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On F-Frobenius-Euler polynomials and their matrix approach

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Abstract

In this article, the generalized F-Frobenius-Euler polynomials $H_{n,F}^{(\alpha)}(x;\mu)$ are introduced, through their generating function, and properties are established for these generalized polynomials. In addition, we define the generalized polynomial Fibo-Frobenius-Euler matrix $\mathcal{H}_n^{(\alpha)}(x, F, \mu)$. Factorizations of the Fibo-Frobenius-Euler polynomial matrix are established with the generalized Fibo-Pascal matrix and the Fibonacci matrix. The inverse of the Fibo-Frobenius-Euler matrix is also found.

Keywords: Euler polynomials, F-Frobenius-Euler polynomials, Euler matrix, generalized Euler matrix, generalized Pascal matrix, Fibonacci matrix, Lucas matrix.

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1. Introduction

The generalized Frobenius-Euler polynomials $H_n^{(\alpha)}(x;\mu)$ of order α in the variable x are defined by the generating function (see [14])

$$\left(\frac{1-u}{e^z-\mu}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;\mu)\frac{z^n}{n!}, \qquad |z| < |\log(\mu)|, \tag{1.1}$$

where $\mu \in \mathbb{C}$ and $\alpha \in \mathbb{Z}$. Notice that $H_n^{(1)}(x;\mu) = H_n(x;\mu)$ denotes the classical Frobenius-Euler polynomials and $H_n^{(\alpha)}(0;\mu) = H_n^{(\alpha)}(\mu)$ denotes the Frobenius-Euler numbers of order α. $H_n(x;-1) = E_n(x)$ denotes the Euler polynomials (see [8, 10]).

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With the help of wxMaxima, it is possible to provide some illustrative examples for the generalized Frobenius-Euler polynomials are

$$\begin{aligned}
H_0^{(\alpha)}(x;\mu) &= 1,\\
H_1^{(\alpha)}(x;\mu) &= x-\frac{\alpha}{1-\mu},\\
H_2^{(\alpha)}(x;\mu) &= x^2-\frac{2\alpha}{1-\mu}x+\frac{\alpha(\alpha+\mu)}{(1-\mu)^2},\\
H_3^{(\alpha)}(x;\mu) &= x^3-\frac{3}{1-\mu}\alpha x^2+\frac{3(\alpha^2+\alpha\mu)}{(1-\mu)^2}x-\frac{\alpha^3+3\mu\alpha^2+(\mu^2+u)\alpha}{(1-\mu)^3}.
\end{aligned}$$

For broad information on old literature and new research trends about these classes of polynomials, we recommend to the interested reader (see [1, 4, 9, 11, 12]).

By using (1.1), we have:

$$H_n^{(\alpha+\beta)}(x+y;\mu)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x;\mu)H_{n-k}^{(\beta)}(y;\mu). \tag{1.2}$$

It is known that (see [8, p. 395 Proposition 2.1]) $H_n^{(0)}(x;\mu)=x^n$, making the substitution $\beta=0$ into (1.2) and interchanging x and y, we get

$$H_n^{(\alpha)}(x+y;\mu)=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(y;\mu)x^{n-k},$$

and, as an immediate consequence, yields the relations

$$H_n(x+y;\mu)=\sum_{k=0}^{n}\binom{n}{k}H_k(y;\mu)x^{n-k},\qquad H_n(x;\mu)=\sum_{k=0}^{n}\binom{n}{k}H_k(\mu)\,x^{n-k}.$$

Over the past few decades, there has been an interest in Pascal matrices in mathematical literature (see [2, 15, 16]). Some Pascal-Like arrays have been defined using numbers and polynomials such as the Bernoulli, Euler, q-Bernoulli, and q-Euler among others (see [3, 6, 8–10, 15–17]). Such a matrix representation provides a powerful computational tool for deriving identities and an explicit formula related to the sequence. Also, it is possible to obtain from a matrix representation of a particular counting sequence, an identity and an explicit formula for the general term of the sequence. Particularly interesting are those contexts in which such a matrix representation is related to special classes of polynomials, namely, Bernoulli polynomials, Euler polynomials, Bell polynomials, Jacobi polynomials, Laguerre polynomials their generalizations and q-analogues, and so on. In this study, we are interested in defining the F-Frobenius-Euler polynomials and numbers. Also, we introduce matrices whose entries are the F-Frobenius-Euler polynomials and numbers.

