by 4, we get

$$\left\|f(x) - \frac{1}{4}f(2x), w\right\| \leqslant \frac{\epsilon}{4} \|x\|^{r+s+t} \|w\|^u \tag{3.17}$$

for all $x, w \in A$. Now replacing x by $2x$ in the inequality (3.17), we have

$$\left\|f(2x) - \frac{1}{4}f(4x), w\right\| \leqslant \frac{\epsilon}{4} 2^{r+s+t} \|x\|^{r+s+t} \|w\|^u \tag{3.18}$$

for each $x, w \in A$. Multiplying $\frac{1}{4}$ to the above equation (3.18), we get

$$\left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x), w\right\| \leqslant \frac{1}{4}.\frac{\epsilon}{4}.2^{r+s+t} \|x\|^{r+s+t} \|w\|^u. \tag{3.19}$$

By using (3.17) and (3.19), we can reach

$$\begin{aligned}\left\|f(x) - \frac{1}{4^2}f(2^2x), w\right\| = \left\|f(x) - \frac{1}{16}f(4x), w\right\| = \left\|f(x) - \frac{1}{4}f(2x), w\right\| + \left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x), w\right\| \\ \leqslant \frac{\epsilon}{4} \|x\|^{r+s+t} \|w\|^u + \frac{1}{4}.\frac{\epsilon}{4}.2^{r+s+t} \|x\|^{r+s+t} \|w\|^u\end{aligned}$$

for all $x, w \in A$. Thus also we have

$$\left\|f(x) - \frac{1}{4^2}f(2^2x), w\right\| \leqslant \frac{\epsilon}{4}. \|x\|^{r+s+t} \left\{1 + \frac{1}{4}2^{r+s+t}\right\} \|w\|^u$$

for each $x, w \in A$. Then by using induction method on n, we obtain that

$$\begin{aligned}\left\|f(x)-\frac{1}{4^n}f(2^nx),w\right\| &\leqslant \frac{\epsilon}{4}\|x\|^{r+s+t}\|w\|^u\sum_{k=0}^{n-1}\frac{1}{4^k}.2^{(r+s+t)k}\\ &\leqslant \frac{\epsilon}{4}\|x\|^{r+s+t}\|w\|^u\left[\frac{1-2^{(r+s+t-2)n}}{1-2^{r+s+t-2}}\right]\\ &\leqslant \frac{\epsilon}{4}\|x\|^{r+s+t}\|w\|^u\left[\frac{4}{4-2^{r+s+t}}\right]\leqslant \frac{\|x\|^{r+s+t}}{4-2^{r+s+t}}\|w\|^u\epsilon\end{aligned} \tag{3.20}$$

for all $x, w \in A$. Using (3.20), for $m, n \in \mathbb{N}$ and for all $x \in A$, we have

$$\begin{aligned}\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx),w\right\| &= \left\|\frac{1}{4^{m-n+n}}f(2^{m-n+n}x)-\frac{1}{4^n}f(2^nx),w\right\|\\ &= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}}f(2^{m-n}.2^nx)-f(2^nx),w\right\|\\ &\leqslant \frac{1}{4^n}.\frac{\epsilon}{4}\|w\|^u\sum_{k=0}^{m-n-1}\frac{1}{4^k}2^{(r+s+t)k}\|2^nx\|^{r+s+t}\\ &\leqslant \frac{\epsilon}{4}\|w\|^u\sum_{k=0}^{m-n-1}2^{(r+s+t-2)(n+k)}\|x\|^{r+s+t}\end{aligned}$$

for all $x, w \in A$. Thus, we have

$$\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx),w\right\| \leqslant \frac{\epsilon}{4}\|w\|^u 2^{(r+s+t-2)n}\left(\frac{1-2^{(r+s+t-2)(m-n)}}{1-2^{r+s+t-2}}\right)\|x\|^{r+s+t} \tag{3.21}$$

for all $x, w \in A$. Then $\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx),w\right\| = 0$ as m, n approach to infinity, for all $x, w \in A$. Therefore, $\left\{\frac{1}{4^n}f(2^nx)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. We define a quadratic function $Q : A \to A$ by

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^nx),$$

for all $x \in A$. Then by using equation (3.20), we will reach

$$\lim_{n\to\infty}\left\|f(x)-\frac{1}{4^n}f(2^nx),w\right\| \leqslant \frac{\epsilon}{4}\left[\frac{\|x\|^{r+s+t}}{1-2^{r+s+t-2}}\right]\|w\|^u \tag{3.22}$$

for all $x, w \in A$. Therefore, from the equations (3.21) and (3.22), we get

$$\|g(x)-Q(x)-g(0),w\| \leqslant \left[\frac{\|x\|^{r+s+t}}{4-2^{r+s+t}}\right]\|w\|^u\epsilon \tag{3.23}$$

for all $x, w \in A$. Next, we shall to prove that the quadratic function Q satisfies the functional equation (1.8). Now, for all $x, w \in A$, one can have

$$\begin{aligned}\|D_Q(x,y,z),w\| &= \lim_{n\to\infty}\frac{1}{4^n}\|D_f(2^nx,2^ny,2^nz),w\|\\ &= \lim_{n\to\infty}\frac{\epsilon}{4^n}\left(\|2^nx\|^r\|2^ny\|^s\|2^nz\|^t\right)\|w\|^u\end{aligned}$$

