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Numerical finite difference approximations of a coupled parabolic system with blow-up

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Abstract

This paper is concerned with the numerical blow-up time for a coupled system of two one-dimensional semilinear parabolic equations with zero Dirichlet boundary conditions. Firstly, we derive the semi-discrete problem and prove that the blow-up solution and numerical blow-up time of the semi-discrete problem are convergent to the theoretical ones, as we refine the space-time grids. Secondly, we derive two fully discrete formulas of standard finite difference methods: the explicit Euler and implicit Euler schemes, with non-fixed time-stepping formula. In addition, we investigate the consistency, stability and convergence of the proposed schemes. Finally, we conduct two numerical experiments to show the accuracy and efficiency of the proposed schemes. Namely, for each experiment, we use the proposed schemes to calculate the numerical blow-up time, error bounds and the numerical order of convergence for blow-up times.

Keywords: Blow-up solutions, blow-up time, semilinear heat equation, Euler explicit (implicit), finite difference schemes.

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1. Introduction

It is well known that many phenomena in the world can be described using partial differential equations. Therefore, since the last decades, the analytical and numerical solutions of partial differential equations have been studied by many authors, see for instance [17, 20]. One of the remarkable phenomena in time-dependent problems is the blow-up; which has been considered by many authors (for a single equation and systems), see for instance [18, 21, 22].

In general, [19], for a time-dependent equation,defined in $\Omega \subset \mathbb{R}^n$, one can say that the classical solution $u(x,t)$ blows up in L^∞-norm or blows up (for short), if there exists $T < \infty$, called the blow-up time, such that u is well defined for all $0 < t < T$, while it becomes unbounded in L^∞-norm, when t approaches to T, that is:

$$\sup_{x\in\Omega} |u(x,t)| \to \infty \quad \text{as} \quad t \to T^-.$$

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For a system of two coupled semilinear heat equations, namely

$$u_t = u_{xx} + F(u,v), \quad v_t = v_{xx} + G(u,v),$$

for $(x,t) \in \Omega \times (0,T)$, $\Omega \subset R^n$, where $F, G : R^2 \to R$, it can be said that a solution (u,v) blows up in finite time, if there exists $T < \infty$ such that either u or v blows up at $t = T$, this means

$$\sup_{x\in\Omega} |u(x,t)| \to \infty \quad \text{or} \quad \sup_{x\in\Omega} |v(x,t)| \to \infty, \quad \text{as } t \to T^-,$$

while

$$\sup_{x\in\Omega}\{|u(x,t)| + |v(x,t)|\} \leqslant C < \infty, \quad t < T.$$

Moreover, we say that u, v blow up simultaneously, if both u, v blow up at T. One of the studied problems, is a one dimensional coupled reaction-diffusion system with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t = u_{xx} + f(v), \quad v_t = v_{xx} + g(u), \quad 0 < x < 1, \quad 0 < t < T, \\ u(x,t) = 0, \quad v(x,t) = 0, \quad x = 0,1, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < 1, \end{cases} \tag{1.1}$$

where $u_0(x) \in C^2(R)$, $v_0(x) \in C^2(R)$, satisfying $u_0(0) = u_0(1) = 0$, $v_0(0) = v_0(1) = 0$; $f, g \in C^1(R) \cap C^2(R\backslash\{0\})$, are super linear functions, positive and increasing in $(0,1)$, and $1/f, 1/g$ are integrable at infinity. Moreover, f', g', f'', g'' are positive functions in $(0,\infty)$. Hence f, g are both locally Lipschitz continuous, i.e., $\forall S > 0$, there exists $L_1, L_2 > 0$, such that

$$|f(S_1) - f(S_2)| \leqslant L_1 |S_1 - S_2|, \quad |g(S_1) - g(S_2)| \leqslant L_2 |S_1 - S_2|, \tag{1.2}$$

for $S_1, S_2 \in [0,S]$.

By standard parabolic theory, the local existence and uniqueness of non-negative classical solutions to problem (1.1) are guaranteed [13]. In addition, for many types of the functions f and g, if the initial functions (u_0, v_0) are suitably large, then $T < \infty$, [8, 9]. Moreover, only simultaneous blow-up can occur and that is because the system in (1.1) is coupled.

In fact, problem (1.1) has been used to describe physical models arising in many fields of sciences, for instance, the chemical concentration, the temperature and in the chemical reaction process [6]. The coupled reaction-diffusion systems defined in a ball (or in a one-dimensional space) with homogeneous Dirichlet boundary conditions have been studied in [7, 14, 26, 28].

In [7], under some assumptions on f and g, it is showed that the blow-up can only occur simultaneously, at a single point. Moreover, as applications to that result, two special cases of f, g were considered: the power forms and the exponential forms, namely,

$$f(v) = v^p, \quad g(u) = u^q, \quad p, q > 1, \tag{1.3}$$

$$f(v) = e^v, \quad g(u) = e^u, \quad p, q > 1. \tag{1.4}$$

Later, for problem (1.1) with (1.3), [14] showed that the upper and lower blow-up rate estimates of this problem are as follows:

$$c_1(T-t)^{-\alpha} \leqslant u(x,t) \leqslant c_2(T-t)^{-\alpha}, t \in (0,T), \quad c_3(T-t)^{-\beta} \leqslant u(x,t) \leqslant c_4(T-t)^{-\beta}, t \in (0,T),$$

where $Ci > 0, i = 1,2,3,4$ and $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$, whereas, for problem (1.1) with (1.4), it has been proven that the upper (lower) blow-up rate estimates are as follows [28]:

$$\begin{aligned} \log c - \log[q(T-t)] \leqslant qu(x,t) \leqslant \log C - \log[q(T-t)], t \in (0,T), \\ \log c - \log[p(T-t)] \leqslant pv(x,t) \leqslant \log C - \log[p(T-t)], t \in (0,T). \end{aligned} \tag{1.5}$$

Since in most cases the blow-up time cannot be estimated or evaluated theoretically, many authors have studied the numerical blow-up time, see for instance [1–5, 10–12, 15, 16, 23, 24]. According to [1], it has been shown that the blow-up solution and numerical blow-up time of the semi discrete problem of (1.1) converge to the theoretical values as we refine the grids. Moreover, two numerical schemes (explicit and linear implicit Euler) have been used to compute the blow-up solution and estimate the blow-up time for the single equation

$$\begin{aligned} u_t &= u_{xx} + u^p, \ \ 0 < x < 1, \ 0 < t < T, \\ u(x,t) &= 0, \ \ x = 0,1, \\ u(x,0) &= u_0(x), \ \ 0 < x < 1, \end{aligned} \tag{1.6}$$

where $p = 2$ and $u_0(x) = 20\sin(\pi x)$ using the time-step formula:

$$k_n = \begin{cases} \min\left(\frac{h^2}{2}, \frac{h^\alpha}{\|U_h^n\|_\infty}\right), & \text{for explicit scheme}, \\ \frac{h^\alpha}{\|U_h^n\|_\infty}, & \text{for implicit scheme}, \end{cases}$$

for $n \geqslant 0, \alpha > 0$, where h is the space-step and U_h^n is the vector of numerical solution of the discrete problem. In fact, the reason behind dealing with this type of time-steps rather than fixed time-steps is to ensure that the time-step goes to zero as time is approaching the blow up time. Hence, in this way, we avoid any possible instability, which may occur near blow-up time.

The authors [24] used the explicit and linear implicit Euler finite difference schemes to compute the numerical blow-up solution and estimate the blow-up times for problem (1.6), where $p = 3, 4, 5$, with $u_0(x) = 100\left(x - x^2\right)$. In order to increase the order of numerical convergence and get more accurate results, a special time-steps formula, dependent on p, was used with these schemes:

$$k_n = \begin{cases} \min\left(\frac{h^2}{3}, \frac{h^\alpha}{(\|U_h^n\|_\infty)^p}\right), & \text{for explicit scheme}, \\ \frac{h^\alpha}{(\|U_h^n\|_\infty)^p}, & \text{for implicit scheme}, \end{cases}$$

for $n \geqslant 0, \alpha > 0$. Moreover, the numerical simulations were carried out to support the numerical findings and to confirm the known theoretical blow-up results.

Recently, [4] studied the finite difference approximation for axisymmetric (radial) solutions of a parabolic system with blow-up:

$$\begin{cases} u_t = \Delta u + v^p, & v_t = \Delta v + u^q, \ x \in \Omega, \ 0 < t < T, \\ u(x,t) = 0, & v(x,t) = 0, \ x \in \partial\Omega, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), \ x \in \Omega, \end{cases}$$

where $p, q > 1$, $\Omega \subset R^n$. The authors used an explicit algorithm, in which uniform temporal grids are used, for the computation of the blow-up time and blow-up behaviors. In addition to the convergence of the numerical blow-up time, it has been also studied various blow-up behaviors numerically, including the blowup set, blow-up rate and blow-up in Lp -norm. Moreover, the relation between blow-up of the exact solution and that of the numerical solution was also analyzed and discussed.

In this work, we concern with the numerical approximations and blow-up time of the reaction diffusion system (1.1). For this purpose, we propose two efficient finite difference schemes.

The work is divided into five sections. Firstly, the semi-discrete approximation problem is derived. Secondly, some theories related to convergence of blow-up solutions and blow-up times of the semi-discrete problem, to theoretical ones, are proved. Thirdly, two fully discrete finite difference approximations to problem (1.1): the explicit and implicit Euler schemes, are proposed. In the fourth section, two numerical experiments are carried out to support the numerical findings. Namely, we estimate the numerical blow-up time, error bounds, CPU, and numerical order of convergence. The results are presented in the form of tables and figures. In the last section, some conclusions and possible future work are stated.