The document is organized as follows. This section, has an auxiliary character and provides some background, as well as some results that will be used throughout the article. In Section 2 we present the generalized Pascal matrix and study some interesting particular cases of this matrix, namely the generalized Fibo-Pascal matrix, the Fibonacci matrix, and the inverse matrix of the latter. In addition, the Fibonacci sequence, the F-Factorial, the Fibonomial coefficient, a theorem analogous to the binomial theorem, and the F-exponential function is defined, results of vital importance in this article. Section 3 shows the definition of the generalized F-Frobenius-Euler polynomials and some properties of these polynomials. Finally, in Section 4 we define the new generalized Fibo-Frobenius-Euler matrix and some factorizations of the polynomial matrix Fibo-Frobenius-Euler in terms of generalized Fibo-Pascal matrices and Fibonacci, respectively. Furthermore, in this section the inverse matrix of the Fibo-Frobenius-Euler matrix and its factorization with the generalized inverse Fibo-Pascal matrix are given.

2. Background and previous results

In this section, we recall some definitions and preliminary results, that will be used in this paper, all matrices are in $M_{n+1}(\mathbb{R})$, the set of all $(n+1)$-square matrices over the real field. Also, $\mathbb{N}_0 =: \{0,1,2,3,\ldots\}$, $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers, for ij any nonnegative integers we adopt the following convention

$$\binom{i}{j} = 0, \text{ whenever } j > i.$$

Let x be any nonzero real number. The generalized Pascal matrix of first kind $P[x] \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see [2, 10]):

$$p_{ij}(x) = \begin{cases} \binom{i}{j} x^{i-j}, & i \geqslant j, \\ 0, & \text{otherwise.} \end{cases}$$

The generalized Fibo-Pascal matrix $\mathcal{P}_n[x,F] = [p_{ij}(x,F)]$ $i,j = 0,1,2,...,n$ is defined by (see [6, p. 9]):

$$p_{ij}(x,F) = \begin{cases} \binom{i}{j}_F x^{i-j}, & i \geqslant j, \\ 0, & \text{otherwise.} \end{cases}$$

In [2, 15, 16] some properties of the generalized Pascal matrix of the first kind are shown, for example, its matrix factorization by special summation matrices, and its bivariate extensions.

The Fibonacci sequence $\{F\}_{n\geqslant 1}$ is defined by (see [6, p. 1]):

$$F_n = \begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_0 = 0, \quad F_1 = 1. \end{cases}$$

The Fibonacci matrix $\mathfrak{F} = [f_{ij}]$ $i,j = 0,1,2,\ldots,n$ is the matrix whose entries are given by (see [7, p. 203]):

$$f_{ij} = \begin{cases} F_{i-j+1}, & i-j+1 \geqslant 0, \\ 0, & i-j+1 < 0. \end{cases}$$

Let $\mathfrak{F}_n^{-1}$ be the inverse of $\mathfrak{F}_n$ and denote by $\tilde{f}_{ij}$ the entries of $\mathfrak{F}_n^{-1}$. In [7] the authors obtained the following explicit expression for $\mathfrak{F}_n^{-1}$:

$$\tilde{f}_{ij} = \begin{cases} 1, & i = j, \\ -1, & i = j+1, j+2, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have the following results. The F-factorial is defined as follows (see [5, p. 754]):

$$F_n! = F_n F_{n-1} F_{n-2} \cdots F_1, \quad F_0! = 1.$$

The Fibonomial coefficients are defined $n \geqslant k \geqslant 1$ as (see [5, p. 755]):

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!},$$

with $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$. The Fibonomial coefficients have the following properties (see [6, p. 2]):

$$\binom{n}{k}_F = \binom{n}{n-k}_F \quad \text{and} \quad \binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F.$$

The binomial theorem for the F-analog is given by (see [6, p. 2]):

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

The F-exponential function e_F^t is defined by (see [6, p. 2]):

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!}. \tag{2.1}$$

3. The F-Frobenius-Euler polynomials and some of their properties

In this section, we obtain an exponential generating function of the F-Frobenius-Euler polynomials. Then we give some properties of the F-Frobenius-Euler polynomials.