$$= \lim_{n\to\infty} 2^{(r+s+t-2)n} \left(\|x\|^r \|y\|^s \|z\|^t\right) \|w\|^u \epsilon = 0$$

for all $w \in A$. Thus $\left\|D_Q(x,y,z),w\right\| = 0$, for all $w \in A$, it gives that $D_Q(x,y,z) = 0$. To complete the proof of this theorem, we have to prove that the uniqueness of the function Q. Let Q' be another quadratic function satisfying (3.15) and (3.23). Since Q and Q' are quadratic, then for all $X \in A$,

$$Q(2^n x) = 4^n Q(x), \qquad Q'(2^n x) = 4^n Q'(x).$$

Hence, for all $x \in A$, we have

$$\begin{aligned} \left\|Q(x) - Q'(x),w\right\| &= \frac{1}{4^n}\left\|Q(2^n x) - Q'(2^n x),w\right\| \\ &= \frac{1}{4^n}\left[\|Q(2^n x) - f(2^n x),w\| + \left\|f(2^n x) - Q'(2^n x),w\right\|\right] \\ &= \frac{2}{4^n}\left[\frac{\|2^n x\|^{r+s+t}}{4-2^{r+s+t}}\right]\epsilon\|w\|^u = 2\epsilon\|w\|^u\left(\frac{2^{(r+s+t-2)n}\|x\|^{r+s+t}}{4-2^{r+s+t}}\right) = 0 \ \text{ as } n\to\infty \end{aligned}$$

for all $w \in A$. Therefore, $\|Q(x) - Q'(x),w\| = 0$. Then $Q(x) = Q'(x)$ for all $x \in A$. Hence the proof of the theorem. □

Corollary 3.6. *If* $\theta \in [0,\infty)$, $0 < r,s,t < 2$, $u > 0$ *and if* $g : A \to A$ *satisfies*

$$\|D_g(x,y,z),w\| \leqslant \theta\|x\|^r\|y\|^s\|z\|^t\|w\|^u$$

for all $x,y,z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the Jensen's type quadratic-quadratic mapping* (1.8) *such that*

$$\|g(x) - Q(x) - g(0),w\| \leqslant \frac{\theta\|x\|^{r+s+t}}{4-2^{r+s+t}}\|w\|^u,$$

for all $x,w \in A$.

Theorem 3.7. *Let* $\epsilon \geqslant 0$, $r,s,t > 2$, $u > 0$ *and if* $g : A \to A$ *is a function which satisfies the following inequality*

$$\|D_g(x,y,z),w\| \leqslant \epsilon\|x\|^r\|y\|^s\|z\|^t\|w\|^u$$

for all $x,y,z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the Jensen's quadratic-quadratic mapping* (1.8) *such that*

$$\|g(x) - Q(x) - g(0),w\| \leqslant \frac{\|x\|^{r+s+t}}{2^{r+s+t}-4}\epsilon\|w\|^u,$$

for all $x,w \in A$.

Proof. By equation (3.16) of Theorem 3.5, we have

$$\|f(2x) - 4f(x),w\| \leqslant \epsilon\|x\|^{r+s+t}\|w\|^u \tag{3.24}$$

for each $x,w \in A$. Now replacing x by $\left(\frac{x}{2}\right)$ in (3.24), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right),w\right\| \leqslant 2^{-(r+s+t)}\|x\|^{r+s+t}\|w\|^u\epsilon \tag{3.25}$$

for all $x,w \in A$. Again replacing x by $\left(\frac{x}{2}\right)$ in (3.25), we have

$$\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right),w\right\| \leqslant 2^{-2(r+s+t)}\|x\|^{r+s+t}\|w\|^u\epsilon \tag{3.26}$$

for each $x, w \in A$. Combining (3.25) and (3.26), we will have

$$\begin{aligned}\left\|f(x) - 4^2 f\left(\frac{x}{2^2}\right), w\right\| &= \left\|f(x) - 16f\left(\frac{x}{4}\right), w\right\| \\ &= \left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| + 4\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \\ &\leqslant 2^{-(r+s+t)}\|x\|^{r+s+t}\|w\|^u \epsilon + 42^{-2(r+s+t)}\|x\|^{r+s+t}\|w\|^u \epsilon \\ &= 2^{-(r+s+t)}\|x\|^{r+s+t}\|w\|^u \epsilon\left[1 + 4.2^{-(r+s+t)}\right]\end{aligned}$$

