J. Math. Computer Sci., 32 (2024), 348–357

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On Reich and Chaterjea type cyclic weakly contraction mappings in metric spaces

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Abstract

This paper signifies the existence and uniqueness of fixed points for some classes of mappings on general settings. Indeed, we prove existence and uniqueness results for Reich and Chatterjea type cyclic contractions using the perception of sequentially convergence mappings in metric spaces. We also present an example to illustrate our results.

Keywords: Fixed point, cyclic contraction, best proximity points, complete metric space, cyclic Chatterjea-Reich type contraction.

2020 MSC: 47H10, 54E50, 47H09.

1. Introduction and preliminaries

The Banach Contraction Principle (BCP) has undergone significant extensions and generalizations since its inception, see [1, 3, 4, 7, 13], etc. One of the notable extension of the BCP was obatined by Kannan [6] which states that a self-mapping T of a complete metric space (X, d) has a unique fixed point in X, if for all $x, y \in X$ and $\alpha \in \left(0, \frac{1}{2}\right)$,

$$d(Tx, Ty) \leqslant \alpha[d(x, Tx) + d(y, Ty)].$$

In 1971, Reich [11] unified the BCP and Kannan fixed point result. He [12] further generalized his theorem by replacing the contraction constants with monotonically decreasing functions. Another important extension of the BCP was obtained by Chatterjea [3] which states that a self-mapping T of a complete metric space (X, d) has a unique fixed point in X, if for all $x, y \in X$ and $\alpha \in \left(0, \frac{1}{2}\right)$,

$$d(Tx, Ty) \leqslant \alpha[d(x, Ty) + d(y, Tx)].$$

In 2003, the idea of cyclic representation of mappings was introduced by Kirk et al. [8] as follows.

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doi: 10.22436/jmcs.032.04.05

Received: 2023-07-24 *Revised:* 2023-08-23 *Accepted:* 2023-08-25

Definition 1.1 ([8]). Let X be a nonempty set and let $T : X \to X$ be a mapping. If

(1) $X_i, i = 1, 2, 3, \ldots, m$ are non-empty sets;
(2) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m, T(X_m) \subset X_1$,

then $X = \bigcup\limits_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T.

Eventually, they defined cyclical contraction and presented fixed point results for such mappings. Later, in 2013, Chandok and Postalache [2] defined a version of cyclic weakly contraction of Chatterjea type in the setting of metric spaces. Recently, [5], Das et al. discussed the iterative algorithm and theoretical treatment for existence of solution of (k, z)-Riemann-Liouville fractional integral equations.

Definition 1.2. Let Φ be the set of all monotone increasing functions $\nu : [0, \infty) \to [0, \infty)$ with $\nu(t) = 0$ iff $t = 0$. Suppose Ω represents the set of all lower semi continuous functions $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $\omega(t_1, t_2) > 0$ for $t_1, t_2 \in (0, \infty)$ and $\omega(0, 0) = 0$.

Definition 1.3 ([2]). Let (X, d) be a metric space and $X_1, X_2, \ldots, X_m$ be non empty subsets of X and $Y = \bigcup\limits_{i=1}^{m} X_i$. An operator $G : Y \to Y$ is called a Chatterjea type cyclic weakly contraction if

(a) $\bigcup\limits_{i=1}^{m} X_i$ is a cyclical representation of Y with respect to G;
(b) $\nu(d(Gx, Gy)) \leqslant \nu\left(\frac{1}{2}[d(x, Gy) + d(y, Gx)]\right) - \omega(d(x, Gy), d(y, Gx))$ for all $x \in X_i, y \in X_{i+1}$, and $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$.

In 2016, Malceskii [9] proved some results on fixed point for Kannan type contractions and Chatterjea type contractions using notion of sequentially convergent mappings. In this paper, we have extended Reich type fixed point results and weakly cyclical Chatterjea type fixed point results using the notion of sequentially convergence mapping.

Definition 1.4 ([10]). Let (X, d) be a metric space. A mapping $G : X \to X$ is said to be sequentially convergent if for every sequence (y_r) in X, convergence of (Gy_r) implies convergence of (y_r).

2. Extension of Reich type contraction mapping

Theorem 2.1. *Let (X, d) be a complete metric space and let $\mathcal{H} : X \to X$ be a continuous, injection, and sequentially convergent mapping. Suppose $G : X \to X$ is a mapping such that $x, y \in X$ with $x \neq y$,*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x, y))d(\mathcal{H}x, \mathcal{H}Gx) + b(d(x, y))d(\mathcal{H}y, \mathcal{H}Gy) + c(d(x, y))d(\mathcal{H}x, \mathcal{H}y), \tag{2.1}$$

where a, b, c are monotonic decreasing functions defined from $(0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) + c(t) < 1$. Then, there exists a unique fixed point of G.