2. The semidiscrete problem

Let I be a positive integer, and $x_i = ih$, $0 \leqslant i \leqslant I$, where $h = \frac{1}{I}$, then we can adjective by the solution: $(U_h(t), V_h(t))$,

$$U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T, \quad V_h(t) = (V_0(t), V_1(t), \dots, V_I(t))^T.$$

By replacing the second-order space derivative in problem (1.1) by the standard second-order central finite-difference operator δ^2 [25], we obtain the semidiscrete problem:

$$\frac{d}{dt}U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + f(V_i), \tag{2.1}$$

$$\frac{d}{dt}V_i = \frac{V_{i+1} - 2V_i + V_{i-1}}{h^2} + g(U_i), \tag{2.2}$$

$$U_0(t) = U_I(t) = 0,\ V_0(t) = V_I(t) = 0, \tag{2.3}$$

$$U_i(0) = U_0(x_i),\ V_i(0) = V_0(x_i), \quad 0 \leqslant i \leqslant I. \tag{2.4}$$

Definition 2.1. Let (U_h, V_h) be a nonnegative solution to problem (2.1)–(2.4). We say that (U_h, V_h) archives blow-up simultaneously in finite time, if there exists $T_h < \infty$ such that:

$$\|U_h(t)\|_\infty < \infty \text{ and } \|V_h(t)\|_\infty < \infty \text{ for } t \in [0, T_h),$$
$$\|U_h(t)\|_\infty \to \infty, \|V_h(t)\|_\infty \to \infty \text{ as } t \to T_h,$$

where $\|U_h(t)\|_\infty = \max_{0\leqslant i\leqslant 1} |U_i(t)|$ and $\|V_h(t)\|_\infty = \max_{0\leqslant i\leqslant 1} |V_i(t)|$.

Theorem 2.2. *Let $f, g \in C(R, R)$ be locally Lipschtiz continuous functions. If $U, V, W_1, W_2 \in C^1([0, T], R^{I+1})$ such that:*

$$\left\{ \begin{array}{l} \frac{dU_i(t)}{dt} - \delta^2 U_i - f(V_i) \leqslant \frac{d}{dt}W_{1i} - \delta^2 W_{1i} - f(W_{2i}) \\ \frac{dV_i(t)}{dt} - \delta^2 V_i - g(U_i) \leqslant \frac{d}{dt}W_{2i} - \delta^2 W_{2i} - g(W_{2i}) \end{array} \right\},$$

I be a positive integer for $1 \leqslant i \leqslant I-1$, $t \in [0, T]$, and

$$U_0(t) \leqslant W_{10}(t), U_I(t) \leqslant W_{1I}(t), \quad V_0(t) \leqslant W_{20}(t), V_I(t) \leqslant W_{2I}(t), \quad U_i(0) \leqslant W_{1i}(0), V_i(0) \leqslant W_{2i}(0),$$

then

$$U(t) \leqslant W_1(t), \quad V(t) \leqslant W_2(t), \quad t \in [0, T].$$

Corollary 2.3. *Let (U, V) and (W_1, W_2), where $U, V, W_1, W_2 \in C^1([0, T], R^{I+1})$, be a lower and upper solution of the discrete problem, respectively. Then*

$$U(t) \leqslant W_1(t), \quad V(t) \leqslant W_2(t), \quad t \in [0, T].$$

Theorem 2.4. *Let (U_h, V_h) be a nonnegative solution of the semidiscrete problem, where the reaction functions f, g are locally Lipchitz continuous such that $f, g > 0$ in $(0, \infty)$. If (U_h, V_h) achieves blow-up at time: $T_h < \infty$, then T_h is bounded from below.*

Proof. Suppose that (W_1, W_2) is the solution of the system:

$$\frac{dW_1}{dt} = f(W_2), \quad \frac{dW_2}{dt} = g(W_1). \tag{2.5}$$

Clearly (W_1, W_2) is an upper solution of the semi-discrete problem (1.4), and by Theorem 2.2 we obtain

$$U_i \leqslant W_{1i}, \quad V_i \leqslant W_{2i}, \quad \forall t \in [0, T].$$

But, it is known that the solution of system (2.5) blows up in a finite time T [4]. Thus $T \leqslant T_h$. □

Corollary 2.5. *Let* (U_h, V_h) *be a nonnegative solution of the semidiscrete problem* (2.1)-(2.4)*, where* $f = v^p, g = u^q, p, q > 1$*. If* (U_h, V_h) *achieves blow-up at* $T_h < \infty$*, then*

$$\int_{\|V_0\|}^{\infty} \frac{ds}{G_2(s)} = \int_{\|U_0\|}^{\infty} \frac{ds}{G_1(s)} = T \leqslant T_h,$$

where

$$G_1(s) = \left[\frac{p+1}{q+1}S^{q+1} - (p+1)C_0\right]^{\frac{p}{p+1}}, G_2(s) = \left[\frac{q+1}{p+1}S^{p+1} - (q+1)C_0\right]^{\frac{q}{q+1}}, C_0 = \frac{1}{q+1}u_0^{q+1} - \frac{1}{p+1}v_0^{p+1}.$$

Proof. The proof can be obtained easily, based on Theorem 2.4 and using some results from [4]. □

Proposition 2.6. *Let* (U_h, V_h) *be a non-negative solution the discrete problem. Define*

$$Q_1(t) = \sum_{i=1}^{I-1} hU_i(t)\Upsilon_i, Q_2(t) = \sum_{i=1}^{I-1} hV_i(t)\Upsilon_i,$$

where

$$\Upsilon_i = \frac{\sin(\pi ih)}{\sum_{m=1}^{I-1} h\sin(\pi mh)}, \quad 0 \leqslant i \leqslant I, \quad -\delta^2\Upsilon_i = \lambda_n\Upsilon_i, \quad \Upsilon_0 = \Upsilon_I = 0, \quad \lambda_n = \left(\frac{4}{h^2}\right)\sin^2\left(\frac{\pi n}{2}\right).$$

If f, g *are convex functions* ($f'', g'' \geqslant 0$)*, then*

$$\frac{d}{dt}Q_1 \geqslant f(Q_2) - \lambda n Q_1, \quad \frac{d}{dt}Q_2 \geqslant g(Q_1) - \lambda n Q_2.$$

Proof. Multiplying the discrete equation by $h\Upsilon_i$, $(i = 1, \dots, I-1)$ and taking the summation, yields

$$\sum_{i=1}^{I-1} h\left(\frac{dU_i}{dt} - \delta^2 U_i\right)\Upsilon_i = \sum_{i=1}^{I-1} hf(V_i)\Upsilon_i, \qquad \sum_{i=1}^{I-1} h\left(\frac{dV_i}{dt} - \delta^2 V_i\right)\Upsilon_i = \sum_{i=1}^{I-1} hg(U_i)\Upsilon_i.$$

Thus

$$\frac{dQ_1}{dt} + \lambda_n Q_1 = \sum_{i=1}^{I-1} hf(V_i)\Upsilon_i, \qquad \frac{dQ_2}{dt} + \lambda_n Q_2 = \sum_{i=1}^{I-1} hg(U_i)\Upsilon_i.$$

Since f, g are convex and $\sum_{i=1}^{I-1} h\Upsilon_i = 1$ by Jensen's inequality, we obtain

$$\frac{dQ_1}{dt} + \lambda n Q_1 \geqslant f(Q_2), \quad \frac{dQ_2}{dt} + \lambda n Q_2 \geqslant g(Q_1),$$

Thus

$$\frac{dQ_1}{dt} \geqslant f(Q_2) - \lambda\eta Q_1, \quad \frac{dQ_2}{dt} \geqslant g(Q_1) - \lambda\eta Q_2.$$

□

Theorem 2.7. *Assume the following:*

a) $f, g \in C^1([0,\infty])$ *are convex* ($f'', g'' \geqslant 0$)*,* $f(z) > 0$ *in* $(0,\infty)$*, and* $\forall \varepsilon > 0$*, we have*

$$\int_{\varepsilon}^{\infty} \frac{dZ}{f(Z)} < \infty, \quad \int_{\varepsilon}^{\infty} \frac{dZ}{g(Z)} < \infty;$$

b) $\exists Z_0 \geqslant 0$ *such that*

  (a) $\left(\frac{f(z)}{Z}\right) \geqslant \lambda_n$, $\left(\frac{g(z)}{Z}\right) \geqslant \lambda_n$ *for* $Z \in [Z_0, \infty)$;

(b) $\lim_{Z\to\infty}\frac{Z}{f(Z)}=0,\ \lim_{\ Z\to\infty}\frac{Z}{g(Z)}=0;$

c) *the initial conditions are non-negative and such that*

$$U_h(0)\neq 0,\ V_h(0)\neq 0,\ Q_1(0)\geqslant Z_0,\ Q_2(0)\geqslant Z_0,\ f\left(Q_2(0)\right)-\lambda n Q_1(0)>0,\ g\left(Q_1(0)\right)-\lambda n Q_2(0)\geqslant 0,$$

where

$$\begin{aligned} Q_1(t) &= \sum_{i=1}^{I-1} hU_i(t)\Upsilon_i, \quad Q_2(t)=\sum_{i=1}^{I-1} hV_i(t),\Upsilon_i,\\ \Upsilon_i &= \frac{\sin(\pi i h)}{\sum_{m=1}^{I-1}\sin(\pi m h)}, \quad -\delta^2\Upsilon_i=\lambda_n\Upsilon_i, \quad i=1,2,\ldots,I-1, \quad \Upsilon_0=\Upsilon_I=0,\\ \lambda_n &= \left(\frac{4}{h^2}\right)\sin^2\left(\frac{\pi h}{2}\right). \end{aligned}$$

Then the non-negative solution (U_h,V_h) *of the discrete problem achieves blow-up at time* T_h *with*

$$T_h \leqslant \int_{Q_1(0)}^{\infty}\frac{dZ_1}{f\left(Z_2\right)-\lambda_n Z_1}, \quad \forall Z_2\in[Z_0,\infty), \tag{2.6}$$

or

$$T_h \leqslant \int_{Q_2(0)}^{\infty}\frac{dZ_2}{g\left(Z_1\right)-\lambda_n Z_2}, \quad \forall Z_1\in[Z_0,\infty). \tag{2.7}$$

Proof. Let (R_1,R_2) be the solution to Cauchy problem:

$$\tfrac{dR_1}{dt}=f\left(R_2\right)-\lambda_n R_1, \qquad \tfrac{dR_2}{dt}=g\left(R_1\right)-\lambda_n R_2, \quad R_1(0)=Q_1(0), \quad R_2(0)=Q_2(0).$$