for each $x, w \in A$. Now, we will apply the induction method on n, we get that

$$\begin{aligned}\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &\leqslant 2^{-(r+s+t)}\|x\|^{r+s+t}\|w\|^u \epsilon \sum_{k=0}^{n-1} 4^k.2^{-(r+s+t)k} \\ &= \epsilon\|w\|^u\|x\|^{r+s+t}\sum_{k=0}^{n-1} 2^{k[2-(r+s+t)]-(r+s+t)} \\ &\leqslant \epsilon\|w\|^u\|x\|^{r+s+t}\left(2^{-(r+s+t)}\left[\frac{1-2^{[2-(r+s+t)]n}}{1-2^{2-(r+s+t)}}\right]\right)\end{aligned} \tag{3.27}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\begin{aligned}\left\|4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &= 4^n\left\|4^{m-n} f\left(\frac{x}{2^{m-n}.2^n x}\right) - f\left(\frac{x}{2^n}\right), w\right\| \\ &\leqslant 4^n \epsilon\|w\|^u \sum_{k=0}^{m-n-1}\left[2^{k[2-(r+s+t)]-(r+s+t)}\left\|\frac{x}{2^n}\right\|^{r+s+t}\right] \\ &= \epsilon\|w\|^u \sum_{k=0}^{m-n-1}\left[2^{(n+k)[2-(r+s+t)]-(r+s+t)}\|x\|^{r+s+t}\right] \\ &= 2^{n[2-(r+s+t)]-(r+s+t)}\left[\frac{1-2^{[2-(r+s+t)](m-n)}}{1-2^{2-(r+s+t)}}\right]\|x\|^{r+s+t}\epsilon\|w\|^u \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for all $x \in A$ and $w \in A$. Therefore, $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Since A is a 2-Banach space, the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is 2-converges, for all $x \in A$. We define a mapping $Q : A \to A$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right),$$

for all $x \in A$. Now, with the help of equation (3.27), we get

$$\lim_{n\to\infty}\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| \leqslant \epsilon\|w\|^u\left(\frac{2^{-(r+s+t)}}{1-2^{2-(r+s+t)}}\right)\|x\|^{r+s+t}$$

for all $x, w \in A$. Hence,

$$\|g(x) - Q(x) - g(0), w\| \leqslant \left(\frac{\epsilon}{2^{r+s+t} - 4}\right)\|x\|^{r+s+t}\|w\|^u,$$

for all $x, w \in A$. The further part of the proof is similar to the above Theorem 3.5, which completes the Theorem. □

Corollary 3.8. *Let* $\theta \in [0, \infty)$, $r, s, t > 2$, $u > 0$ *and* $g : A \to A$ *be a function which satisfies*

$$\|D_g(x, y, z), w\| \leqslant \theta \|x\|^r \|y\|^s \|z\|^t \|w\|^u$$

for all $x, y, z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the Jensen's quadratic-quadratic mapping* (1.8) *such that*

$$\|g(x) - Q(x) - g(0), w\| \leqslant \left[\frac{\theta}{2^{r+s+t} - 4}\right] \|x\|^{r+s+t} \|w\|^u,$$

for all $x, w \in A$.

4. Stability of a Jensen's type quadratic-quadratic mapping for functions $g : (A, \|.,.\|) \to (A, \|.,.\|)$

In this section, we will study similar problems which we have discussed in the last section for the function $g : A \to A$, where $(A, \|.,.\|)$ is a 2-Banach space. First, we will establish the Hyers-Ulam-Gavruta stability of a Jensen's quadratic-quadratic functional equation (1.8), which is controlled by the product of powers of norms on 2-Banach spaces.

Theorem 4.1. *For every* $\epsilon \geqslant 0$, $0 < s < 2$ *and a function* $g : A \to A$ *satisfying the inequality*

$$\|D_g(x, y, z), w\| \leqslant \epsilon \|x, w\|^s \|y, w\|^s \|z, w\|^s$$

for all $x, y, z \in A$, *then there exists a unique quadratic function* $Q : A \to A$ *satisfying the functional equation* (1.8) *such that*

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon \|x, w\|^{3s}}{4 - 2^{3s}}$$

for all $x, w \in A$.

Proof. Let us assume that the function $f : A \to A$ be defined by $f(x) = g(x) - g(0)$ for all $x \in A$. Then $f(0) = 0$. Consider

$$\begin{aligned} \|D_f(x, y, z), w\| = \Bigg\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) + f\left(\frac{x-y}{2} + z\right) + f\left(\frac{x-y}{2} - z\right) \\ - f(x) - f(y) - 4f(z), w\Bigg\| \leqslant \epsilon \|x, w\|^s \|y, w\|^s \|z, w\|^s \end{aligned} \tag{4.1}$$

for each $x, w \in A$. Setting $x = y = z$ in (4.1), we have

$$\|f(2x) - 4f(x), w\| \leqslant \epsilon \|x, w\|^{3s} \tag{4.2}$$

for each $x, w \in A$. Dividing the above equation (4.2) by 4, we get

$$\left\|f(x) - \frac{1}{4}f(2x), w\right\| \leqslant \frac{\epsilon}{4} \|x, w\|^{3s} \tag{4.3}$$

for each $x, w \in A$. Now replacing x by $2x$ in the inequality (4.3), we have

$$\left\|f(2x) - \frac{1}{4}f(4x), w\right\| \leqslant \frac{\epsilon}{4}.2^{3s} \|x, w\|^{3s} \tag{4.4}$$

for each $x, w \in A$. Again multiplying the above inequation (4.4) by $\frac{1}{4}$, then one can have

$$\left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x), w\right\| \leqslant \frac{\epsilon}{4}.\frac{2^{3s}}{4} \|x, w\|^{3s}. \tag{4.5}$$