Proof. Let $x_0 \in X$. Define x_r such that $x_{r+1} = Gx_r$ for $r = 0, 1, 2, 3 \ldots$. Then, by (2.1), we get

$$\begin{aligned} d(\mathcal{H}x_{r+1}, \mathcal{H}_r) = d(\mathcal{H}Gx_r, \mathcal{H}Gx_{r-1}) &\leqslant a(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}Gx_r) + b(d(x_r, x_{r-1}))d(\mathcal{H}x_{r-1}, \mathcal{H}Gx_{r-1}) \\ &\quad + c(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1}) \\ &= a(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + b(d(x_r, x_{r-1}))d(\mathcal{H}x_{r-1}, \mathcal{H}x_r) \\ &\quad + c(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1}). \end{aligned}$$

Let $\lambda(d(x_i, x_{i-1})) = \frac{b(d(x_i,x_{i-1}))+c(d(x_i,x_{i-1}))}{1-a(d(x_i,x_{i-1}))}$ for $i = 1, 2, \ldots$. Then $0 < \lambda < 1$ and

$$d(\mathcal{H}x_{r+1}, \mathcal{H}_r) \leqslant \lambda(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1}) \leqslant \lambda(d(x_r, x_{r-1}))\lambda(d(x_{r-1}, x_{r-2})) \cdots \lambda(d(x_1, x_0))d(\mathcal{H}x_1, \mathcal{H}x_0).$$

If $\lambda(t) = \max\{\lambda(d(x_r, x_{r-1})), \lambda(d(x_{r-1}, x_{r-2})), \dots, \lambda(d(x_1, x_0))\}$, then the above inequality reduces to

$$d(\mathcal{H}x_{r+1}, \mathcal{H}_r) \leqslant (\lambda(t))^r d(\mathcal{H}x_1, \mathcal{H}x_0).$$

Now, for all $p, q \in \mathbb{N}$ with $p > q$, we have

$$d(\mathcal{H}x_p, \mathcal{H}_q) \leqslant d(\mathcal{H}x_p, \mathcal{H}_{p-1}) + \cdots + d(\mathcal{H}x_{q+1}, \mathcal{H}_q) \leqslant \frac{(\lambda(t'))^q}{1-\lambda(t')} d(\mathcal{H}x_1, \mathcal{H}x_0),$$

where $\lambda(t') = \max\{\lambda(t_1), \lambda(t_2), \dots, \lambda(t_{p-q})\}$. So, $\{\mathcal{H}x_r\}$ is a Cauchy sequence. Since X is complete, the sequence $\{\mathcal{H}x_r\}$ converges to some point in X. Further, $\mathcal{H}$ being sequentially convergent, the sequence $\{x_r\}$ is also convergent. Therefore $\lim_{r\to\infty} x_r = z$ for some $z \in X$. Since $\mathcal{H}$ is continuous, we have

$$\lim_{r\to\infty} \mathcal{H}x_r = \mathcal{H}z.$$

Again by (2.1), we get

$$\begin{aligned} d(\mathcal{H}Gz, \mathcal{H}z) &\leqslant d(\mathcal{H}Gz, \mathcal{H}x_r) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z) \\ &= d(\mathcal{H}Gz, \mathcal{H}G^r x_0) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z) \\ &\leqslant a(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}Gz) + b(d(z, G^{r-1}x_0))d(\mathcal{H}G^{r-1}x_0, \mathcal{H}G^r x_0) \\ &\quad + c(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}G^{r-1}x_0) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z) \\ &\leqslant a(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}Gz) + b(d(z, G^{r-1}x_0))d(\mathcal{H}Gx_{r-1}, \mathcal{H}Gx_r) \\ &\quad + c(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}Gx_{r-1}) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z). \end{aligned}$$

Making $r \to \infty$, gives

$$d(\mathcal{H}Gz, \mathcal{H}z) \leqslant a(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}Gz).$$

Since $a(t) < 1$, we get $d(\mathcal{H}Gz, \mathcal{H}z) = 0 \Longrightarrow \mathcal{H}Gz = \mathcal{H}z$. Further, since H is injective, $Gz = z$. Thus, G has fixed point.