By Proposition (2.6), we have

$$\frac{dQ_1}{dt}\geqslant f\left(Q_2\right)-\lambda_n Q_1, \quad \frac{dQ_2}{dt}\geqslant g\left(Q_1\right)-\lambda_n Q_2.$$

By the maximum principle, we obtain

$$0\leqslant R_1(t)\leqslant Q_1(t), \quad 0\leqslant R_2(t)\leqslant Q_2(t).$$

By the assumptions b and c, we have

$$\begin{aligned} &\lim_{Z_1\to Z_2}\left(\frac{f\left(Z_2\right)}{Z_1}\right)\geqslant\lambda_n, && \lim_{Z_2\to Z_1}\left(\frac{g\left(Z_1\right)}{Z_2}\right)\geqslant\lambda_n \text{ for } Z_1,Z_2\in[Z_0,\infty),\\ &\lim_{Z_1,Z_2\to\infty}\frac{Z_1}{f\left(Z_2\right)}=0, && \lim_{Z_1,Z_2\to\infty}\frac{Z_2}{g\left(Z_1\right)}=0. \end{aligned}$$

Therefore, $R_1(t),R_2(t)$ are non-decreasing, and they blow up simultaneously, in finite time, and

$$t=\int_{Q_1(0)}^{R_1(t)}\frac{dZ_1}{f\left(R_2\right)-\lambda_n Z_1}=\int_{Q_2(0)}^{R_2(t)}\frac{dZ_2}{g\left(R_1\right)-\lambda_n Z_2}. \tag{2.8}$$

Now

$$\frac{1/f\left(Z_2\right)}{1/f\left(Z_2\right)-\lambda_n Z_1}=\frac{1}{f\left(Z_2\right)}\left[\frac{f\left(Z_2\right)-\lambda n Z_1}{1}\right]=1-\lambda_n\frac{Z_1}{f\left(Z_2\right)}.$$

Thus

$$\lim_{Z_1,Z_2\to\infty} \frac{1/f(Z_2)}{1/f(Z_2)-\lambda_n Z_1} = 1-\lambda_n \lim_{Z_1,Z_2\to\infty} \frac{Z_1}{f(Z_2)} = 1.$$

Similarly, we can show that

$$\lim_{Z_1,Z_2\to\infty} \frac{1/g(Z_1)}{1/g(Z_1)-\lambda_n Z_2} = 1.$$

It follows that

$$\lim_{Z_1,Z_2\to\infty} \frac{1}{f(Z_2)-\lambda_n Z_1} = \lim_{Z_1,Z_2\to\infty} \frac{1}{f(Z_2)} < \infty, \qquad \lim_{Z_1,Z_2\to\infty} \frac{1}{g(Z_1)-\lambda_n Z_2} = \lim_{Z_1,Z_2\to\infty} \frac{1}{g(Z_1)} < \infty.$$

So, we obtain

$$T = \int_{Q_1(0)}^{\infty} \frac{dZ_1}{f(Z_2)-\lambda_n Z_1} = \int_{Q_2(0)}^{\infty} \frac{dZ_2}{g(Z_1)-\lambda_n Z_2} < \infty.$$

Hence, (U_h, V_h) achieves blow-up at time $T_h \leqslant T$ and (2.6) and (2.7) are held. □

Lemma 2.8. *Define the problem $-w_{xx} = \lambda_w,\ 0 < x < 1,\ w(0) = w(1) = 0$ with the corresponding eigenvalue $\lambda = \pi^2$ and is normalized so that*

$$\int_0^1 w(x)dx = 1.$$

Then

$$\max_{0\leqslant i\leqslant I} |\Upsilon_i - w(x_i)| = 0\left(h^2\right), \quad h \to 0,$$

where Υ_i is defined as in Proposition 2.6.

The next theorem shows that the solution of the semi-discrete system (2.1)-(2.4) converges to the exact solution of problem (1.1) as space-step approaches zero.

Theorem 2.9. *Assume the following.*

(a) *The reaction functions $f, g \in C^1([0,\infty])$ and problem of system (1.1) has a solution $(u,v), u,v \in C^{4,1}([0,1] \times [0,T])$.*

(b) *The initial condition $\left(U_h^0, V_h^0\right)$ satisfies*

$$\left\|U_h^0 - u_h(0)\right\|_\infty = O(1), h \to 0, \quad \left\|V_h^0 - v_h(0)\right\|_\infty = O(1), h \to 0.$$

Then, for h sufficiently small, the semi-discrete problem (1.2) has a unique solution: (U_h, V_h), $U_h, V_h \in C^1([0,T])$, R^{J+1} such that

$$\max_{t\in[o,T]} \|U_h(t) - u_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \ h \to 0,$$

$$\max_{t\in[o,T]} \|V_h(t) - v_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \ h \to 0,$$

where $\|e(0)\|_\infty = \max\left\{\left\|U_h^0 - u_h(0)\right\|_\infty, \left\|V_h^0 - v_h(0)\right\|_\infty\right\}$.

Proof. Let $t(h)$ be the greatest value such that $t(h) \leqslant T_h$ and for $0 \leqslant t \leqslant t(h)$,

$$\|U_h(t) - u_h(t)\|_\infty < 1, \quad \|V_h(t) - v(t)\|_\infty < 1.$$

Note that because of (b), $t(h) > 0$ for h sufficiently small. Next, we set $t^*(h) = \min[t(h), T]$ and define

$$e_{uh} = U_h - u_h, \quad e_{vh} = V_h - v_h.$$

If $t \in [0, t^*(h)]$, then by problem (1.1), we have

$$\frac{d}{dt}e_{ui} - \delta^2 e_{ui}(t) \leqslant |f(V_i(t)) - f(V(x_i,t))| + \frac{h^2}{12}K_1,$$
$$\frac{d}{dt}e_{vi} - \delta^2 e_{vi}(t) \leqslant |g(U_i(t)) - g(U(x_i,t))| + \frac{h^2}{12}K_2.$$

Let $K = \max_{1\leqslant i\leqslant I-1}\left\{\frac{K_1}{12}, \frac{K_2}{12}\right\}$, where $\|U_{xxxx}\| \leqslant K_1$, $\|V_{xxxx}\| \leqslant K_2$. Using the mean value theorem,

$$\frac{d}{dt}e_{ui} - \delta^2 e_{ui}(t) \leqslant M|e_{vi}| + Kh^2, \qquad \frac{d}{dt}e_{vi} - \delta^2 e_{vi}(t) \leqslant M|e_{ui}| + Kh^2,$$

where

$$M = \text{Max}[M_1, M_2], |f'(u)| \leqslant M_1, \quad |f'(v)| \leqslant M_2.$$

Define the functions

$$w_1 = \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right], \qquad w_2 = \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right],$$

where

$$\|e(0)\|_\infty = \max\{\|e_u(0)\|_\infty, \|e_v(0)\|_\infty\}.$$

Clearly,

$$\frac{dw_{1i}}{dt} - \delta^2 w_{1i} \geqslant M|w_{2i}| + Kh^2, \qquad \frac{dw_{2i}}{dt} - \delta^2 w_{2i} \geqslant M|w_{1i}| + Kh^2,$$

for $1 \leqslant i \leqslant I-1$. So by Theorem 2.4, we obtain

$$e_{ui}(t) \leqslant w_{1i}(t), e_{vi}(t) \leqslant w_{2i}(t).$$

Thus

$$e_{ui}(t) \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right], \qquad e_{vi}(t) \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right],$$

for $t \in [0, t^*(h)]$. By using the same argument for $-e_u, -e_v$, we obtain

$$\|U_h(t) - u_h(t)\|_\infty \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right], \ \|V_h(t) - v_h(t)\|_\infty \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right].$$

We need to show that $t^*(h) = T$, assume that it is not true, then for some small h, where $t^*(h) = t(h) < T$, we have

$$1 = \|U_h(t) - u_h(t)\|_\infty \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right] \to 0,$$
$$1 = \|V_h(t) - v_h(t)\|_\infty \leqslant \exp[(M+1)t]\left[\|e(0)\|_\infty + Kh^2\right] \to 0,$$

which is impossible, so, $t^*(h) = T$. Thus

$$\max_{t\in[0,T]} \|U_h(t) - u_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \ h \to 0,$$
$$\max_{t\in[0,T]} \|V_h(t) - v_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \ h \to 0,$$

where $\|e(0)\|_\infty = \max[\|e_u(0)\|_\infty, \|e_v(0)\|_\infty]$. □

The next theorem shows that the blow-up time of the semi-discrete system (2.1)-(2.4) converges to the blow-up time of problem (1.1) as space-step approaches zero.

Theorem 2.10. *Assume the following.*

(a) *The functions* $f, g \in C^1([0,\infty), R)$ *and* $f(Z), g(Z) > 0$ *is in* $(0,\infty)$.

(b) *There exists* $\bar{Z} \geqslant 0$, *such that*

1. $\left(\frac{f(Z)}{Z}\right) \geqslant \pi^2$, $\left(\frac{g(Z)}{Z}\right) \geqslant \pi^2$, *for* $Z \in [\bar{Z}, \infty)$;
2. $\lim_{Z\to\infty} \frac{Z}{f(Z)} = 0$, $\lim_{Z\to\infty} \frac{Z}{g(Z)} = 0$;
3. $f(\bar{Z}) - \pi^2 \bar{Z} > 0$, $g(\bar{Z}) - \pi^2\bar{Z} > 0$.