Definition 3.1. For real or complex α, $\mu \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}_0$, the generalized F-Frobenius-Euler polynomials $H_{n,F}^{(\alpha)}(x, \mu)$ are defined by the following generatrix function:

$$\left(\frac{1-\mu}{e_F^z - \mu}\right)^{\alpha} e_F^{xz} = \sum_{n=0}^{\infty} H_{n,F}^{(\alpha)}(x;\mu)\frac{z^n}{F_n!}, \qquad |z| < \frac{\ln|\mu|}{\ln|e_F|}. \tag{3.1}$$

Clearly $H_{n,F}^{(1)}(x, \mu) = H_{n,F}(x, \mu)$ and $H_{n,F}^{(1)}(0, \mu) = H_{n,F}(\mu)$ are the F-Frobenius-Euler polynomials and the F-Frobenius-Euler numbers, respectively.

Theorem 3.2. *Let $\{H_{n,F}^{(\alpha)}(x;\mu)\}_{n\geqslant 0}$ be the sequence the generalized F-Frobenius-Euler polynomials. They satisfy the following relation*

$$H_{n,F}^{(\alpha+\beta)}(x+y;\mu) = \sum_{k=0}^{n} \binom{n}{k}_F H_{k,F}^{(\alpha)}(x;\mu) H_{n-k,F}^{(\beta)}(y;\mu). \tag{3.2}$$

Proof. From (3.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} H_{n,F}^{(\alpha+\beta)}(x+y;\mu)\frac{z^n}{F_n!} &= \left(\frac{1-\mu}{e_F^z-\mu}\right)^{\alpha+\beta} e_F^{(x+y)z} \\
&= \sum_{n=0}^{\infty} H_{n,F}^{(\alpha)}(x;\mu)\frac{z^n}{F_n!} \sum_{n=0}^{\infty} H_{n,F}^{(\beta)}(y;\mu)\frac{z^n}{F_n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}_F H_{k,F}^{(\alpha)}(x;\mu) H_{n-k,F}^{(\beta)}(y;\mu)\frac{z^n}{F_n!}.
\end{aligned}$$

□

Comparing the coefficients of z^n, we get the result (3.2).

Corollary 3.3. *Let $\{H_{n,F}^{(\alpha)}(x;\mu)\}_{n\geqslant 0}$ be the sequence the generalized F-Frobenius-Euler polynomials. Then the following statements hold*

1. *Special values: for every $n \in \mathbb{N}_0$,*

$$H_{n,F}^{(0)}(x;\mu) = x^n. \tag{3.3}$$

2. *Summation formulas:*

$$\begin{aligned} H^{(\alpha)}_{n,F}(x+y;\mu) &= \sum_{k=0}^{n}\binom{n}{k}_F H^{(\alpha)}_{k,F}(y;\mu)x^{n-k}, & H^{(\beta)}_{n,F}(x+y;\mu) &= \sum_{k=0}^{n}\binom{n}{k}_F H^{(\beta)}_{n-k,F}(y;\mu)x^{k}, \\ H_{n,F}(x+y;\mu) &= \sum_{k=0}^{n}\binom{n}{k}_F H_{k,F}(y;\mu)x^{n-k}, & H_{n,F}(x;\mu) &= \sum_{k=0}^{n}\binom{n}{k}_F H_{n-k,F}(\mu)x^{k}. \end{aligned} \tag{3.4}$$

Proof. For (3.3), by $\alpha = 0$ in (3.1) and considering (2.1), we obtain the result. For (3.4), let $\beta = 0$ be in (3.2) then:

$$H^{(\alpha)}_{n,F}(x+y;\mu) = \sum_{k=0}^{n}\binom{n}{k}_F H^{(\alpha)}_{k,F}(x;\mu)H^{(0)}_{n-k,F}(y;\mu).$$