Next, we show that fixed point of G is unique. Assume that G has two fixed points in X, say z and z'. Then by (2.1), we have

$$\begin{aligned} d(\mathcal{H}z, \mathcal{H}z') = d(\mathcal{H}Gz, \mathcal{H}Gz') &\leqslant a(d(z,z'))d(\mathcal{H}z, \mathcal{H}Gz) + b(d(z,z'))d(\mathcal{H}z', \mathcal{H}Gz') + c(d(z,z'))d(\mathcal{H}z, \mathcal{H}z') \\ &= c(d(z,z'))d(\mathcal{H}z, \mathcal{H}z'). \end{aligned}$$

Since $c < 1$, $d(\mathcal{H}z, \mathcal{H}z') = 0 \Longrightarrow \mathcal{H}z = \mathcal{H}z'$, which further implies $z = z'$. □

Corollary 2.2. *Let (X, d) be a complete metric space. Let $\mathcal{H} : X \to X$ be an injection, which is continuous and sequentially convergent mapping. Let $G : X \to X$ be a mapping such that*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x,y))d(\mathcal{H}x, \mathcal{H}Gx) + b(d(x,y))d(\mathcal{H}y, \mathcal{H}Gy),$$

where $x, y \in X, x \neq y$ and a, b are monotonic decreasing functions defined from $(0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) < 1$. Then, G has a unique fixed point.

Remark 2.3. If $a(t) = b(t) = \alpha$ and $c(t) = \beta$, and if a, b, c are defined from $(0, \infty)$ into $(0, 1)$, then Theorem 2.1 reduces to Theorem 1 of Malčeskii ([9, p. 2]).

Remark 2.4. If $\mathcal{H}(x) = x$ in Theorem 2.1, then it reduces to Theorem 3 of Reich ([12, p. 2]).

Theorem 2.5. *Let (X, d) be a complete metric space. Let $\mathcal{H} : X \to X$ be a continuous, injection, and sequentially convergent mapping. Let $G : X \to X$ be a mapping such that $x, y \in X$ with $x \neq y$,*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x,y))d(\mathcal{H}x, \mathcal{H}Gy) + b(d(x,y))d(\mathcal{H}y, \mathcal{H}Gx) + c(d(x,y))d(\mathcal{H}x, \mathcal{H}y), \tag{2.2}$$

where a, b, c are monotonic decreasing functions defined from $(0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) + c(t) < 1$ and $a(t) \geqslant b(t)$. Then, there exists a unique fixed point of G.

Proof. Let $x_0 \in X$ be arbitrary. Define x_r such that $x_{r+1} = Gx_r$ for $r = 0, 1, 2, 3 \dots$. Then, by (2.2), we have

$$\begin{aligned}
&d(\mathcal{H}x_{r+1}, \mathcal{H}_r)\\
&\quad = d(\mathcal{H}Gx_r, \mathcal{H}Gx_{r-1})\\
&\quad \leqslant a(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}Gx_{r-1}) + b(d(x_r, x_{r-1}))d(\mathcal{H}x_{r-1}, \mathcal{H}Gx_r) + c(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1})\\
&\quad = a(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_r) + b(d(x_r, x_{r-1}))d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1}) + c(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1})\\
&\quad \leqslant b(d(x_r, x_{r-1}))d(\mathcal{H}x_{r-1}, \mathcal{H}x_r) + b(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + c(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1}).
\end{aligned}$$

If $\lambda(d(x_j, x_{j-1})) = \frac{b(d(x_j,x_{j-1}))+c(d(x_j,x_{j-1}))}{1-b(d(x_j,x_{j-1}))} < 1$ for $j = 1, 2, \dots$, then

$$\begin{aligned}
d(\mathcal{H}x_{r+1}, \mathcal{H}_r) &\leqslant \lambda(d(x_r, x_{r-1}))d(\mathcal{H}x_r, \mathcal{H}x_{r-1})\\
&\leqslant \lambda(d(x_r, x_{r-1}))\lambda(d(x_{r-1}, x_{r-2})) \cdots \lambda(d(x_1, x_0)).d(\mathcal{H}x_1, \mathcal{H}x_0).
\end{aligned}$$

Let $\lambda(t) = \max\{\lambda(d(x_r, x_{r-1})), \lambda(d(x_{r-1}, x_{r-2})), \dots, \lambda(d(x_1, x_0))\}$. Then the above inequality reduces to

$$d(\mathcal{H}x_{r+1}, \mathcal{H}_r) \leqslant (\lambda(t))^r d(\mathcal{H}x_1, \mathcal{H}x_0).$$