(c) *There exists* $T < \infty$ *such that* $u, v \in C^{4,2}([0,1] \times [0,T))$ *and*

$$\lim_{t\to T} \int_0^1 u(x,t)w(x)dx = \infty, \qquad \lim_{t\to T} \int_0^1 v(x,t)w(x)dx = \infty,$$

where (u, v) *is the solution of the original problem and* W *is defined as in Lemma 2.8.*

If $\|U_h(0) - u_h(0)\|_\infty = 0(1), h \to 0$, *then the solution of the discrete problem achieves blow-up, for* h *sufficiently small at* T_h *and*

$$\lim_{h\to 0} T_h = T.$$

Proof. For each h, we denote by $[0, T_h)$, the maximal interval, where U_h is defined. If $T_h \underset{h\to 0}{\longrightarrow} T$, then U_h, V_h blow up for h sufficiently small. To prove that, we need to show

$$\liminf_{h\to 0} T_h \geqslant T, \tag{2.9}$$

$$\limsup_{h\to 0} T_h \leqslant T. \tag{2.10}$$

First, if we suppose that (2.9) does not hold, then

$$T' = \liminf_{h\to 0} T_h < T.$$

Set $M = \frac{(T-T')}{2}$, there exists a sequence $\{h_s\}_{s=0}^{\infty}$ with $h_s \underset{s\to\infty}{\longrightarrow} 0$ such that $T_h < T' + M < T$. By Theorem 2.9, for h small enough, U_h, V_h are defined and bounded in $[0, T' + M]$. Thus, we have a contradiction, because $[0, T_h)$ is the maximal interval, and

$$T_h < T' + M.$$

So, (2.9) holds.

Next, we suppose that (2.10) is not held. Set $T'' = \lim_{h\to 0} (\sup T_h) > T$. If $T'' = \infty$, we choose T'' as a fixed constant such that $T'' > T$. Set $M = \frac{(T''-T)}{2}$, there exists a sequence $\{h_s\}_{s=0}^{\infty}$ with $h_s \underset{h\to 0}{\longrightarrow} \infty$ such that

$$T < T + M < T_h. \tag{2.11}$$

The next aim is to show that this is impossible. By the assumptions, we have for $\varepsilon > 0$,

$$\int_\varepsilon^\infty \frac{dZ}{f(Z)} < \infty, \quad \int \frac{dZ}{g(Z)} < \infty.$$

By assumption (b), we have

$$\int_y^\infty \frac{dZ_1}{f(Z_2) - \pi^2 Z_1} < \infty, \quad \int_y^\infty \frac{dZ_2}{g(Z_1) - \pi^2 Z_2} < \infty, \quad y \in (\bar{Z}, \infty).$$

Hence, there exists $R \in (\bar{Z}, \infty)$ such that

$$\int_y^\infty \frac{dZ_1}{f(Z_2) - \pi^2 Z_1} < \frac{M}{2}, \quad \int_y^\infty \frac{dZ_2}{f(Z_1) - \pi^2 Z_2} < \frac{M}{2}, \quad y \in [R, \infty). \tag{2.12}$$

On the other hand, by assumption (c), there exists T_0 with $0 < T_0 < T$ such that, $\forall t \in [T_0, T)$,

$$\int_0^1 u(x,t)w(x)dx \geqslant 2R.$$

Next, we define $T_1 = \frac{T+T_0}{2}$, by Theorem 2.9, the semi discrete problem has a solution (U_h, V_h) defined in $[0, T_1]$, for h sufficiently small. In addition, for $t \in [0, T_1]$,

$$\begin{aligned}
&\left|\sum_{i=1}^{I-1} hU_i(t)\Upsilon_i - \int_0^1 u(x,t)w(x)dx\right| \\
&= \left|\sum_{i=1}^{I-1} hU_i(t)\Upsilon_i \pm \sum_{i=1}^{I-1} hU_i w(x_i) \pm \sum_{i=1}^{I-1} hu(x_i,t)w(x_i) - \int_0^1 u(x,t)w(x)dx\right| \\
&\leqslant \left|\sum_{i=1}^{I-1} hU_i(t)(\Upsilon_i - w(x_i))\right| \\
&\quad \Upsilon_i + \left|\sum_{I=1}^{I-1} h(U_i(t) - u(x_i,t))w(x_i)\right| + \left|\sum_{I=1}^{I-1} hu(x_i,t)w(x_i) - \int_0^1 u(x,t)w(x)dx\right|.
\end{aligned}$$

By Theorem 2.9 and equation (2.8), the first and second terms are bounded and approach zero, while, by composite-trapezoidal formula, the third term is equal zero, when $h \to 0$, for $t \in [0, T_1]$. Thus

$$\lim_{h\to 0}\left[\sum_{i=1}^{I-1} hU_i(t)\Upsilon_i - \int_0^1 u(x,t)w(x)dx\right] = 0.$$

So, by above and (2.12), it follows that

$$Q_1(t) > \int_0^1 u(x,t)w(x)dx > \int_0^1 u(x,t)w(x)dx - R > R. \tag{2.13}$$

Similarly, we can show that

$$Q_2(t) > \int_0^1 v(x,t)w(x)dx > \int_0^1 v(x,t)w(x)dx - R > R.$$

By Theorem 2.7, with taking $U_h(T_0)$ as initial condition, and using $\lambda_n \leqslant \pi^2$ and (2.12) and (2.13), it follows that

$$T_h \leqslant T_0 + \int_{Q_1(T_0)}^\infty \frac{dZ_1}{f(Z_2) - \lambda_n Z_1}, \quad \leqslant T_0 + \int_{Q_1(T_0)}^\infty \frac{dZ_1}{f(Z_2) - \pi^2 Z_1} \leqslant T_0 + \frac{M}{2} < T + \frac{M}{2}.$$

Similarly, one can show that

$$T_h \leqslant T_0 + \int_{Q_2(T_0)}^\infty \frac{dZ_2}{g(Z_1) - \pi^2 Z_2} \leqslant T_0 + \frac{M}{2} < T + \frac{M}{2}.$$

Thus

$$T_h < T + \frac{M}{2} < T + M,$$

which is a contradiction with (2.11), with h being small. Therefore, both (2.9) and (2.10) are held. Hence $T_h \underset{h\to 0}{\longrightarrow} T$. □

3. Finite difference schemes

In this section, we derive the explicit (implicit) fully-discrete finite difference formulas for the problem (1.1), by approximating the time derivative in problem (1.2), using the forward (backward) finite difference formula ([25]). We consider that U_i^n and V_i^n, are the approximate values of $u(x_i, t_n)$ and $v(x_i, t_n)$, respectively, where $t_{n+1} = t_n + k_n$, I be a positive integer, and consider the grid: $x_i = ih$, $0 \leqslant i \leqslant I$, $h = \frac{1}{I}$, $t_{n+1} = t_n + k_n$, $x_{i+1} = x_i + h$, and h is the space-step, and k_n is the time-step.

3.1. *Explicit Euler scheme*

We approximate u_t, v_t by forward finite difference formulas at the mesh-point (x_i, t_n) as follows:

$$u_t|_i^n = \frac{U_i^{n+1} - U_i^n}{k_n} + O(k_n), \quad v_t|_i^n = \frac{V_i^{n+1} - V_i^n}{k_n} + O(k_n),$$

where u_{xx}, v_{xx} are approximated as using 2^d order central finite difference formula as follows:

$$u_{xx} \mid_i^n = \frac{U_i^{n+1} - 2U_i^n + U_{i-1}^n}{h^2} + O\left(h^2\right), \qquad v_{xx} \mid_i^n = \frac{V_i^{n+1} - 2V_i^n + V_{i-1}^n}{h^2} + O\left(h^2\right).$$

By substituting all of these formulas in the system (1.1), we get:

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{k_n} &= \frac{U_i^n - 2U_i^n + U_{i-1}^n}{h^2} + f(V_i^n), \\ \frac{V_i^{n+1} - V_i^n}{k_n} &= \frac{V_i^n - 2V_i^n + V_{i-1}^n}{h^2} + g(U_i^n). \end{aligned} \tag{3.1}$$

The last difference equations can be written as follows:

$$U_i^{n+1} = (1 - 2r_h^n) U_i^n + r_h^n \left(U_{i+1}^n + U_{i-1}^n\right) + k_n f(V_i^n), \tag{3.2}$$

$$V_i^{n+1} = (1 - 2r_h^n) V_i^n + r_h^n \left(V_{i+1}^n + V_{i-1}^n\right) + k_n g(U_i^n), \tag{3.3}$$

$$I = 1, 2, 3, \dots, I-1, \quad n = 0, 1, 2, \dots,$$

$$U_h^n = \left(U_1^n, U_2^n, \dots\dots., U_{1-I}^n\right), \quad V_h^n = \left(V_1^n, V_2^n, \dots\dots. V_{1-I}^n\right), \quad r_n = \frac{k_n}{h^2}.$$

In addition, in order to guarantee that the stability (convergence) condition of explicit scheme is satisfied, the following non-fixed time-step formula is considered:

$$k_n = \min\left(\frac{h^2}{3}, \frac{h^\alpha}{\|U_h^n\|}, \frac{h^\alpha}{\|V_h^n\|}\right), \quad \alpha \geqslant 1. \tag{3.4}$$

3.1.1. *Local truncation error of explicit Euler scheme*

Theorem 3.1. *Let* $\left(T_{ui}^n, T_{vi}^n\right)$ *be the local truncation error of the explicit Euler formulas* (3.1) *and* (3.2) *at the mesh point* (x_i, t_n)*. Then, there exist* $C_1, C_2, C_3, C_4 > 0$*, such that*

$$|T_{ui}^n| \leqslant C_1 k + C_2 h^2, \quad |T_{vi}^n| \leqslant C_3 k + C_4 h^2,$$

where $k = \max_{n \in N} k_n$*, i.e.,* $T_{ui}^n = O\left(k + h^2\right), T_{vi}^n = O\left(k + h^2\right)$.

Proof. Substituting the exact solution $u_i^n = u(x_i, t_n), v_i^n = v(x_i, t_n)$ in the explicit Euler formula (3.2), yields that

$$T_{ui}^n = \left(u_i^{n+1} - u_i^n\right) - \frac{k}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - kf(v_i^n) = k\left[\frac{\partial u_i^n}{\partial t} + O(k)\right] - k\left[\frac{\partial^2 u_i^n}{\partial x^2} + O\left(h^2\right)\right] - kf(v_i^n).$$

It follows that

$$= k\left[\frac{\partial u_i^n}{\partial t} - \frac{\partial^2 u_i^n}{\partial x^2} - f\left(v_i^n\right)\right] + k\left[O(k) + O\left(h^2\right)\right].$$

From problem (1.1), with assuming all the partial derivatives are bounded at the meshpoint (x_i, t_n), we obtain $\left|T_{ui}^n\right| \leqslant C_1 k + C_2 h^2$, i.e., $T_{ui}^n = O\left(k + h^2\right), C > 0$. Similarly, we can show that $T_{vi}^n = O\left(k + h^2\right), C > 0$. □

3.1.2. The stability analysis of Explicit Euler Method

In order to investigative the stability for the Explicit Euler formulas, we need to recall the following definition.