Exchanging x for y we obtain:

$$H^{(\alpha)}_{n,F}(x+y;\mu) = \sum_{k=0}^{n}\binom{n}{k}_F H^{(\alpha)}_{k,F}(y;\mu)H^{(0)}_{n-k,F}(x;\mu).$$

From (3.3) we have, $H^{(0)}_{n-k,F}(x;\mu) = x^{n-k}$, therefore

$$H^{(\alpha)}_{n,F}(x+y;\mu) = \sum_{k=0}^{n}\binom{n}{k}_F H^{(\alpha)}_{k,F}(y;\mu)x^{n-k}.$$

□

Theorem 3.4. *Let* $\{H_{n,F}(x;\mu)\}_{n\geqslant 0}$ *be the sequence of the F-Frobenius-Euler polynomials. They satisfy the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}_F H_{k,F}(\mu) - \mu H_{n,F}(\mu) = 0, \quad n \geqslant 1, \quad \textit{and} \quad H_{0,F}(\mu) = 1.$$

Proof. From (3.1), we obtain

$$\begin{aligned} 1-\mu = (e_F^z - \mu)\sum_{n=0}^{\infty} H_{n,F}(\mu)\frac{z^n}{F_n!} &= \sum_{n=0}^{\infty}\frac{z^n}{F_n!}\sum_{n=0}^{\infty} H_{n,F}(\mu)\frac{z^n}{F_n!} - \mu\sum_{n=0}^{\infty} H_{n,F}(\mu)\frac{z^n}{F_n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}_F H_{k,F}(\mu) - \mu H_{n,F}(\mu)\right\}\frac{z^n}{F_n!}. \end{aligned} \tag{3.5}$$

Expanding (3.5) for $n = 0$ and rearranging the series, we obtain the result. □

For any $n \in \mathbb{N}$, the first few generalized F-Frobenius-Euler polynomials $H^{(\alpha)}_{n,F}(x;\mu)$ are given as:

$$H^{(\alpha)}_{0,F}(x;\mu) = 1, \qquad H^{(\alpha)}_{1,F}(x;\mu) = x - \frac{F_\alpha}{1-\mu}, \qquad H^{(\alpha)}_{2,F}(x;\mu) = x^2 - \frac{F_\alpha}{1-\mu}x + \frac{F_\alpha^2 - F_\alpha F_{\alpha-1} + F_\alpha\mu - F_\alpha}{(1-\mu)^2}.$$

Similarly, for $\alpha = 1$, we obtain the first few F-Frobenius-Euler polynomials given by:

$$\begin{aligned} H_{0,F}(x;\mu) &= 1, \\ H_{1,F}(x;\mu) &= x - \frac{1}{1-\mu}, \\ H_{2,F}(x;\mu) &= x^2 - \frac{1}{1-\mu}x + \frac{\mu}{(1-\mu)^2}, \end{aligned}$$

$$\begin{aligned}
H_{3,F}(x;\mu) &= x^3 - \frac{2}{1-\mu}x^2 + \frac{2\mu}{(1-\mu)^2}x - \frac{\mu^2+2\mu-1}{(1-\mu)^3},\\
H_{4,F}(x;\mu) &= x^4 - \frac{3}{1-\mu}x^3 + \frac{6\mu}{(1-\mu)^2}x^2 - \frac{3\left(\mu^2+2\mu-1\right)}{(1-\mu)^3}x + \frac{\mu^3+9\mu^2+-3\mu-1}{(1-\mu)^4},\\
H_{5,F}(x;\mu) &= x^5 - \frac{5}{1-\mu}x^4 + \frac{15\mu}{(1-\mu)^2}x^3 - \frac{15\left(\mu^2+2\mu-1\right)}{(1-\mu)^3}x^2\\
&\quad + \frac{5(\mu^3+9\mu^2+-3\mu-1}{(1-\mu)^4}x - \frac{\mu^4+36\mu^3+21\mu^2-34\mu+51}{(1-\mu)^5}.
\end{aligned}$$