Using (4.3) and (4.5), we will have

$$\left\|f(x)-\frac{1}{4^2}f(2^2x),w\right\| = \left\|f(x)-\frac{1}{16}f(4x),w\right\|$$
$$= \left\|f(x)-\frac{1}{4}f(2x),w\right\| + \left\|\frac{1}{4}f(2x)-\frac{1}{16}f(4x),w\right\| \leqslant \frac{\epsilon}{4}\|x,w\|^{3s} + \frac{\epsilon}{4}.\frac{2^{3s}}{4}\|x,w\|^{3s}$$

for all $x, w \in A$. Then the above inequality can be written as

$$\left\|f(x)-\frac{1}{4^2}f(2^2x),w\right\| \leqslant \left[1+\frac{2^{3s}}{4}\right]\frac{\epsilon}{4}\|x\|^{3s}, \tag{4.6}$$

for each $x, w \in A$. By applying induction on n and using (4.6), we arrive at

$$\left\|f(x)-\frac{1}{4^n}f(2^nx),w\right\| \leqslant \frac{\epsilon}{4}\|x,w\|^{3s}\sum_{k=0}^{n-1}\frac{2^{3sk}}{4^k} \leqslant \frac{\epsilon}{4}\|x,w\|^{3s}\left[\frac{1-2^{(3s-2)n}}{1-2^{3s-2}}\right] \leqslant \frac{\epsilon}{4-2^{3s}}\|x,w\|^{3s} \tag{4.7}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx),w\right\| = \left\|\frac{1}{4^{m-n+n}}f(2^{m-n+n}x)-\frac{1}{4^n}f(2^nx),w\right\|$$
$$= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}}f(2^{m-n}.2^nx)-f(2^nx),w\right\|$$
$$\leqslant \frac{1}{4^n}\left[\frac{\epsilon}{4}\|2^nx,w\|^{3s}\right]\sum_{k=0}^{m-n-1}\frac{2^{3sk}}{4^k} \leqslant \frac{\epsilon}{4}\|x,w\|^{3s}\sum_{k=0}^{m-n-1}2^{(3s-2)(n+k)},$$

for all $x, w \in A$. Therefore, we have

$$\left\|\frac{1}{4^m}f(2^mx)-\frac{1}{4^n}f(2^nx),w\right\| \leqslant \frac{\epsilon}{4}\|x,w\|^{3s}\left[\frac{2^{(3s-2)n}\left(1-2^{(3s-2)(m-n)}\right)}{1-2^{3s-2}}\right] \tag{4.8}$$

for all $x, w \in A$. Then the above inequality (4.8) vanishes as m, n approach to infinity, for all $x, w \in A$. Therefore, $\left\{\frac{1}{4^n}f(2^nx)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Now, we will define a function $Q : A \to A$ by

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^nx),$$

for all $x \in A$. Then by using equation (4.7), we get

$$\lim_{n\to\infty}\left\|f(x)-\frac{1}{4^n}f(2^nx),w\right\| \leqslant \frac{\epsilon}{4}\|x,w\|^{3s}\left(\frac{1}{1-2^{3s-2}}\right) \tag{4.9}$$

for all $x, w \in A$. Therefore, from (4.8) and (4.9), we get

$$\|g(x)-Q(x)-g(0),w\| \leqslant \frac{\epsilon\|x,w\|^{3s}}{4-2^{3s}} \tag{4.10}$$

for all $x, w \in A$. Next, we have to show that the quadratic function Q satisfies the functional equation (1.8). Now, for all $x, w \in A$, we will have

$$\|D_Q(x,y,z),w\| = \lim_{n\to\infty}\frac{1}{4^n}\|D_f(2^nx,2^ny,2^nz),w\|$$

$$= \lim_{n\to\infty} \frac{\epsilon}{4^n} \left(\|2^n x, w\|^s \|2^n y, w\|^s \|2^n z, w\|^s\right)$$
$$= \lim_{n\to\infty} 2^{(3s-2)n} \epsilon \left(\|x, w\|^s \|y, w\|^s \|z, w\|^s\right) = 0$$

for all $w \in A$. Thus $\left\|D_Q(x, y, z), w\right\| = 0$, for all $w \in A$. Also we have $D_Q(x, y, z) = 0$. Finally, to complete the proof, we have to prove that the function Q is unique. Suppose that Q' is an another quadratic function satisfying (4.1) and (4.10). Since Q and Q' are quadratic, then for all $X \in A$,

$$Q(2^n x) = 4^n Q(x), \qquad Q'(2^n x) = 4^n Q'(x).$$

Hence, for all $x \in A$, we have

$$\begin{aligned}
\left\|Q(x) - Q'(x), w\right\| &= \frac{1}{4^n} \left\|Q(2^n x) - Q'(2^n x), w\right\| \\
&= \frac{1}{4^n} \left[\|Q(2^n x) - f(2^n x), w\| + \left\|f(2^n x) - Q'(2^n x), w\right\|\right] \\
&= \frac{1}{4^n} \left[\frac{\epsilon \|2^n x, w\|^{3s}}{4 - 2^{3s}} + \frac{\epsilon \|2^n x, w\|^{3s}}{4 - 2^{3s}}\right] \\
&= \frac{1}{4^n} \left[\frac{2\epsilon \|2^n x, w\|^{3s}}{4 - 2^{3s}}\right] = \frac{2}{4^n} \left[\frac{2^{3ns} \epsilon \|x, w\|^{3s}}{4 - 2^{3s}}\right] \\
&= \frac{2\epsilon . 2^{n(3s-2)} \|x, w\|^{3s}}{4 - 2^{3s}} = 0 \ \text{ as } n \to \infty
\end{aligned}$$

for all $w \in A$. Therefore, $\|Q(x) - Q'(x), w\| = 0$. Then $Q(x) = Q'(x)$ for all $x \in A$. This completes the theorem. □