Next, for all $p, q \in \mathbb{N}, p > q$,

$$d(\mathcal{H}x_p, \mathcal{H}_q) \leqslant d(\mathcal{H}x_p, \mathcal{H}_{p-1}) + \cdots + d(\mathcal{H}x_{q+1}, \mathcal{H}_q) \leqslant \frac{(\lambda(t'))^q}{1 - \lambda(t')} d(\mathcal{H}x_1, \mathcal{H}x_0),$$

where $\lambda(t') = \max\{\lambda(t_1), \lambda(t_2), \dots, \lambda(t_{p-q})\}$. Thus, $\{\mathcal{H}x_r\}$ is a Cauchy sequence and hence convergent. Further, $\mathcal{H}$ being sequentially convergent, $\{x_r\}$ is convergent. So, $\lim\limits_{r\to\infty} x_r = z$ for some $z \in X$,

$$\lim_{r\to\infty} \mathcal{H}x_r = \mathcal{H}z.$$

Thus

$$\begin{aligned}
d(\mathcal{H}Gz, \mathcal{H}z) &\leqslant d(\mathcal{H}Gz, \mathcal{H}x_r) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z)\\
&= d(\mathcal{H}Gz, \mathcal{H}G^r x_0) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z)\\
&\leqslant a(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}G^r x_0) + b(d(z, G^{r-1}x_0))d(\mathcal{H}G^{r-1}x_0, \mathcal{H}Gz)\\
&\quad + c(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}G^{r-1}x_0) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z)\\
&\leqslant a(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}x_r) + b(d(z, G^{r-1}x_0))d(\mathcal{H}x_{r-1}, \mathcal{H}Gz)\\
&\quad + c(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}x_{r-1}) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) + d(\mathcal{H}x_{r+1}, \mathcal{H}z).
\end{aligned}$$

Making $r \to \infty$, gives

$$d(\mathcal{H}Gz, \mathcal{H}z) \leqslant b(d(z, G^{r-1}x_0))d(\mathcal{H}z, \mathcal{H}Gz).$$

Since $b < 1$, $d(\mathcal{H}Gz, \mathcal{H}z) = 0$. Since H is injective, we have $Gz = z$. Thus, G has fixed point. Uniqueness of fixed point can be proved easily. □

Corollary 2.6. *Let* (X, d) *be a complete metric space. Let* $\mathcal{H} : X \to X$ *be an injection which is continuous and sequentially convergent mapping. Let* $G : X \to X$ *be a mapping such that* $x, y \in X$ *with* $x \neq y$,

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x, y))d(\mathcal{H}x, \mathcal{H}Gy) + b(d(x, y))d(\mathcal{H}y, \mathcal{H}Gx),$$

where a, b *are monotonic decreasing functions defined from* $(0, \infty)$ *into* $[0, 1)$ *such that* $a(t) + b(t) < 1$*. Then,* G *has a unique fixed point.*

Remark 2.7. If $a(t) = b(t) = \alpha$ and $c(t) = \beta$, and if a, b, c are defined from $(0, \infty)$ into $(0, 1)$, then Theorem 2.5 reduces to Theorem 2 of Malčeskii ([9, p. 2]).

Theorem 2.8. *Let (X, d) be a complete metric space. Let V and W be two nonempty subsets of X. Let $\mathcal{H} : V \cup W \to V \cup W$ be a continuous, injection, and sequentially convergent mapping such that $\mathcal{H}(V) \subseteq W$ and $\mathcal{H}(W) \subseteq V$. Let $G : V \cup W \to V \cup W$ be a mapping such that $G(V) \subseteq W$, $G(W) \subseteq V$ and*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x, y))d(\mathcal{H}x, \mathcal{H}Gx) + b(d(x, y))d(\mathcal{H}y, \mathcal{H}Gy) + c(d(x, y))d(\mathcal{H}x, \mathcal{H}y),$$

for $x \in V, y \in W$, $x \neq y$, where a, b, c are monotonic decreasing functions defined from $(0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) + c(t) < 1$. Then, there exists a unique fixed point of G.

Theorem 2.9. *Let (X, d) be a complete metric space and let V and W be two nonempty subsets of X. Let $\mathcal{H} : V \cup W \to V \cup W$ be a continuous, injection, and sequentially convergent mapping such that $\mathcal{H}(V) \subseteq W$ and $\mathcal{H}(W) \subseteq V$. Let $G : V \cup W \to V \cup W$ be a mapping such that $G(V) \subseteq W$, $G(W) \subseteq V$, and*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant a(d(x, y))d(\mathcal{H}x, \mathcal{H}Gy) + b(d(x, y))d(\mathcal{H}y, \mathcal{H}Gx) + c(d(x, y))d(\mathcal{H}x, \mathcal{H}y),$$

for $x \in V, y \in W$ with $x \neq y$, where a, b, c are monotonic decreasing functions defined from $(0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) + c(t) < 1$ and $a(t) \geqslant b(t)$. Then, there exists a unique fixed point of G.