On the other hand, for any $n \in \mathbb{N}$, the first few F-Frobenius-Euler numbers are given as:

$$\begin{aligned}
&H_{0,F}(\mu) = 1, && H_{1,F}(\mu) = -\frac{1}{1-\mu}, && H_{2,F}(\mu) = \frac{\mu}{(1-\mu)^2},\\
&H_{3,F}(\mu) = -\frac{\mu^2+2\mu-1}{(1-\mu)^3}, && H_{4,F}(\mu) = \frac{\mu^3+9\mu^2-3\mu-1}{(1-\mu)^4}, && H_{5,F}(\mu) = -\frac{\mu^4+36\mu^3+21\mu^2-34\mu+51}{(1-\mu)^5}.
\end{aligned}$$

4. The Fibo-Frobenius-Euler polynomials matrix

In this section, we will introduce the new family of Fibo-Frobenius-Euler matrices and some algebraic properties of these, including factorizations with the generalized Fibo-Pascal matrix and the Fibonacci matrix.

Definition 4.1. Let $H^{(\alpha)}_{n,F}(x,\mu)$, the first n the generalized F-Frobenius-Euler polynomials, the new $(n+1)\times(n+1)$ generalized Fibo-Frobenius-Euler polynomial matrix, $\mathcal{H}^{(\alpha)}_n(x,F,\mu) = [h^{(\alpha)}_{ij}(x,F,\mu)]$ $i,j = 0,1,2,\dots,n$, is defined by:

$$h^{(\alpha)}_{ij}(x,F,\mu) = \begin{cases} \binom{i}{j}_F H^{(\alpha)}_{i-j,F}(x,\mu), & i \geqslant j,\\ 0, & \text{otherwise}. \end{cases}$$

For $\alpha = 1$, $\mathcal{H}^{(1)}_n(x,F,\mu) = \mathcal{H}_n(x,F,\mu)$ and $\mathcal{H}_n(0,F,\mu) = \mathcal{H}_n(F,\mu)$ are called the Fibo-Frobenius-Euler polynomial and thr matrix Fibo-Frobenius-Euler matrix, respectively. Let us consider $n = 3$. It follows from Definition 4.1 that $\mathcal{H}_3(x,F,\mu)$ is

$$\begin{bmatrix}
1 & 0 & 0 & 0\\
x-\frac{1}{1-\mu} & 1 & 0 & 0\\
x^2-\frac{1}{1-\mu}x+\frac{\mu}{(1-\mu)^2} & x-\frac{1}{1-\mu} & 1 & 0\\
x^3-\frac{2}{1-\mu}x^2+\frac{2\mu}{(1-\mu)^2}x-\frac{\mu^2+2\mu-1}{(1-\mu)^3} & 2\left[x^2-\frac{1}{1-\mu}x+\frac{\mu}{(1-\mu)^2}\right] & 2\left(x-\frac{1}{1-\mu}\right) & 1
\end{bmatrix}.$$

Theorem 4.2. *For $\mathcal{H}^{(\alpha)}_n(x,F,\mu)$, the new family of generalized Fibo-Frobenius-Euler matrices in variable x, the following formulas hold:*

$$\mathcal{H}^{(\alpha+\beta)}_n(x+y,F,\mu) = \mathcal{H}^{(\alpha)}_n(x,F,\mu)\mathcal{H}^{(\beta)}_n(y,F,\mu) = \mathcal{H}^{(\alpha)}_n(y,F,\mu)\mathcal{H}^{(\beta)}_n(x,F,\mu).$$