Definition 3.2 (The stability condition, [30]). Set $E_u^n = \left(e_{ui}^n, i = 1, 2, \ldots, I-1\right), E_v^n = \left(e_{vi}^n, i = 1, 2, \ldots, I-1\right)$, $e_{ui}^n = u_i^n - U_i^n, e_{vi}^n = v_i^n - V_i^n$, where $\left(u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right)\right)$ and $\left(U_i^n, V_i^n\right)$ are the exact and numerical solutions of a one-dimensional time-dependent coupled system of two PDEs, respectively. For any arbitrary initial rounding error: $\left(E_u^0, E_v^0\right)$, we say that the numerical solution (the difference approximation) is stable, if there exists a positive number μ independent on the space-step (h) and time-step (k), such that

$$\left\|E_u^n\right\| \leqslant \mu \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, \quad E_v^n \leqslant \mu \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\},$$

where $\left\|E_u^n\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{ui}^n\right|, \ \left\|E_v^n\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{vi}^n\right|, \ n = 0, 1, 2, \ldots$.

Theorem 3.3. *The explicit Euler scheme* (3.2), (3.3) *is stable, if* $(1 - 2r) \geqslant 0$, *where* $k = \max_{n \in N} k_n$, $r = k/h^2$.

Proof. To prove this theorem, we use the maximum error stability-technique [29]. Let $e_{ui}^n = u_i^n - U_i^n$, $e_{vi}^n = v_i^n - V_i^n$, $\left(u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right)\right)$ is the exact solution of problem (1.1). To end this, we apply the Mathematical induction. For $n = 0$, set $\left\|E_u^1\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{ui}^1\right| = \left|e_{uj}^1\right|, \left\|E_v^1\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{vi}^1\right| = \left|e_{um}^1\right|$. By substituting e_{uj}^n in the explicit formula (3.2), we obtain

$$e_{uj}^1 = (1 - 2r)e_{uj}^0 + r\left(e_{uj+1}^0 + e_{uje-1}^0\right) + k\left(f\left(v_i^0\right) - f\left(V_j^0\right)\right).$$

Since $(1 - 2r) \geqslant 0$, we have

$$\left|e_{uj}^1\right| \leqslant (1 - 2r)\left|e_{uj}^0\right\| + r\left(\left|e_{uj+1}^0\right| + \left|e_{uj-1}^0\right|\right) + k\left|\, f\left(v_i^0\right) - f\left(V_j^0\right)\,\right|.$$

From (1.2), it follows that

$$\begin{aligned} \left|e_{uj}^1\right| &\leqslant (1 - 2r)\left\|E_u^0\right\| + 2r\left\|E_u^0\right\| + kL_1\left|v_i^0 - V_j^0\right|, \\ \left\|E_u^1\right\| &\leqslant \left\|E_u^0\right\| + kL_1\left\|E_v^v\right\| \leqslant (1 + kL_1)\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}. \end{aligned}$$

Similarly, we can show that

$$\left\|E_v^1\right\| \leqslant \left\|E_v^0\right\| + kL_2\left\|E_u^v\right\| \leqslant (1 + kL_2)\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^v\right\|\right\}.$$

Now, we suppose that

$$\begin{aligned} \left\|E_u^s\right\| &\leqslant (1 + kL)^s\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, s = 1, 2, 3, \ldots n, \\ \left\|E_v^s\right\| &\leqslant (1 + kL)^s\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, s = 1, 2, 3, \ldots n, \end{aligned}$$

where $L = \mathrm{Max}\{L_1, L_2\}$. For $n + 1$, set $\left\|E_u^{n+1}\right\| = \max_{1 \leqslant ivi}\left|e_{ui}^{n+1}\right| = \left|\ e_{uj}^{s+1}\right|. \left\|E_v^{n+1}\right\| = \max_{1xixy}\left|e_{vi}^{n+1}\right| = \left|\ e_{um}^{n+1}\right|$. By substituting e_{uj}^n in the explicit formula (3.2), we obtain

$$e_{uj}^{n+1} = (1 - 2r)e_{uj}^n + r\left(\ e_{uj+1}^n + e_{uj-1}^n\right) + k\left(f\left(v_j^n\right) - f\left(V_j^n\right)\right).$$

Since $(1-2r) \geqslant 0$, we have

$$\left| e_{uj}^{n+1} \right| \leqslant (1-2r) \left| e_{uj}^{n} \right| + r \left(\left| e_{uj+1}^{n} \right| + \left| e_{uj-1}^{n} \right| \right) + k \left| f\left(v_j^n\right) - f\left(V_i^n\right) \right|.$$

Thus, from (1.2), it follows that

$$\left| e_{uj}^{n+1} \right| \leqslant (1-2r) \left\| E_u^n \right\| + 2r \left\| E_u^n \right\| + kL_1 \left| v_i^n - V_j^n \right|,$$
$$\left\| E_u^{n+1} \right\| \leqslant \left\| E_u^n \right\| + kL \left\| E_v^n \right\| \leqslant (1+kL)^n \text{Max}\{\|E_u^u\|, \|E_v^\circ\|\} + kL(1+kL)^n \text{Max}\left\{ \left\| E_v^0 \right\|, \left\| E_v^0 \right\| \right\}.$$

Thus $\left\| E_u^{n+1} \right\| \leqslant (1+kL)^{n+1} \text{Max}\left\{ \left\| E_u^0 \right\|, \left\| E_v^0 \right\| \right\}$,

$$\leqslant \exp((n+1)kl) \text{Max}\{\|E_u^\circ\|, \|E_v^\circ\|\} = \exp\left(t_{n+1}L\right) \text{Max}\left\{ \left\| E_u^0 \right\|, \left\| E_v^0 \right\| \right\}.$$

Similarly, we can show that

$$\left\| E_v^{n+1} \right\| \leqslant \exp\left(t_{n+1}L\right) \text{Max}\left\{ \left\| E_u^0 \right\|, \left\| E_v^0 \right\| \right\}.$$

So, this implies the stability for explicit scheme, if $r \leqslant 1/2$. □

3.1.3. Convergence analysis of explicit Euler scheme

Theorem 3.4. *The explicit Euler formulas* (3.2) *and* (3.3) *are convergent with:* $O\left(k+h^2\right)$, *if* $(1-2r) \geqslant 0$, *where* $k = \max_{n \in N} k_n, r = k/h^2$.

Proof. Set $e_{ui}^n = u_i^n - U_i^n, e_{vi}^n = v_i^n - V_i^n$ $\left(u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right)\right)$ is the exact solution of problem (1.1). Assume that $e_{ui}^0 = 0, e_{vi}^0 = 0, \forall i = 0, 1, \ldots, I$. We aim to show that there exists $C > 0$, such that

$$e_{ui}^{n+1} \leqslant C\left(k+h^2\right), \quad e_{vi}^{n+1} \leqslant C\left(k+h^2\right), \quad n = 0, 1, \ldots.$$

To end this, we use the mathematical induction technique. For $n = 1$, we set $\left| e_{uj}^1 \right| = \max_{1 \leqslant i \leqslant I-1} \left| e_{ui}^1 \right|$. Substituting e_{uj}^1 in the explicit formula (3.2) yields that

$$e_{uj}^1 = (1-2r) e_{uj}^0 + r\left(e_{uj+1}^0 + e_{uj-1}^0\right) + k\left(f\left(v_i^0\right) - f\left(V_i^0\right)\right) + T_i^0 = k\left(f\left(v_i^0\right) - f\left(V_i^0\right)\right) + T_i^0.$$

From (1.2), we obtain $\left| e_{uj}^1 \right| \leqslant Lk \left| e_{vj}^0 \right| + \left| T_{ui}^0 \right| = \left| T_{ui}^0 \right| \leqslant C\left(k+h^2\right)$, where $L = \text{Max}\{L_1, L_2\}$. Hence

$$\left| e_{ui}^1 \right| \leqslant C\left(k+h^2\right), i = 1, 2, \ldots, I-1.$$

Similarly, we can show that

$$\left| e_{vi}^1 \right| \leqslant C\left(k+h^2\right), \quad i = 1, 2, \ldots, I-1.$$

Suppose that $\left| e_{ui}^s \right| \leqslant C_1\left(k+h^2\right), \left| e_{vi}^s \right| \leqslant C_1\left(k+h^2\right), s = 0, 1, 2, \ldots, n, C_1 > 0$. For $n+1$, we set $\left| e_{uj}^{n+1} \right| = \max_{1 \leqslant i \leqslant I-1} \left| e_{ui}^{n+1} \right|$. Substituting e_{uj}^{n+1} in the explicit formula (3.2) yields that

$$e_{uj}^{n+1} = (1-2r) e_{uj}^n + r\left(e_{uj+1}^n + e_{uj-1}^n\right) + k\left(f\left(v_i^n\right) - f\left(V_i^n\right)\right) + T_{ui}^n.$$

Thus

$$\left| e_{uj}^{n+1} \right| \leqslant (1-2r) \left\| E_u^n \right\| + 2r \left\| E_u^n \right\| + k \left| f\left(v_j^n\right) - f\left(V_j^n\right) \right| + \left| T_{uj}^n \right|.$$

From (1.2), we have

$$\left\| E_u^{n+1} \right\| \leqslant \left\| E_u^n \right\| + kL \left| e_{vj}^n \right| + \left| T_j^n \right| \leqslant \left\| E_u^n \right\| + kL \left\| E_v^n \right\| + \left| T_j^n \right|$$
$$\leqslant (1+kL) C_1\left(k+h^2\right) + C\left(k+h^2\right) = \left[(1+kL) C_1 + C\right]\left(k+h^2\right).$$