3. Extension of cyclic Chatterjea type mapping

Let $X_1, X_2, \ldots, X_p$ be non-empty closed subsets of X such that $Y = \bigcup_{i=1}^{p} X_i$.
Consider the two self mappings $\mathcal{H} : Y \to Y$ and $G : Y \to Y$ satisfying the following properties:

(P1) $\bigcup_{i=1}^{p} X_i$ is cyclic with respect to $\mathcal{H}$ and G, that is, for $1 \leqslant i \leqslant p$,

$$\mathcal{H}(X_i) \subseteq X_{i+1} \text{ and } G(X_i) \subseteq X_{i+1};$$

(P2) for any $x \in X_i, y \in X_{i+1}$,

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) \leqslant \nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx)), \tag{3.1}$$

where $\nu \in \Phi$ and $\omega \in \Omega$.

Theorem 3.1. *If $\mathcal{H}$ and G are self mappings in Y satisfying properties (P1) and (P2), then, for any $x_0 \in Y$, $d(\mathcal{H}x_{r+1}, \mathcal{H}x_r) \to 0$, where $x_{r+1} = Gx_r$.*

Proof. Let $x_0 \in Y$. Define $\{x_r\}$ such that $x_{r+1} = Gx_r$ for $r = 0, 1, 2, 3 \ldots$. Then by (3.1) we have

$$\begin{aligned}
\nu(d(\mathcal{H}x_{r+1}, \mathcal{H}x_r)) &= \nu(d(\mathcal{H}Gx_r, \mathcal{H}Gx_{r-1})) \\
&\leqslant \nu\left(\frac{1}{2}[d(\mathcal{H}x_r, \mathcal{H}Gx_{r-1}) + d(\mathcal{H}x_{r-1}, \mathcal{H}Gx_r)]\right) \\
&\quad - \omega(d(\mathcal{H}x_r, \mathcal{H}Gx_{r-1}), d(\mathcal{H}x_{r-1}, \mathcal{H}Gx_r)) \\
&= \nu\left(\frac{1}{2}[d(\mathcal{H}x_r, \mathcal{H}x_r) + d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1})]\right) - \omega(d(\mathcal{H}x_r, \mathcal{H}x_r), d(Hx_{r-1}, \mathcal{H}x_{r+1})) \\
&\leqslant \nu\left(\frac{1}{2}d(\mathcal{H}x_{r-1}, \mathcal{H}Gx_r)\right) - \omega(0, d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1})).
\end{aligned} \tag{3.2}$$

This implies that

$$\nu(d(\mathcal{H}x_{r+1}, \mathcal{H}x_r)) \leqslant \nu\left(\frac{1}{2}d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1})\right).$$

Since ν is a monotone increasing function, we have

$$d(\mathcal{H}x_{r+1}, \mathcal{H}x_r) \leqslant \frac{1}{2}d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1}) \leqslant \frac{1}{2}[d(\mathcal{H}x_{r-1}, \mathcal{H}x_r) + d(\mathcal{H}x_r, \mathcal{H}x_{r+1})] \leqslant d(\mathcal{H}x_{r-1}, \mathcal{H}x_r).$$

Thus, $\{d(\mathcal{H}x_r, \mathcal{H}x_{r+1})\}$ is a monotonic decreasing sequence. So, it is convergent. Let $d(\mathcal{H}x_r, \mathcal{H}x_{r+1}) \to c$, for some $c \geqslant 0$. Then

$$\lim_{r\to\infty} d(\mathcal{H}x_{r-1}, \mathcal{H}x_{r+1}) = 2c.$$

Since ν is continuous and ω is lower semi-continuous, from (3.2),

$$\nu(c) \leqslant \nu(c) - \omega(0, 2c) \quad \text{as} \quad r \to \infty.$$

This implies $\omega(0, 2c) = 0 \Longrightarrow c = 0$. Hence $d(\mathcal{H}x_{r+1}, \mathcal{H}x_r) \to 0$. □

Theorem 3.2. *Corresponding to every $\lambda > 0$, a number $r \in \mathbb{N}$ can be found such that $d(\mathcal{H}x_m, \mathcal{H}x_n) < \lambda$ whenever $m, n \geqslant r$ with $m - n \equiv 1 \pmod k$.*

Proof. On the contrary, let us assume that there exists a $\lambda > 0$ for which a natural number r can be found such that $d(\mathcal{H}x_{m_r}, \mathcal{H}x_{n_r}) \geqslant \lambda$ for some $m_r > n_r \geqslant r$ with $m_r - n_r \equiv 1 \pmod k$. Let $r > 2k$. Now, corresponding to $n_r \geqslant r$, choose the smallest $m_r > n_r$ satisfying $m_r - n_r \equiv 1 \pmod k$ and

$$d(\mathcal{H}x_{m_r}, \mathcal{H}x_{n_r}) \geqslant \lambda.$$

Therefore, $d(Hx_{m_{r-k}}, Hx_{n_r}) < \lambda$. Now,

$$\lambda \leqslant d(\mathcal{H}x_{m_r}, \mathcal{H}x_{n_r}) \leqslant d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_{r-k}}) + \sum_{i=1}^{k} d(\mathcal{H}x_{m_{r-i}}, \mathcal{H}x_{m_{r-i+1}}) < \lambda + \sum_{i=1}^{k} d(\mathcal{H}x_{m_{r-i}}, \mathcal{H}x_{m_{r-i+1}}).$$