Proof. From Definition 4.1, taking $i > j$, we have

$$\begin{aligned}
\mathcal{H}_n^{(\alpha+\beta)}(x+y, F, \mu) &= \binom{i}{j}_F H_{i-j,F}^{(\alpha+\beta)}(x+y;\mu) \\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F H_{k,F}^{(\beta)}(y;\mu) H_{i-j-k,F}^{(\alpha)}(x;\mu) \\
&= \sum_{k=0}^{i-j} \binom{i}{j}_F \binom{i-j}{k}_F H_{i-j-k,F}^{(\alpha)}(x;\mu) H_{k-j,F}^{(\beta)}(y;\mu) \\
&= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j}_F H_{i-k,F}^{(\alpha)}(x;\mu) H_{k,F}^{(\beta)}(y;\mu) \\
&= \sum_{k=j}^{i} \binom{i}{k}_F H_{i-k,F}(x;\mu) \binom{k}{j}_F H_{k-j,F}^{(\beta)}(y;\mu) \\
&= \mathcal{H}_n^{(\alpha)}(x, F, \mu) \mathcal{H}_n^{(\beta)}(y, F, \mu),
\end{aligned}$$

which proves the first equality of the Theorem. The second equality can be obtained in a similar way. □

Corollary 4.3. *Let $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$. For α_k real or complex numbers, the matrices $\mathcal{H}_n^{(\alpha_j)}(x_j, \mu)$ comply with the following product formula, $j = 1, 2, \ldots, k$,*

$$\mathcal{H}_n^{(\alpha_1+\alpha_2+\cdots+\alpha_k)}(x_1 + x_2 + \cdots + x_k, F, \mu) = \mathcal{H}_n^{(\alpha_1)}(x_1, F, \mu)\mathcal{H}_n^{(\alpha_2)}(x_2, F, \mu) \cdots \mathcal{H}_n^{(\alpha_k)}(x_k, F, \mu), \tag{4.1}$$

particularly,

$$\mathcal{H}^{(k\alpha)}(kx, F, \mu) = \left[\mathcal{H}^{(\alpha)}(x, F, \mu)\right]^k, \tag{4.2}$$

$$\mathcal{H}_n^{(k)}(kx, F, \mu) = [\mathcal{H}_n(x, F, \mu)]^k, \tag{4.3}$$

$$\mathcal{H}_n^{(k)}(F, \mu) = [\mathcal{H}_n(F, \mu)]^k. \tag{4.4}$$

Proof. To prove (4.1) let's apply induction on k, to obtain (4.2) we take $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$ and $x_1 = x_2 = \cdots = x_k = x$ in (4.1), to prove (4.3) we take $\alpha = 1$ in (4.2) and to obtain (4.4) we take $x = 0$ in (4.3).

□

Theorem 4.4. *The Fibo-Frobenius-Euler polynomial matrix, $\mathcal{H}_n(x, F, \mu)$, satisfies the following product formula:*

$$\mathcal{H}_n(x+y, F, \mu) = \mathcal{P}_n[x, F]\mathcal{H}_n(y, F, \mu) = \mathcal{P}_n[y, F]\mathcal{H}_n(x, F, \mu).$$

Particularly,

$$\mathcal{H}_n(x, F, \mu) = \mathcal{P}_n[x, F]\mathcal{H}_n(F, \mu). \tag{4.5}$$

Proof. From Definition 4.1, we have

$$\begin{aligned}
\mathcal{H}_n(x+y, F, \mu) &= \binom{i}{j}_F H_{i-j,F}(x+y;\mu) \\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F H_{k,F}(y;\mu) x^{i-j-k}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{i}{j}_F \sum_{k=j}^{i} \binom{i-j}{k-j}_F H_{k-j,F}(y;\mu)x^{i-k} \\
&= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j} H_{k-j,F}(y;\mu)x^{i-k} \\
&= \sum_{k=j}^{i} \binom{i}{k}_F x^{i-k} \binom{k}{j}_F H_{k-j,F}(y;\mu) = \mathcal{P}_n[x,F]\mathcal{H}_n(y,F,\mu).
\end{aligned}
$$

□

Let $\mathcal{D}_n(F,\mu) \in M_{n+1}(\mathbb{R})$ be the matrix whose entries are defined by

$$
d_{ij}(F,\mu) = \begin{cases} \binom{i}{j}_F \dfrac{1}{1-\mu}, & i > j, \\ 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}
$$