It follows that $\left\| E_u^{n+1} \right\| \leqslant C\left(k+h^2\right), n = 0, 1, \ldots$. Similarly, we can show that

$$\left\| E_v^{n+1} \right\| \leqslant C\left(k+h^2\right), \quad n = 0, 1, \ldots.$$

□

3.2. Implicit Euler scheme

We approximate u_t, v_t by backward finite difference formulas at the mesh-point (x_i, t_{n+1}) as follows ([25]):

$$\begin{cases} \frac{\partial u}{\partial t}\big|_i^{n+1} = \frac{1}{k_n}\left(U_i^{n+1} - U_i^n\right) + O\left(k_n\right), \\ \frac{\partial v}{\partial t}\big|_i^{n+1} = \frac{1}{k_n}\left(V_i^{n+1} - V_i^n\right) + O\left(k_n\right), \end{cases}$$

while u_{xx}, v_{xx} are approximated as using 2^d order central finite difference formulas follows:

$$u_{xx} \mid_i^{n+1} = \frac{U_i^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + O\left(h^2\right), \quad v_{xx} \mid_i^{n+1} = \frac{V_i^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} + O\left(h^2\right).$$

By substituting all of these formulas in the system (1.1), we get:

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{k_n} &= \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + k_n f\left(V_i^n\right),\ 1 \leqslant i \leqslant I-1, \\ \frac{V_i^{n+1} - V_i^n}{k_n} &= \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} + k_n g\left(U_i^n\right),\ 1 \leqslant i \leqslant I-1. \end{aligned}$$

Thus

$$\left(1 + 2r_h^n\right) U_i^{n+1} - r_h^n \left(U_{i+1}^{n+1} + U_{i-1}^{n+1}\right) = U_i^n + k_n f\left(V_i^n\right), \tag{3.5}$$

$$\left(1 + 2r_h^n\right) V_i^{n+1} - r_h^n \left(V_{i+1}^{n+1} + V_{i-1}^{n+1}\right) = V_i^n + k_n g\left(U_i^n\right), \tag{3.6}$$

$1 \leqslant i \leqslant I-1,\ n = 0, 1, 2, \ldots$. In addition, since the implicit Euler method is unconditionally stable, we choose the non-fixed time-stepping formula as follows:

$$k_n = \min\left(\frac{h^\alpha}{\|u_h^n\|}, \frac{h^\alpha}{\|v_h^n\|}\right), \quad \alpha \in R. \tag{3.7}$$

We can write the last equations in matrix form as follows:

$$\left(I - r_h^n H\right) U_h^{n+1} = U_h^n + k_n F\left(V_h^n\right), \tag{3.8}$$

$$\left(I - r_h^n H\right) V_h^{n+1} = V_h^n + k_n G\left(U_h^n\right), \tag{3.9}$$

where $H = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}_{(I-1)\times(I-1)}$, $F\left(V_h^n\right) = \left(f\left(V_1^n\right), f\left(V_2^n\right), \ldots, f\left(V_{I-1}^n\right)\right)^T; G\left(U_h^n\right) = \left(g\left(U_1^n\right), g\left(U_2^n\right), \ldots, g\left(U_{I-1}^n\right)\right)^T$.

3.2.1. Local truncation error of implicit Euler scheme

Theorem 3.5. *Let $\left(T_{ui}^n, T_{vi}^n\right)$ be the local truncation error of the implicit Euler formulas (3.5) and (3.6) at the mesh point (x_i, t_{n+1}). Then, there exist $C_1, C_2, C_3, C_4 > 0$, such that $\left|T_{ui}^{n+1}\right| \leqslant C_1 k + C_2 h^2, \left|T_{vi}^{n+1}\right| \leqslant C_3 k + C_4 h^2$, where $k = \max_{n \in N} k_n$, i.e., $T_{ui}^{n+1} = O\left(k + h^2\right), T_{vi}^{n+1} = O\left(k + h^2\right)$.*

Proof. Substituting the exact solution $u_i^n = u(x_i, 0), v_l^n = v(x_i, 0)$ in the implicit Enler formula (3.5), yields that

$$\begin{aligned} T_{ui}^{n+1} &= \left(u_i^{n+1} - u_i^n\right) - \frac{k}{h^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i=1}^{n+1}\right] - kf\left(v_i^n\right) \\ &= k\left[\frac{\partial u_i^{n+1}}{\partial t} + O(k)\right] - k\left[\frac{\partial^2 u_i^{n+1}}{\partial x^2} + O\left(h^2\right)\right] - k\left[f\left(v_i^{n+1}\right) + O(k)\right]. \end{aligned}$$

It follows that

$$T_{ui}^{n+1} = k\left[\frac{\partial u_i^{n+1}}{\partial t} - \frac{\partial^2 u_i^{n*1}}{\partial x^2} - f\left(v_i^{n+1}\right)\right] - k\left[O(k) + O\left(h^2\right)\right].$$

From equation (1.1), with assuming all the partial derivatives are bounded at the meshpoint (x_i, t_{n+1}), we obtain $\left|T_{ui}^{n+1}\right| \leqslant C_1 k + C_2 h^2$, i.e., $\left|T_i^{n+1}\right| = O\left(k + h^2\right), C > 0$. Similarly, we can show that there exist $C_3, C_4 > 0$ such that

$$\left|T_{vi}^{n+1}\right| \leqslant C_3 k + C_4 h^2.$$

□

3.2.2. The stability analysis of implicit Euler method

Theorem 3.6. *The implicit Euler scheme (3.5)-(3.6) is unconditionally stable.*

Proof. To prove this theorem, we use the maximum error stability-technique [29]. Let $e_{ui}^n = u_i^n - U_i^n$, $e_{vi}^n = v_i^n - V_i^n$, $\left(u_i^n = u\left(x_i, t_n\right), v_i^n = v\left(x_i, t_n\right)\right)$ be the exact solution of problem (1.1). To end this, we apply the Mathematical induction. For $n = 0$, set $\left\|E_u^1\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{ui}^1\right| = \left|e_{uj}^1\right|, k = \max_{n \in N} k_n, r = k/h^2$, we have

$$\begin{aligned}\left|e_{uj}^1\right| = (1+2r)\left|e_{uj}^1\right| - r\left(\left|e_{uj}^1\right| + \left|e_{uj}^1\right|\right) &\leqslant (1+2r)\left|e_{uj}^1\right| - r\left(\left|e_{uj+1}^1\right| + \left|e_{uj-1}^1\right|\right) \\ &\leqslant \left|(1+2r)e_{uj}^1 - r\left(e_{uj+1}^1 + e_{uj-1}^1\right)\right| = \left|e_{uj}^0 + k\left(f\left(v_j^0\right) - f\left(V_j^0\right)\right)\right|.\end{aligned}$$

Thus, from (1.2), we obtain $\left|e_{uj}^1\right| \leqslant \left|e_j^0\right| + k\left|f\left(v_j^0\right) - f\left(V_j^0\right)\right| \leqslant \left|e_j^0\right| + kL_1\left|v_j^0 - V_j^0\right|$. It follows that

$$\left\|E_u^1\right\| \leqslant \left\|E_u^0\right\| + kL_1\left\|E_v^0\right\| \leqslant (1 + kL_1)\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}.$$

Similarly, we can show that

$$\left\|E_v^1\right\| \leqslant \left\|E_v^0\right\| + kL_2\left\|E_u^0\right\| \leqslant (1 + kL_2)\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}.$$

Now, we suppose that

$$\begin{aligned}&\left\|E_u^s\right\| \leqslant (1+kL)^s\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_u^0\right\|\right\}, \ s = 1, 2, 3, \ldots, n, \\ &\left\|E_v^s\right\| \leqslant (1+kL)^s\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, \ s = 1, 2, 3, \ldots, n,\end{aligned}$$

where $L = \mathrm{Max}\{L_1, L_2\}$. For $n+1$, set $\left\|E_{ij}^{n+1}\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{ui}^{n+1}\right| = \left|e_{uj}^{n+1}\right|$, $\left\|E_v^{h+1}\right\| = \max_{1 \leqslant i \leqslant I}\left|e_{vi}^{n*1}\right|$, $k = \max_{n \in N} k_k, r = k/h^2$, we have

$$\begin{aligned}\left|e_{uj}^{n+1}\right| &= (1+2r)\left|e_{uj}^{n+1}\right| - r\left(\left|e_{uj}^{n+1}\right| + \left|e_{uj}^{n+1}\right|\right) \\ &\leqslant (1+2r)\left|e_{uj}^{n+1}\right| - r\left(\left|e_{uj+1}^{n+1}\right| + \left|e_{uj-1}^{n+1}\right|\right) \\ &\leqslant \left|(1+2r)e_{uj}^{n+1} - r\left(e_{uj+1}^{n+1} + e_{uj=1}^{n+1}\right)\right| = \left|e_{uj}^n + k\left(f\left(v_j^n\right) - f\left(V_j^n\right)\right)\right|.\end{aligned}$$

Thus $\left|e_{uf}^{n+1}\right| \leqslant \left|e_{uj}^n\right| + k\left|f\left(v_j^n\right) - f\left(V_j^n\right)\right| \leqslant \left|e_{uj}^n\right| + kL\left|v_j^n - V_j^n\right|$. It follows that

$$\begin{aligned}\left\|E_u^{n+1}\right\| \leqslant \left\|E_u^n\right\| + kL\left\|E_v^n\right\| &\leqslant (1+kL)^n\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\} + kL(1+kL)^n\mathrm{Max}\left\{\left\|E_v^0\right\|, \left\|E_v^0\right\|\right\} \\ &\leqslant (1+kL)^{n+1}\,\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\} \\ &\leqslant \exp((n+1)kL)\mathrm{Max}\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\} \\ &= \exp\left(t_{n+1}L\right)\mathrm{Max}\left\{\left\|E_i^0\right\| \cdot \left\|E_v^0\right\|\right\}.\end{aligned}$$

Therefore, the implicit Euler scheme (3.5)-(3.6) is unconditionally stable. □

3.2.3. Convergence analysis of implicit Euler scheme

Theorem 3.7. *The implicit Euler formula (3.5)-(3.6) is convergent, if* $r > 0$ *with* $O\left(k+h^2\right)$, *where* $k = \max_{n\in N} k_n$.