Using Theorem 3.1, $\lim_{r\to\infty} d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_r}) = \lambda$. Also, by the triangular inequality,

$$\begin{aligned} \lambda \leqslant d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_r}) &\leqslant d(\mathcal{H}x_{n_r}, \mathcal{H}x_{n_{r+1}}) + d(\mathcal{H}x_{n_{r+1}}, \mathcal{H}x_{m_{r+1}}) + (\mathcal{H}x_{m_{r+1}}, \mathcal{H}x_{m_r}) \\ &\leqslant d(\mathcal{H}x_{n_r}, \mathcal{H}x_{n_{r+1}}) + d(\mathcal{H}x_{n_{r+1}}, \mathcal{H}x_{n_r}) + d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_r}) \\ &\quad + d(\mathcal{H}x_{m_r}, \mathcal{H}x_{m_{r+1}}) + d(\mathcal{H}x_{m_{r+1}}, \mathcal{H}x_{m_r}). \end{aligned}$$

Again, by Theorem 3.1,

$$\lim_{r\to\infty} d(\mathcal{H}x_{n_{r+1}}, \mathcal{H}x_{m_{r+1}}) = \lambda.$$

Further,

$$d(\mathcal{H}x_{n_r}, \mathcal{H}Gx_{m_r}) = d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_{r+1}}) \leqslant d(\mathcal{H}x_{n_r}, \mathcal{H}x_{m_r}) + d(\mathcal{H}x_{m_r}, \mathcal{H}x_{m_{r+1}}).$$

Making $r \to \infty$, gives

$$\lim_{r\to\infty} d(\mathcal{H}x_{n_r}, \mathcal{H}Gx_{m_r}) = \lambda.$$

Similarly, we have

$$\lim_{r\to\infty} d(\mathcal{H}x_{m_r}, \mathcal{H}Gx_{n_r}) = \lambda.$$

Now,

$$\lambda \leqslant d(\mathcal{H}x_{n_{r+1}}, \mathcal{H}x_{m_{r+1}}).$$

Since ν is monotone increasing,

$$\begin{aligned} \nu(\lambda) &\leqslant \nu(d(\mathcal{H}x_{n_{r+1}}, \mathcal{H}x_{m_{r+1}})) \\ &= \nu(d(\mathcal{H}Gx_{n_r}, \mathcal{H}Gx_{m_r})) \end{aligned}$$

$$\leqslant \nu\left(\frac{1}{2}[d(\mathcal{H}x_{n_r},\mathcal{H}Gx_{m_r})+d(\mathcal{H}x_{m_r},\mathcal{H}Gx_{n_r})]\right)-\omega(d(\mathcal{H}x_{n_r},\mathcal{H}Gx_{m_r}),d(\mathcal{H}x_{m_r},\mathcal{H}Gx_{n_r})).$$

Making $r\to\infty$, gives

$$\nu(\lambda)\leqslant \nu\left(\frac{1}{2}(\lambda+\lambda)\right)-\omega(\lambda,\lambda)=\nu(\lambda)-\omega(\lambda,\lambda).$$

Thus $\omega(\lambda,\lambda)\leqslant 0$, which is not possible as $\lambda>0$. Hence the result. □

Theorem 3.3. *Let $\mathcal{H}: Y\to Y$ be a continuous, injective, and sequentially convergent mapping. Let $G: Y\to Y$ be a self mapping such that $\mathcal{H}$ and G satisfy properties* (P1) *and* (P2)*. Then, G has a unique fixed point in $\cap X_i$.*

Proof. By virtue of Theorem 3.1, an integer $r_0\in\mathbb{N}$ can be found such that

$$d(\mathcal{H}x_{r+1},\mathcal{H}x_r)\leqslant \frac{\lambda}{2k}, \forall r>r_0.$$