Theorem 4.5. *The matrix $\mathcal{D}_n(F,\mu)$ is the inverse matrix of $\mathcal{H}_n(F,\mu)$, that is*

$$
\mathcal{H}_n^{-1}(F,\mu) = \mathcal{D}_n(F,\mu). \tag{4.6}
$$

Proof. Let

$$
\sum_{k=0}^{n} \frac{1}{1-\mu} \binom{n}{k}_F H_{n-k,F}(\mu) = \delta_{n,0},
$$

where $\delta_{n,0}$ is the Kronecker delta (see [13, p. 107]). Let us show that $\mathcal{H}_n(F,\mu)\mathcal{D}_n(F,\mu) = \mathbb{I}_n$, in effect

$$
\begin{aligned}
(\mathcal{H}_n(F,\mu)\mathcal{D}_n(F,\mu))_{ij} &= \sum_{k=j}^{i} h_{ik}(F,\mu) d_{kj}(F,\mu) \\
&= \sum_{k=j}^{i} \binom{i}{k}_F H_{i-k,F}(\mu) \binom{k}{j}_F \frac{1}{1-\mu} \\
&= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j}_F \frac{1}{1-\mu} H_{i-k,F}(\mu) \\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F \frac{1}{1-\mu} H_{i-j-k,F} = \binom{i}{j}_F \delta_{i-j,0}.
\end{aligned}
$$

This completes the demonstration. □

Definition 4.6. For $n \geqslant 2$, the inverse of the generalized Fibo-Pascal matrix $\mathbf{V}_n[x,F] = (v_{ij}(x,F)$ is defined by (see [6, p. 9-10]):

$$
v_{ij}(x,F) = \begin{cases} b_{i-j+1} \binom{i}{j}_F x^{i-j}, & i \geqslant j, \\ 0, & \text{otherwise,} \end{cases}
$$

where $b_1 = 1$ and $b_n = \sum_{k=0}^{n-1} b_k \binom{n}{k}_F$.

Corollary 4.7. *Considering the above Theorems and Definition, we have the following result:*

$$
\mathcal{H}_n^{-1}(x,F,\mu) = \mathcal{H}_n^{-1}(F,\mu)\mathbf{V}_n[x,F] = \mathcal{D}_n(F,\mu)\mathbf{V}_n[x,F].
$$

Proof. From (4.5) we have $\mathcal{H}_n^{-1}(x,F,\mu)=\mathcal{H}_n^{-1}(F,\mu)\mathcal{P}_n^{-1}[x,F]$ and applying (4.6) we have what we want to prove. □

For $0\leqslant i,j\leqslant n$ and α a real or complex number, let $\mathcal{M}_n(x,\mu,F)=[m_{ij}(x,\mu,F)]$, $i,j=0,1,2,\dots,n$ be the $(n+1)\times(n+1)$ matrix whose entries are given by

$$m_{ij}(x,\mu,F)=\binom{i}{j}_F H_{i-j,F}(x;\mu)-\binom{i-1}{j}_F H_{i-j-1,F}(x;\mu)-\binom{i-2}{j}_F H_{i-j-2,F}(x;\mu),$$

we denote $\mathcal{M}_n(0,\mu,F)=\mathcal{M}_n(\mu,F)$. Similarly, let $\mathcal{N}_n(x,\mu,F)=[n_{ij}(x,\mu,F)]_{i,j=0,1,2,\dots,n}$ be the $(n+1)\times(n+1)$ matrix whose entries are given by

$$n_{ij}(x,\mu,F)=\binom{i}{j}_F H_{i-j,F}(x,\mu)-\binom{i}{j+1}_F H_{i-j-1,F}(x;\mu)-\binom{i}{j+2}_F H_{i-j-2,F}(x;\mu),$$

we denote $\mathcal{N}_n(0,\mu,F)=\mathcal{N}_n(\mu,F)$. From the definitions of $\mathcal{M}_n(x,\mu,F)$ and $\mathcal{N}_n(x,\mu,F)$, we see that