Proof. Set $e^n_{ui} = u^n_i - U^n_i, e^n_{vi} = v^n_i - V^h_i$ $\left(u^n_i = u\left(x_i, t_k\right), v^n_i = v\left(x_i, t_n\right)\right)$ is the exact solution of problem (1.1). Assume that $e^0_{ui} = 0, e^0_{vi} = 0, \forall i = 0, 1, \ldots, I$. We aim to show that there exists $C > 0$, such that

$$e^{n+1}_{ui} \leqslant C\left(k+h^2\right), \quad e^{n+1}_{vi} \leqslant C\left(k+h^2\right), \quad n = 0, 1, \ldots.$$

To end this, we use the mathematical induction technique. For $n = 1$, we set $\left|e^1_{uj}\right| = \max_{1\leqslant i\leqslant I}\left|e^1_{ui}\right|$. Thus

$$\begin{aligned}\left|e^1_{uj}\right| = (1+2r)\left|e^1_{uj}\right| - r\left(\left|e^1_{uj}\right| + \left|e^1_{uj}\right|\right) &\leqslant (1+2r)\left|e^1_{uj}\right| - r\left(\left|e^1_{uj+1}\right| + \left|e^1_{uj-1}\right|\right)\\ &\leqslant \left|(1+2r)e^1_{uj} - r\left(e^1_{uj+1} + e^1_{uj-1}\right)\right|\\ &= \left|e^0_{uj} + k\left(f\left(v^0_i\right) - f\left(V^0_i\right)\right) + T^0_i\right| = \left|k\left(f\left(v^0_i\right) - f\left(V^0_i\right)\right) + T^0_j\right|.\end{aligned}$$

From (1.2), we have

$$\left|e^1_{uj}\right| \leqslant kL\left|v^0_j - V^0_j\right| + \left|T^0_j\right|,$$

where $L = \mathrm{Max}\{L_1, L_2\}$. We obtain

$$\left|e^1_{uj}\right| \leqslant Lk\left|e^0_{vj}\right| + \left|T^0_i\right| = \left|T^0_i\right| \leqslant C\left(k+h^2\right), \qquad \left|e^1_{ui}\right| \leqslant C\left(k+h^2\right), i = 1, 2, \ldots, I-1.$$

Similarly, we can show that

$$\left|e^1_{vi}\right| \leqslant C\left(k+h^2\right), i = 1, 2, \ldots, I-1.$$

Suppose that $\left|e^S_{ui}\right| \leqslant C_1\left(k+h^2\right)$, $\left|e^S_{vi}\right| \leqslant C_1\left(k+h^2\right)$, $s = 0, 1, 2, \ldots, n,\ C_1 > 0$. For $n+1$, we set $\left|e^{n+1}_{uj}\right| = \max_{1\leqslant i\leqslant I-1}\left|e^{n+1}_{ui}\right|$,

$$\begin{aligned}\left|e^{n+1}_{uj}\right| &= (1+2r)\left|e^{n+1}_{uj}\right| - r\left(\left|e^{n+1}_{uj}\right| + \left|e^{n+1}_{uj}\right|\right)\\ &\leqslant (1+2r)\left|e^{n+1}_{uj}\right| - r\left(\left|e^{n+!}_{uj+1}\right| + \left|e^{n+!}_{uj-1}\right|\right)\\ &\leqslant \left|(1+2r)e^{n+1}_{uj} - r\left(e^{n+1}_{uj+1} + e^{n+1}_{uj-1}\right)\right| = \left|e^n_{uj} + k\left(f\left(v^n_j\right) - f\left(V^n_j\right)\right) + T^{n+1}_j\right|.\end{aligned}$$

Thus

$$\left|e^{n+1}_{uj}\right| \leqslant \left|e^n_{uj}\right| + k\left|f\left(v^n_j\right) - f\left(V^n_j\right)\right| + \left|T^{n+1}_j\right|.$$

By (1.2), we obtain

$$\left|e^{n+1}_{uj}\right| \leqslant \left|e^n_{uj}\right| + kL\left|v^n_j - V^n_j\right| + \left|T^{n+1}_j\right|.$$

It follows that

$$\begin{aligned}\left\|E^{n+1}_u\right\| &\leqslant \left\|E^n_u\right\| + kL\left\|E^n_v\right\| + \left|T^{n+1}_j\right|\\ &\leqslant C_1\left(k+h^2\right) + kLC_1\left(k+h^2\right) + C\left(k+h^2\right)\\ &\leqslant (1+kL)C_1\left(k+h^2\right) + C\left(k+h^2\right) = \left[(1+kL)C_1 + C\right]\left(k+h^2\right).\end{aligned}$$

It follows that $\left\|E^{n+1}_u\right\| \leqslant C\left(k+h^2\right), n = 0, 1, \ldots, i = 1, 2, \ldots, I-1$. Similarly, we can show that

$$\left\|E^{n+1}_v\right\| \leqslant C\left(k+h^2\right), \quad n = 0, 1, \ldots.$$

□

Definition 3.8. The solution of the explicit, implicit Euler schemes blows up simultaneously in a finite time T_h, if the following conditions hold:

1. $\left\|U^n_h\right\|_\infty \to \infty, \left\|V^n_h\right\|_\infty \to \infty$, as $n \to \infty$;
2. $T_h = \sum^\infty_n k_n$.

Remark 3.9.

1. Since the matrix $A = \left(I - r^n_h H\right)$ is diagonally dominant with positive real diagonal entries, then it is positive definite and nonsingular [27], hence the linear systems (3.8) and (3.9) are uniquely solvable.

2. The blow-up time of the discrete solution is considered the numerical blow-up time of problem (1.1).
3. The numerical blow-up time depends on space step h and also on the choice of time steps k_n.
4. It is well known that, for each fixed time interval $[0, t]$, explicit (implicit) Euler numerical schemes give approximate solutions with rate of convergence, $O\left(k+h^2\right)$ where $k = \max_n k_n$, while with the time-steps formulas (3.4) and (3.7), the rate of convergence becomes $o\left(h^{\alpha}\right)$, as $h \to 0$, for $\alpha \leqslant 2$. The same order of convergence might be expected to the numerical blow-up time.

4. Numerical experiments

In this section, we estimate the numerical blow-up times for two numerical experiments, with different space steps, using the proposed finite difference schemes (explicit Euler implicit Euler). All the numerical computations codes are written in Matlab (R2020a) software. Moreover, we measure the order of convergence to the numerical blow-up time, for each of these methods. The numerical blow-up time is taken, once you find $m \in \mathbb{N}$ such that the condition $\left\|U_h^m\right\|_\infty \geqslant 10^{15}$ holds. In addition, the value $T_h = t_m = \sum_{n=0}^{m} k_n$ is considered the numerical blow-up time to the studied problems. $E_h = |T_{2h} - T_h|$ is the error bonds between T_{2h} and T_h. For each example, we include some tables to present the numerical results obtained from using the proposed schemes (explicit Euler and implicit Euler), with the mesh size: $I = \{20, 40, 80, 160, 320\}$, and $\alpha = 1, 2$. Each table shows the number of iterations, when numerical blow-up occurs, the numerical blow-up times, and the central processing unit times (CPUTs) in seconds, the errors-bounds of numerical blow-up times. Finally, in order to examine experimentally, the rate of numerical convergence for the numerical blow-up times, we should take different mesh sizes with some values of α, and we use the formula $S_h = \frac{\log(E_{2h}/E_h)}{\log 2}$. Additionally, some numerical simulations are carried out to support the numerical results.

4.1. Examples

Example 4.1. Consider the following system:

$$\begin{cases} u_t = u_{xx} + v^4, \quad v_t = v_{xx} + u^5, x \in (0,1), t \in (0,T), \\ u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, t \in (0,T), \\ u(x,0) = 70\left(x - x^2\right), v(x,0) = 80\left(x - x^2\right), \quad x \in (0,1). \end{cases}$$

Table 1: Example 4.1, explicit Euler scheme, $\alpha = 1$.

h	m	T_h	cput	E_h	S_h
1/20	5	7.8225E − 04	0.050367	···	···
1/40	5	2.2906E − 04	0.063373	5.5319E − 04	···
1/80	6	7.7405E − 05	0.100925	1.5165E − 04	1.8670
1/160	6	3.7762E − 05	0.177312	3.9643E − 05	1.9356
1/320	9	1.9175E − 05	0.327089	1.8587E − 05	1.0928

Table 2: Example 4.1, explicit Euler scheme, $\alpha = 2$.

h	m	T_h	cput	E_h	S_h
1/20	6	5.2726E − 05	0.054967	···	···
1/40	8	1.9621E − 05	0.083085	3.3105E − 05	···
1/80	18	1.1851E − 05	0.093601	7.7700E − 06	2.0911
1/160	71	1.0042E − 05	0.176785	1.8090E − 06	2.1027
1/320	366	9.6511E − 06	0.285960	3.9090E − 07	2.2103

Table 3: Example 4.1, implicit Euler scheme, $\alpha = 1$.

h	m	T_h	cput	E_h	S_h
1/20	5	7.8252E − 04	0.322480	$\cdots$	$\cdots$
1/40	5	2.2915E − 04	0.421447	5.5337E − 04	$\cdots$
1/80	6	7.7432E − 05	0.463314	1.5172E − 04	1.8668
1/160	6	3.7766E − 05	1.469750	3.9666E − 05	1.9354
1/320	9	1.9178E − 05	3.851017	1.8588E − 05	1.0935

Table 4: Example 4.1, implicit Euler scheme, $\alpha = 2$.

h	m	T_h	cput	E_h	S_h
1/20	5	5.2742E − 05	0.139162	$\cdots$	$\cdots$
1/40	8	1.9623E − 05	0.202235	3.3119E − 05	$\cdots$
1/80	18	1.1852E − 05	0.453932	7.7710E − 06	2.0915
1/160	71	1.0042E − 05	1.192606	1.8100E − 06	2.1021
1/320	366	9.6511E − 06	3.832013	3.9090E − 07	2.2111

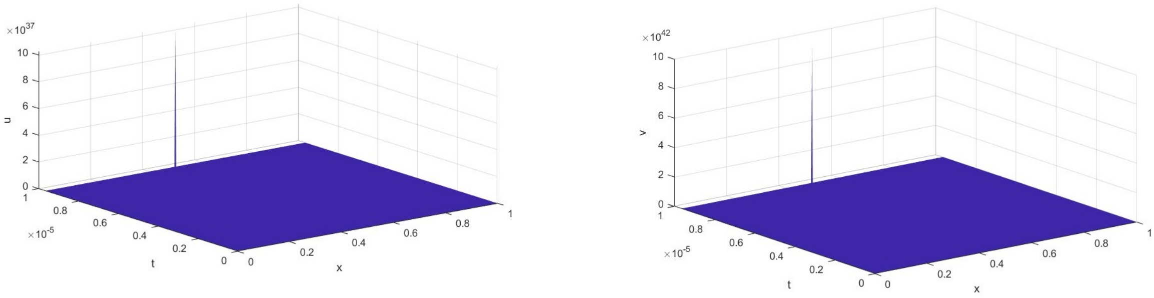

Figure 1: Evolution of numerical blow-up solution over time arising from using Explicit for Example 4.1, with $h = 320, \alpha = 2$.