Also, by Theorem 3.2, a positive integer r_1 exists such that for $m,n\geqslant r_1$ with $m-n\equiv 1 \pmod k$,

$$d(\mathcal{H}x_m,\mathcal{H}x_n)\leqslant \frac{\lambda}{2}.$$

Let $s>m\geqslant \max(r_0,r_1)$. Then, an $l\in\{1,2,\dots,k\}$ can be found satisfying $s-m\equiv l \pmod k$. Moreover, taking $j=k-l+1$, we get $s-m+j\equiv 1 \pmod k$. Thus,

$$\begin{aligned}d(\mathcal{H}x_m,\mathcal{H}x_s)&\leqslant d(\mathcal{H}x_s,\mathcal{H}x_{m+j})+d(\mathcal{H}x_{m+j},\mathcal{H}x_{m+j-1})+\dots+d(\mathcal{H}x_{m+1},\mathcal{H}x_m)\\&\leqslant \frac{\lambda}{2}+j\frac{\lambda}{2k}\leqslant \frac{\lambda}{2}+k\frac{\lambda}{2k}=\lambda.\end{aligned}$$

Thus, $\{\mathcal{H}x_r\}$ is a Cauchy sequence in Y. Also, Y, being closed in a complete space, is complete. Hence $\{\mathcal{H}x_r\}$ is convergent. Further, as $\mathcal{H}$ is sequentially convergent, $\{x_r\}$ is also convergent. That is, Y has a member, say z, such that $\lim_{r\to\infty} x_r=z$. Further, G being continuous, gives

$$\lim_{r\to\infty} Gx_r=Gz.$$

Now

$$\begin{aligned}\mu(d(\mathcal{H}Gz,\mathcal{H}z))&=\nu(d(\mathcal{H}Gz,\mathcal{H}x_r))\\&=\nu(d(\mathcal{H}Gz,\mathcal{H}Gx_{r+1}))\\&\leqslant \nu\left(\frac{1}{2}[d(\mathcal{H}z,\mathcal{H}Gx_{r+1})+d(\mathcal{H}x_{r+1},\mathcal{H}Gz)]\right)-\omega(d(\mathcal{H}z,\mathcal{H}Gx_{r+1}),d(\mathcal{H}x_{r+1},\mathcal{H}Gz))\\&=\nu\left(\frac{1}{2}[d(\mathcal{H}z,\mathcal{H}x_r)+d(\mathcal{H}x_{r+1},\mathcal{H}Gz)]\right)-\omega(d(\mathcal{H}z,\mathcal{H}x_r),d(\mathcal{H}x_{r+1},\mathcal{H}Gz)).\end{aligned}$$

Since $\mathcal{H}$ and ν are continuous and ω is lower semi-continuous, the above inequality reduces to

$$\nu(d(\mathcal{H}Gz,\mathcal{H}z))\leqslant \nu\left(\frac{1}{2}d(\mathcal{H}z,\mathcal{H}Gz)\right)-\omega(0,d(\mathcal{H}z,\mathcal{H}Gz)),$$

as $r\to\infty$. This leads to a contradiction unless

$$d(\mathcal{H}z,\mathcal{H}Gz)=0\implies \mathcal{H}z=HGz.$$

This implies $z=Gz$ and $\mathcal{H}$ being injective. Thus, G has a fixed point z. Uniqueness of fixed point can be proved easily. □

If $\nu(t) = t$, then Theorem 3.3 gives the following result.

Corollary 3.4. *Let (X, d) be a complete metric space. Let $X_1, X_2, \dots, X_p$ be non-empty closed subsets of X and $Y = \bigcup_{i=1}^{p} X_i$. Let $H : Y \to Y$ be a continuous, injective, and sequentially convergent mapping. Also, let $G : Y \to Y$ be a self mapping such that property (P1) holds. If, for any $x \in X_i$, $y \in X_{i+1}$ and $\omega \in \Omega$,*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant \frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)] - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx)),$$

then G has a unique fixed point in $\cap X_i$.

If $\omega(s,t) = (\frac{1}{2} - k)(s+t)$, where $k \in [0, \frac{1}{2})$, then Theorem 3.3 gives the following result.

Corollary 3.5. *Let (X, d) be a complete metric space. Let $X_1, X_2, \dots, X_p$ be non-empty closed subsets of X and $Y = \bigcup_{i=1}^{p} X_i$. Let $\mathcal{H} : Y \to Y$ be a continuous, injective, and sequentially convergent mapping. Also, let $G : Y \to Y$ be a self mapping such that property (P1) holds. If, for any $x \in X_i$, $y \in X_{i+1}$ and $\omega \in \Omega$,*

$$d(\mathcal{H}Gx, \mathcal{H}Gy) \leqslant k[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)],$$

where $k \in [0, \frac{1}{2})$, then G has a unique fixed point in $\cap X_i$.

We now present an example to illustrate our results.