Corollary 4.2. *Suppose $\theta \in [0, \infty)$, $0 < s < 2$ and if a function $g : A \to A$ satisfying the inequality*

$$\|D_g(x, y, z), w\| \leqslant \theta \|x, w\|^s \|y, w\|^s \|z, w\|^s$$

for all $x, y, z \in A$, then there exists a unique quadratic function $Q : A \to A$ satisfying the functional equation (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\theta \|x, w\|^{3s}}{4 - 2^{3s}},$$

for all $x, w \in A$.

Theorem 4.3. *Let $\epsilon \geqslant 0$ and $s > 2$. If $g : A \to A$ satisfies the inequality*

$$\|D_g(x, y, z), w\| \leqslant \epsilon \|x, w\|^s \|y, w\|^s \|z, w\|^s$$

for all $x, y, z \in A$, then there exists a unique quadratic mapping $Q : A \to A$ satisfying the functional equation (1.8) such that

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon \|x, w\|^{3s}}{2^{3s} - 4},$$

for all $x, w \in A$.

Proof. By the equation (4.2) of the Theorem 4.1, we have

$$\|f(2x) - 4f(x), w\| \leqslant \epsilon \|x, w\|^{3s} \tag{4.11}$$

for each $x, w \in A$. Now replacing x by $\left(\frac{x}{2}\right)$ in (4.11), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| \leqslant \frac{\epsilon}{2^{3s}} \|x, w\|^{3s} = \epsilon 2^{-3s} \|x, w\|^{3s} \tag{4.12}$$

for all $x, w \in A$. Again replacing x by $\left(\frac{x}{2}\right)$ in (4.12), we have

$$\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \leqslant \epsilon 2^{-2(3s)} \|x, w\|^{3s} \tag{4.13}$$

for each $x, w \in A$. Combining (4.12) and (4.13), we will have

$$\begin{aligned}\left\|f(x) - 4^2 f\left(\frac{x}{2^2}\right), w\right\| &= \left\|f(x) - 16f\left(\frac{x}{4}\right), w\right\| \\ &= \left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| + 4\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \\ &\leqslant \epsilon 2^{-3s} \|x, w\|^{3s} + 4.\epsilon 2^{-2(3s)} \|x, w\|^{3s} = \epsilon \|x, w\|^{3s} \left(2^{-3s} + 4.2^{-2(3s)}\right)\end{aligned}$$

for each $x, w \in A$. By using induction on n, we arrive at

$$\begin{aligned}\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &\leqslant \epsilon \|x, w\|^{3s} \sum_{k=0}^{n-1} 4^k.2^{-3s(k+1)} \\ &= \epsilon \|x, w\|^{3s} \sum_{k=0}^{n-1} 2^{k(2-3s)-3s} \leqslant \epsilon \|x, w\|^{3s} 2^{-3s} \left[\frac{1 - 2^{(2-3s)n}}{1 - 2^{2-3s}}\right]\end{aligned} \tag{4.14}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\begin{aligned}\left\|4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &= \left\|4^{m-n+n} f\left(\frac{x}{2^{m-n+n}}\right) - 4^n f\left(\frac{x}{2^n}\right), w\right\| \\ &= 4^n \left\|4^{m-n} f\left(\frac{x}{2^{m-n}.2^n x}\right) - f\left(\frac{x}{2^n}\right), w\right\| \\ &\leqslant 4^n \epsilon \left\|\frac{x}{2^n}, w\right\|^{3s} \sum_{k=0}^{m-n-1} 2^{k(2-3s)-3s} \\ &= \epsilon \|x, w\|^{3s} 2^{(2-3s)n} \sum_{k=0}^{m-n-1} 2^{k(2-3s)-3s} \\ &= \epsilon \|x, w\|^{3s} \sum_{k=0}^{m-n-1} 2^{(n+k)(2-3s)-3s} \\ &= \epsilon \|x, w\|^{3s} 2^{n(2-3s)-3s} \left[\frac{1 - 2^{(2-3s)(m-n)}}{1 - 2^{2-3s}}\right] \longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for all $x \in A$ and $w \in A$. Therefore, $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Since A is a 2-Banach space, the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is 2-converges, for all $x \in A$. We define a mapping $Q : A \to A$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right),$$

for all $x \in A$. Now, from the equation (4.14), we get

$$\lim_{n\to\infty} \left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| \leqslant \epsilon \|x, w\|^{3s} \left(\frac{2^{-3s}}{1 - 2^{2-3s}}\right) = \frac{3\epsilon \|x, w\|^{3s}}{2^{3s} - 4}$$

for all $x, w \in A$. Hence,

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon \|x, w\|^{3s}}{2^{3s} - 4},$$

for all $x, w \in A$. The remaining part of the proof is similar to the above Theorem 4.1. □

Corollary 4.4. *If* $\theta \in [0,\infty)$ *and* $s > 2$, *and if a mapping* $g : A \to A$ *satisfying the inequality*

$$\|D_g(x,y,z),w\| \leqslant \theta \|x,w\|^s \|y,w\|^s \|z,w\|^s$$

for all $x,y,z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the functional equation* (1.8) *such that*

$$\|g(x) - Q(x) - g(0),w\| \leqslant \frac{\theta \|x,w\|^{3s}}{2^{3s}-4},$$

for all $x,w \in A$.