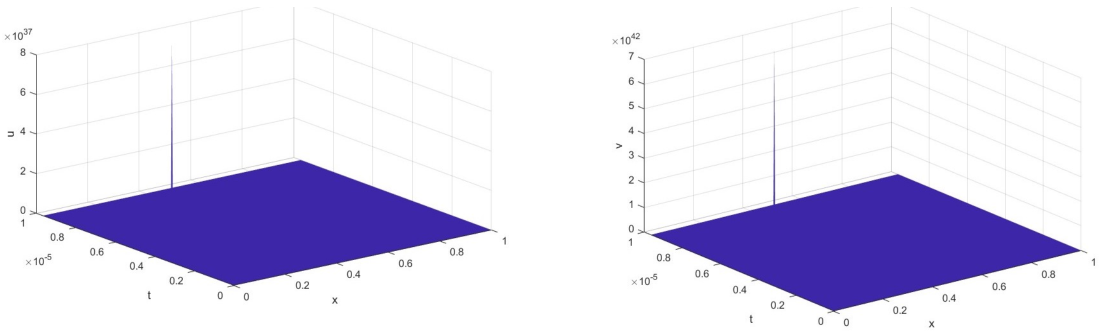

Figure 2: Evolution of numerical blow-up solution over time arising from using Implicit for Example 4.1, with $h = 320, \alpha = 2$.

Example 4.2. Consider the following system:

$$\begin{cases} u_t = u_{xx} + v^3, \quad v_t = v_{xx} + u^4, \quad x \in (0,1), t \in (0,T), \\ u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, t \in (0,T), \\ u(x,0) = 70\sin(\pi x), v(x,0) = 80\sin(\pi x), \quad x \in (0,1). \end{cases}$$

Table 5: Example 4.2, explicit Euler scheme $\alpha = 1$.

h	m	T_h	cput	E_h	S_h
1/20	3	1.9771E − 04	0.141475	···	···
1/40	4	6.9945E − 05	0.207396	1.2777E − 04	···
1/80	4	2.4773E − 05	0.230424	4.5172E − 05	1.5000
1/160	4	8.8020E − 06	0.646888	1.5971E − 05	1.5000
1/320	4	5.1350E − 06	0.281267	3.6670E − 06	2.1228

Table 6: Example 4.2, explicit Euler scheme $\alpha = 2$.

h	m	T_h	cput	E_h	S_h
1/20	4	9.9496E − 06	0.265604	···	···
1/40	4	1.8125E − 06	0.115497	8.1371E − 06	···
1/80	5	3.7184E − 07	0.167696	1.4407E − 06	2.4977
1/160	7	1.2851E − 07	0.301928	2.4333E − 07	2.5658
1/320	17	7.7340E − 08	0.608993	5.1170E − 08	2.2495

Table 7: Example 4.2, implicit Euler scheme, $\alpha = 1$.

h	m	T_h	cput	E_h	S_h
1/20	3	1.9771E − 04	0.344763	···	···
1/40	4	6.9945E − 05	0.583877	1.2777E − 04	···
1/80	4	2.4773E − 05	0.848388	4.5172E − 05	1.5000
1/160	4	8.8020E − 06	0.931657	1.5971E − 05	1.5000
1/320	4	5.1350E − 06	4.605578	3.6670E − 06	2.1228

Table 8: Example 4.2, implicit Euler scheme, $\alpha = 2$.

h	m	T_h	cput	E_h	S_h
1/20	4	9.9496E − 06	0.336013	···	···
1/40	4	1.8126E − 06	0.446961	8.1370E − 06	···
1/80	5	3.7184E − 07	0.633924	1.4408E − 06	2.4976
1/160	7	1.2851E − 07	1.949431	2.4333E − 07	2.5659
1/320	11	9.3199E − 08	2.774117	3.5311E − 08	2.7847

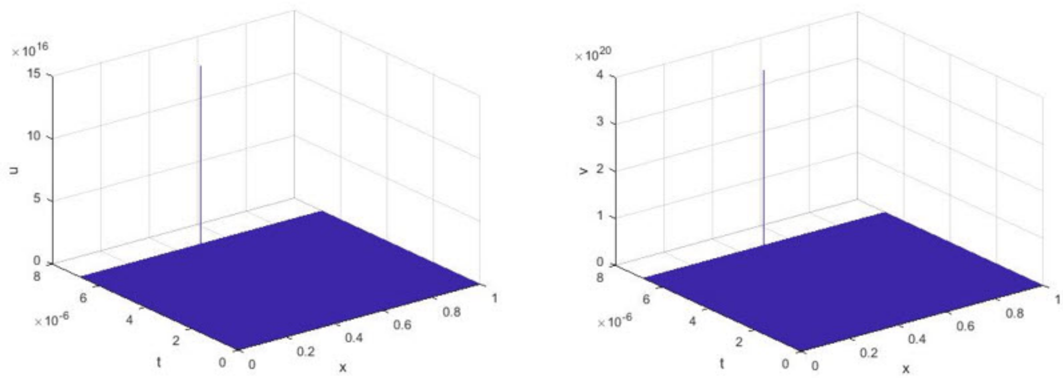

Figure 3: Evolution of numerical blow-up solution over time arising from using Explicit for Example 4.2, with $h = 320$, $\alpha = 2$.

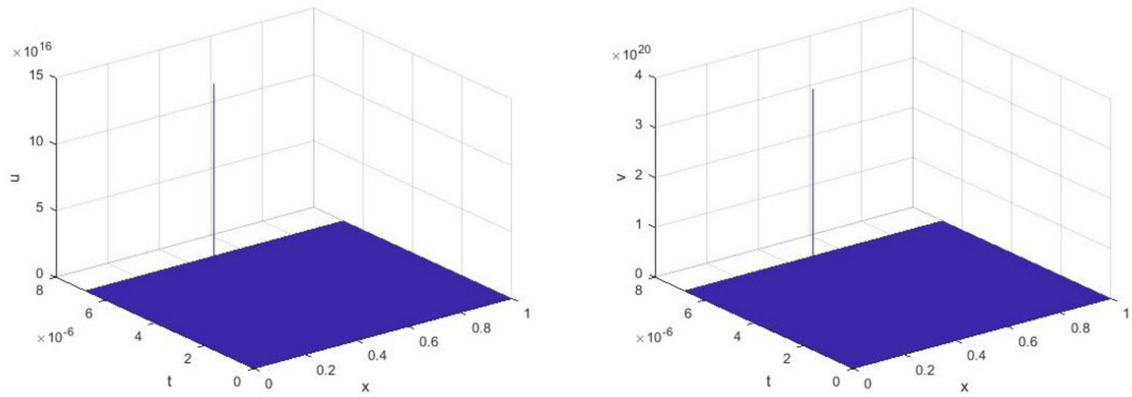

Figure 4: Evolution of numerical blow-up solution over time arising from using Implicit, for Example 4.2, with $h = 320, \alpha = 2$.

4.2. Discussing the results

From the numerical results in Examples 4.1 and 4.2, we can point out the following notes.

1. The numerical blow-up can simultaneously occur at only a single point ($x = 0.5$), and that confirms the known theoretical blow-up results of problem (1.1), see [7].
2. The blow-up time errors-bounds decrease when the space-steps are refined. This indicates that the numerical blow-up times sequence T_h is convergent, as the space-step approaches to zero.
3. The order of convergence of the numerical blow-up times S_h is close to or larger than the value of α, which means, the numerical order of convergence is $O\left(h^{\alpha+\epsilon}\right)$, where $\epsilon > 0$.
4. Due to dealing with the time-stepping formulas (2.1)-(2.4) and (1.6), for large $\boldsymbol{\alpha}$, the required number of iterations to achieve blow-up, increases, comparing with taking a small value to α.
5. We see that the CPU times are increasing, as we refine the spatial step, or if we compare CPUT of implicit method with that of explicit method.
6. Figures 1 and 2 show that, in each of studied problems, the numerical blow-up growth-rates, obtained from using explicit Euler method, is almost the same as that obtained from using implicit Euler method.

5. Conclusions

In this work, two finite-difference algorithms with non-fixed time stepping formulas, used for finding the numerical blow-up solution and estimating the numerical blow-up time of a system of two coupled semeilinear heat equations associated with zero Dirichlet boundary conditions. Firstly, the semi-discrete problem was derived and its convergence regarding blow-up solutions and blow-up time to theoretical ones were investigated. Secondly, two fully discrete finite difference methods are proposed and demonstrated as the obtained results in work. In addition, two numerical experiments have been reported to verify the theoretical results and to determine the rate of convergence of the numerical blow-up times. The numerical results were presented and illustrated in the form of tables and figures. We see that the considered methods were in high order of convergence and with good efficiency. It might be interesting to extend the obtained results to the case of Neumann boundary conditions in the future works.

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