Problem 3.6. Consider a metric space X in $\mathbb{R}$ with the standard metric d defined by $d(x,y) = |x-y|$. Let $X_1 = [0,1]$, $X_2 = [0, \frac{1}{2}]$, and $Y = \bigcup_{i=1}^{2} X_i$. Define a mapping $\mathcal{H} : Y \to Y$ by

$$\mathcal{H}x = \frac{x}{9}, \quad \forall x \in Y.$$

It can easily be seen that F is sequentially convergent, injective, and continuous. Define $\omega : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $\omega(s,t) = \frac{s+t}{9}$ and $\nu : [0,\infty) \to [0,\infty)$ by $\nu(t) = t$. Then $\omega \in \Omega$ and $\nu \in \Phi$. Also, define mapping $G : Y \to Y$ by $Gx = \frac{x}{3}$ for all $x \in Y$. Clearly, $\mathcal{H}$ and G satisfy property (P1). Now, let us prove that H and G also satisfy property (P2). So,

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) = \nu\left(|\mathcal{H}(\frac{x}{3}) - \mathcal{H}(\frac{y}{3})|\right) = \nu\left(\left|\frac{x}{27} - \frac{y}{27}\right|\right) = \left|\frac{x}{27} - \frac{y}{27}\right|, \tag{3.3}$$

and

$$\begin{aligned} &\nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx)) \\ &= \nu\left(\frac{1}{2}[|\mathcal{H}x - \mathcal{H}Gy| + |\mathcal{H}y - \mathcal{H}Gx|]\right) - \omega(|\mathcal{H}x - \mathcal{H}Gy|, |\mathcal{H}y - \mathcal{H}Gx|) = \frac{7}{162}\left[\left|x - \frac{y}{3}\right| + \left|y - \frac{x}{3}\right|\right]. \end{aligned} \tag{3.4}$$

Now, let us examine the following four cases.

Case I: $x \geqslant y$ and $y < \frac{x}{3}$. Then (3.3) gives

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) = \left|\frac{x-y}{27}\right|.$$

Also, (3.4) gives

$$\begin{aligned} &\nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx)) \\ &= \frac{7}{162}\left|\left[x - \frac{y}{3} - y + \frac{x}{3}\right]\right| = \left|\frac{14}{243}(x-y)\right|. \end{aligned}$$

Case II: $x \geqslant y$ and $y \geqslant \frac{x}{3}$. Then (3.3) gives

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) = \left|\frac{x-y}{27}\right|.$$

Also, (3.4) gives

$$\begin{aligned}&\nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx))\\ &= \frac{7}{162}\left|\left[x - \frac{y}{3} + y - \frac{x}{3}\right]\right| = \left|\frac{7}{243}(x+y)\right|.\end{aligned}$$

Case III: $x \leqslant y$ and $x < \frac{y}{3}$. Then (3.3) gives

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) = \left|\frac{y-x}{27}\right|.$$

Also, 3.4 gives

$$\begin{aligned}&\nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx))\\ &= \frac{7}{162}\left|\left[\frac{y}{3} - x + y - \frac{x}{3}\right]\right| = \left|\frac{14}{243}(y-x)\right|.\end{aligned}$$

Case IV: $x \leqslant y$ and $x \geqslant \frac{y}{3}$. Then (3.3) gives

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) = \left|\frac{y-x}{27}\right|.$$

Also, (3.4) gives

$$\begin{aligned}&\nu\left(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]\right) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx))\\ &= \frac{7}{162}\left|\left[x - \frac{y}{3} + y - \frac{x}{3}\right]\right| = \left|\frac{7}{243}(y+x)\right|.\end{aligned}$$

Therefore in all the cases, we get

$$\nu(d(\mathcal{H}Gx, \mathcal{H}Gy)) \leqslant \nu(\frac{1}{2}[d(\mathcal{H}x, \mathcal{H}Gy) + d(\mathcal{H}y, \mathcal{H}Gx)]) - \omega(d(\mathcal{H}x, \mathcal{H}Gy), d(\mathcal{H}y, \mathcal{H}Gx)).$$

Hence $\mathcal{H}$ and G also satisfy property (P2). Since all the conditions of the Theorem 3.3 are fulfilled, G must have a unique fixed point. In this example, $G(0) = 0$ and $0 \in X_1 \cap X_2$.

4. Conclusion

In this article, we expanded the Reich-type fixed point results and weakly cyclical Chatterjea-type fixed point results in metric space using the idea of sequential convergence mapping. Also, we provided existence and uniqueness results for cyclic contractions of the Reich and Chatterjea type using the perception of sequentially convergent mappings in metric spaces. Finally, we demonstrated our findings with the help of an example.

Acknowledgment

The authors thank the reviewers for their helpful recommendations for improving the article's presentation.

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