Finally, we will prove the Hyers-Ulam-Gavruta stability of the Jensen's type functional equation (1.8), which is controlled by the product of different powers of norms on 2-Banach spaces.

Theorem 4.5. *Let* $\epsilon \geqslant 0$ *and* $0 < r,s,t < 2$. *If* $g : A \to A$ *satisfies the inequality*

$$\|D_g(x,y,z),w\| \leqslant \epsilon \|x,w\|^r \|y,w\|^s \|z,w\|^t$$

for all $x,y,z \in A$, *then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the Jensen's type quadratic-quadratic mapping* (1.8) *such that*

$$\|g(x) - Q(x) - g(0),w\| \leqslant \frac{\epsilon \|x,w\|^{r+s+t}}{4-2^{r+s+t}},$$

for all $x,w \in A$.

Proof. Suppose we define a function $f : A \to A$ by $f(x) = g(x) - g(0)$ for all $x \in A$. Then $f(0) = 0$. Assume that

$$\begin{aligned}\|D_f(x,y,z),w\| = \Bigg\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) + f\left(\frac{x-y}{2}+z\right) + f\left(\frac{x-y}{2}-z\right) \\ - f(x) - f(y) - 4f(z), w\Bigg\| \leqslant \epsilon \|x,w\|^r \|y,w\|^s \|z,w\|^t\end{aligned} \tag{4.15}$$

for every $x,w \in A$. Put $x = y = z$ in (4.15), we have

$$\|f(2x) - 4f(x),w\| \leqslant \epsilon \|x,w\|^{r+s+t} \tag{4.16}$$

for each $x,w \in A$. Dividing the above equation (4.16) by 4, we get

$$\left\|f(x) - \frac{1}{4}f(2x),w\right\| \leqslant \frac{\epsilon}{4} \|x,w\|^{r+s+t} \tag{4.17}$$

for all $x,w \in A$. Now replacing x by $2x$ in the inequality (4.17), we have

$$\left\|f(2x) - \frac{1}{4}f(4x),w\right\| \leqslant \frac{\epsilon}{4} 2^{r+s+t} \|x,w\|^{r+s+t} \tag{4.18}$$

for each $x,w \in A$. Multiplying $\frac{1}{4}$ to the above equation (4.18), we get

$$\left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x),w\right\| \leqslant \frac{1}{4}.\frac{\epsilon}{4}.2^{r+s+t} \|x,w\|^{r+s+t}. \tag{4.19}$$

By using (4.17) and (4.19), we can reach

$$\left\|f(x) - \frac{1}{4^2}f(2^2x),w\right\| = \left\|f(x) - \frac{1}{16}f(4x),w\right\| = \left\|f(x) - \frac{1}{4}f(2x),w\right\| + \left\|\frac{1}{4}f(2x) - \frac{1}{16}f(4x),w\right\|$$

$$\leqslant \frac{\epsilon}{4}\|x,w\|^{r+s+t} + \frac{1}{4}.\frac{\epsilon}{4}.2^{r+s+t}\|x,w\|^{r+s+t}$$

for all $x,w \in A$. Thus also we have

$$\left\|f(x) - \frac{1}{4^2}f(2^2x), w\right\| \leqslant \frac{\epsilon}{4}.\|x,w\|^{r+s+t}\left[1+\frac{1}{4}.2^{r+s+t}\right]$$

for each $x,w \in A$. Then by using induction method on n, we obtain that

$$\begin{aligned}\left\|f(x) - \frac{1}{4^n}f(2^nx), w\right\| &\leqslant \frac{\epsilon}{4}.\|x,w\|^{r+s+t}\sum_{k=0}^{n-1}\frac{1}{4^k}.2^{(r+s+t)k}\\ &\leqslant \frac{\epsilon}{4}\left[\frac{1-2^{(r+s+t-2)n}}{1-2^{r+s+t-2}}\right]\|x,w\|^{r+s+t}\\ &\leqslant \frac{\epsilon}{4}\left[\frac{4}{4-2^{r+s+t}}\right]\|x,w\|^{r+s+t} \leqslant \left[\frac{\|x,w\|^{r+s+t}}{4-2^{r+s+t}}\right]\epsilon\end{aligned} \tag{4.20}$$

for all $x,w \in A$. Using (4.20), for $m,n \in \mathbb{N}$ and for all $x \in A$, we have