$$\begin{aligned}
&m_{00}(x,\mu,F)=n_{00}(x,\mu,F)=1,\\
&m_{0j}(x,\mu,F)=n_{0j}(x,\mu,F)=0 \ \text{ for } \ j\geqslant 1,\\
&m_{10}(x,\mu,F)=n_{10}(x,\mu,F)=x-\frac{2-\mu}{1-\mu},\\
&m_{11}(x,\mu,F)=n_{11}(x,\mu,F)=1,\\
&m_{1j}(x,\mu,F)=n_{1j}(x,\mu,F)=0 \ \text{ for } \ j\geqslant 2,\\
&m_{i0}(x,\mu,F)=H_{i,F}(x;\mu)-H_{i-1,F}(x;\mu)-H_{i-2,F}(x;\mu) \ \text{ for } \ i\geqslant 2 \quad ,\\
&n_{i0}(x,\mu,F)=\binom{i}{0}_F H_{i,F}(x;\mu)-\binom{i}{1}_F H_{i-1,F}(x;\mu)-\binom{i}{2}_F H_{i-2,F}(x;\mu) \ \text{ for } \ i\geqslant 2.
\end{aligned}$$

Theorem 4.8. *The Fibo-Frobenius-Euler polynomial matrix, $\mathcal{H}_n(x,F,\mu)$, can be factorized in terms of the Fibonacci matrix $\mathfrak{F}_n$ as*

$$\mathcal{H}_n(x,F,\mu)=\mathcal{F}_n\mathcal{M}_n(x,\mu,F), \tag{4.7}$$

or

$$\mathcal{H}_n(x,F,\mu)=\mathcal{N}_n(x,\mu,F)\mathcal{F}_n. \tag{4.8}$$

Particularly,

$$\mathcal{F}_n\mathcal{M}_n(x,\mu,F)=\mathcal{H}_n(x,F,\mu)=\mathcal{N}_n(x,\mu,F)\mathcal{F}_n, \tag{4.9}$$
$$\mathcal{F}_n\mathcal{M}_n(\mu,F)=\mathcal{H}_n(F,\mu)=\mathcal{N}_n(\mu,F)\mathcal{F}_n. \tag{4.10}$$

Proof. Since the relation (4.7) is equivalent to $\mathfrak{F}^{-1}\mathcal{H}_n(x,F,\mu)=\mathcal{M}_n(x,\mu,F)$, it is possible to follow the proof given in [17, Theorem 4.1], making the corresponding modifications, for obtaining (4.8). The relation (4.9) and (4.10) can be obtained using a similar procedure. □

For $n=3$ we have,

$$\begin{aligned}
\mathcal{F}_3\mathcal{M}_3(x,\mu,F)&=\begin{bmatrix}1&0&0&0\\1&1&0&0\\2&1&1&0\\3&2&1&1\end{bmatrix}\\
&\times\begin{bmatrix}1&0&0&0\\ x-\frac{2-\mu}{1-\mu}&1&0&0\\ x^2-\frac{2-\mu}{1-\mu}x+\frac{2\mu-\mu^2}{(1-\mu)^2}&x-\frac{2-\mu}{1-\mu}&1&0\\ x^3-\frac{3-\mu}{1-\mu}x^2+\frac{3\mu-\mu^2}{(1-\mu)^2}x-\frac{2-5\mu+\mu^2}{(1-\mu)^3}&2x^2-\frac{3-\mu}{1-\mu}x+\frac{3\mu-\mu^2}{(1-\mu)^2}&2x-\frac{3-\mu}{1-\mu}&1\end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x - \frac{1}{1-\mu} & 1 & 0 & 0 \\ x^2 - \frac{1}{1-\mu}x + \frac{\mu}{(1-\mu)^2} & x - \frac{1}{1-\mu} & 1 & 0 \\ x^3 - \frac{2}{1-\mu}x^2 + \frac{2\mu}{(1-\mu)^2}x - \frac{\mu^2+2\mu-1}{(1-\mu)^3} & 2\left[x^2 - \frac{1}{1-\mu}x + \frac{\mu}{(1-\mu)^2}\right] & 2\left(x - \frac{1}{1-\mu}\right) & 1 \end{bmatrix}$$
$$= \mathcal{H}_3(x, F, \mu).$$

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