$$\begin{aligned}\left\|\frac{1}{4^m}f(2^mx) - \frac{1}{4^n}f(2^nx), w\right\| &= \left\|\frac{1}{4^{m-n+n}}f(2^{m-n+n}x) - \frac{1}{4^n}f(2^nx), w\right\|\\ &= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}}f(2^{m-n}.2^nx) - f(2^nx), w\right\|\\ &\leqslant \frac{1}{4^n}.\frac{\epsilon}{4}\sum_{k=0}^{m-n-1}\frac{1}{4^k}2^{(r+s+t)k}\|2^nx,w\|^{r+s+t}\\ &\leqslant \frac{\epsilon}{4}\sum_{k=0}^{m-n-1}2^{(r+s+t-2)(n+k)}\|x,w\|^{r+s+t}\end{aligned}$$

for all $x,w \in A$. Thus, we have

$$\left\|\frac{1}{4^m}f(2^mx) - \frac{1}{4^n}f(2^nx), w\right\| \leqslant \frac{\epsilon 2^{(r+s+t-2)n}}{4}\left(\frac{1-2^{(r+s+t-2)(m-n)}}{1-2^{r+s+t-2}}\right)\|x,w\|^{r+s+t} \tag{4.21}$$

for all $x,w \in A$. Then $\left\|\frac{1}{4^m}f(2^mx) - \frac{1}{4^n}f(2^nx), w\right\| = 0$ as m,n approach to infinity, for all $x,w \in A$. Therefore, $\left\{\frac{1}{4^n}f(2^nx)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. We define a quadratic function $Q : A \to A$ by

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^nx),$$

for all $x \in A$. Then using equation (4.20), we will reach

$$\lim_{n\to\infty}\left\|f(x) - \frac{1}{4^n}f(2^nx), w\right\| \leqslant \frac{\epsilon}{4}\left[\frac{\|x,w\|^{r+s+t}}{1-2^{r+s+t-2}}\right] \tag{4.22}$$

for all $x,w \in A$. Therefore, from (4.21) and (4.22), we get

$$\|g(x) - Q(x) - g(0), w\| \leqslant \frac{\epsilon\|x,w\|^{r+s+t}}{4-2^{r+s+t}} \tag{4.23}$$

for all $x, w \in A$. Next, we shall to prove that the quadratic function Q satisfies the functional equation (1.8). For all $x, w \in A$, one can have

$$\begin{aligned}\left\|D_Q(x,y,z),w\right\| &= \lim_{n\to\infty}\frac{1}{4^n}\left\|D_f(2^n x, 2^n y, 2^n z), w\right\| \\ &= \lim_{n\to\infty}\frac{\epsilon}{4^n}\left(\|2^n x, w\|^r\,\|2^n y, w\|^s\,\|2^n z, w\|^t\right) \\ &= \lim_{n\to\infty}\epsilon 2^{(r+s+t-2)n}\|x,w\|^r\,\|y,w\|^s\,\|z,w\|^t = 0\end{aligned}$$

for all $w \in A$. Thus $\left\|D_Q(x,y,z),w\right\| = 0$, for all $w \in A$, it gives that $D_Q(x,y,z) = 0$. To complete the proof of this theorem, we have to prove that the uniqueness of the function Q. Let Q' be another quadratic function satisfying (4.15) and (4.23). Since Q and Q' are quadratic, then for all $X \in A$,

$$Q(2^n x) = 4^n Q(x), \qquad Q'(2^n x) = 4^n Q'(x).$$

Hence, for all $x \in A$, we have

$$\begin{aligned}\left\|Q(x)-Q'(x),w\right\| &= \frac{1}{4^n}\left\|Q(2^n x)-Q'(2^n x),w\right\| \\ &= \frac{1}{4^n}\left[\|Q(2^n x)-f(2^n x),w\| + \left\|f(2^n x)-Q'(2^n x),w\right\|\right] \\ &= \frac{2}{4^n}\left[\frac{\|2^n x,w\|^{r+s+t}}{4-2^{r+s+t}}\right]\epsilon = 2\epsilon\left(\frac{2^{(r+s+t-2)n}\|x,w\|^{r+s+t}}{4-2^{r+s+t}}\right) = 0 \text{ as } n\to\infty\end{aligned}$$

for all $w \in A$. Therefore, $\|Q(x)-Q'(x),w\| = 0$. Then $Q(x) = Q'(x)$ for all $x \in A$. Hence the proof of the theorem. □

Corollary 4.6. *If $\theta \in [0,\infty)$, $0 < r,s,t < 2$ and let us define $g : A \to A$ that satisfies*

$$\|D_g(x,y,z),w\| \leqslant \theta\,\|x,w\|^r\,\|y,w\|^s\,\|z,w\|^t$$

for all $x,y,z \in A$, then there exists a unique quadratic mapping $Q : A \to A$ satisfying the Jensen's type quadratic-quadratic mapping (1.8) such that

$$\|g(x)-Q(x)-g(0),w\| \leqslant \frac{\theta\,\|x,w\|^{r+s+t}}{4-2^{r+s+t}},$$

for all $x, w \in A$.

Theorem 4.7. *Let $\epsilon \geqslant 0$, $r,s,t > 2$. If $g : A \to A$ is a function satisfying*

$$\|D_g(x,y,z),w\| \leqslant \epsilon\,\|x,w\|^r\,\|y,w\|^s\,\|z,w\|^t$$

for all $x,y,z \in A$, then there exists a unique quadratic mapping $Q : A \to A$ satisfying the Jensen's quadratic-quadratic mapping (1.8) such that

$$\|g(x)-Q(x)-g(0),w\| \leqslant \frac{\epsilon\,\|x,w\|^{r+s+t}}{2^{r+s+t}-4},$$

for all $x, w \in A$.

Proof. By the equation (4.16) of the Theorem 4.5, we have

$$\|f(2x)-4f(x),w\| \leqslant \epsilon\,\|x,w\|^{r+s+t} \tag{4.24}$$

for each $x, w \in A$. Now replacing x by $\left(\frac{x}{2}\right)$ in (4.24), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| \leqslant 2^{-(r+s+t)} \|x, w\|^{r+s+t} \epsilon \tag{4.25}$$

for all $x, w \in A$. Again replacing x by $\left(\frac{x}{2}\right)$ in (4.25), we have

$$\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \leqslant 2^{-2(r+s+t)} \|x, w\|^{r+s+t} \epsilon \tag{4.26}$$

for each $x, w \in A$. Combining (4.25) and (4.26), we will have

$$\begin{aligned}
\left\|f(x) - 4^2 f\left(\frac{x}{2^2}\right), w\right\| &= \left\|f(x) - 16f\left(\frac{x}{4}\right), w\right\| \\
&= \left\|f(x) - 4f\left(\frac{x}{2}\right), w\right\| + 4\left\|f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), w\right\| \\
&\leqslant 2^{-(r+s+t)} \|x, w\|^{r+s+t} \epsilon + 42^{-2(r+s+t)} \|x, w\|^{r+s+t} \epsilon \\
&= 2^{-(r+s+t)} \|x, w\|^{r+s+t} \epsilon \left[1 + 4.2^{-(r+s+t)}\right]
\end{aligned}$$

for each $x, w \in A$. Now, we will apply the induction method on n, we get that

$$\begin{aligned}
\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &\leqslant 2^{-(r+s+t)} \|x\|^{r+s+t} \|w\|^u \epsilon \sum_{k=0}^{n-1} 4^k . 2^{-(r+s+t)k} \\
&= \epsilon \|x, w\|^{r+s+t} \sum_{k=0}^{n-1} 2^{k[2-(r+s+t)]-(r+s+t)} \\
&\leqslant \epsilon \|x, w\|^{r+s+t} \left(2^{-(r+s+t)} \left[\frac{1 - 2^{[2-(r+s+t)]n}}{1 - 2^{2-(r+s+t)}}\right]\right)
\end{aligned} \tag{4.27}$$

for all $x, w \in A$. For $m, n \in \mathbb{N}$ and for $x \in A$, we have

$$\begin{aligned}
\left\|4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), w\right\| &= 4^n \left\|4^{m-n} f\left(\frac{x}{2^{m-n}.2^n x}\right) - f\left(\frac{x}{2^n}\right), w\right\| \\
&\leqslant 4^n \epsilon \sum_{k=0}^{m-n-1} \left[2^{k[2-(r+s+t)]-(r+s+t)} \left\|\frac{x}{2^n}, w\right\|^{r+s+t}\right] \\
&= \epsilon \sum_{k=0}^{m-n-1} \left[2^{(n+k)[2-(r+s+t)]-(r+s+t)} \|x, w\|^{r+s+t}\right] \\
&= 2^{n[2-(r+s+t)]-(r+s+t)} \left[\frac{1 - 2^{[2-(r+s+t)](m-n)}}{1 - 2^{2-(r+s+t)}}\right] \|x, w\|^{r+s+t} \epsilon \\
&\longrightarrow 0 \quad \text{as } n \to \infty
\end{aligned}$$

for all $x \in A$ and $w \in A$. Therefore, $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in A, for all $x \in A$. Since A is a 2-Banach space, the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is 2-converges, for all $x \in A$. We define a mapping $Q : A \to A$ by

$$Q(x) := \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right),$$

for all $x \in A$. Now, with the help of equation (4.27), we get

$$\lim_{n \to \infty} \left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), w\right\| \leqslant \epsilon \left(\frac{2^{-(r+s+t)}}{1 - 2^{2-(r+s+t)}}\right) \|x, w\|^{r+s+t}$$

for all $x, w \in A$. Hence,

$$\|g(x) - Q(x) - g(0), w\| \leqslant \left(\frac{\epsilon}{2^{r+s+t} - 4}\right) \|x, w\|^{r+s+t},$$

for all $x, w \in A$. The further part of the proof is similar to the above Theorem 4.5, which completes the Theorem. □

Corollary 4.8. *Let* $\theta \in [0, \infty)$, $r, s, t > 2$ *and* $g : A \to A$ *be a function which satisfies*

$$\|D_g(x, y, z), w\| \leqslant \theta \|x, w\|^r \|y, w\|^s \|z, w\|^t$$

for all $x, y, z \in A$. *Then there exists a unique quadratic mapping* $Q : A \to A$ *satisfying the Jensen's quadratic-quadratic mapping* (1.8) *such that*

$$\|g(x) - Q(x) - g(0), w\| \leqslant \left[\frac{\theta}{2^{r+s+t} - 4}\right] \|x, w\|^{r+s+t},$$

for all $x, w \in A$.

5. Conclusion

In this paper, we established the Hyers-Ulam-Gavruta stability of a Jensen's type quadratic-quadratic mapping in 2-Banach Spaces by Hyers direct method.

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