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Kernel function with BFGS quasi-newton methods for solving nonlinear semi-definite problems

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Abstract

In this paper, we will improve the practical performance of an interior points algorithm for convex nonlinear semi-definite optimization, where to minimize a nonlinear convex objective function subject to nonlinear convex constraints. We will propose a new method for solving this kind of problem by using a straightforward kernel function and the iterative Newton directions combined with the Broyden-Fletcher-Goldfarb-Shanno (BFGS in short) quasi-Newton method. Further, a best polynomial complexity for solving nonlinear convex problems will be found until now.

Keywords: Algorithm, nonlinear semidefinite optimization, Kernel function, BFGS quasi-Newton method.

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1. Introduction

Interior point methods (IPMs) are called due to the iterations moving inside the feasible region while staying away from the boundary. A feasible IPM can only be executed if a strictly feasible point is found is known. Usually, such a starting point is not readily available and infeasible IPMs are used instead.

In IPMs each of the iterations is strictly feasible for some artificial problem and stays close to the central path of that problem, where closeness is measured by some merit function, see [15] and other researchers. In [17], we presented the first full-Newton step, $O(n)$ IPMs for linear optimization.

The algorithm proposed in this work aims to solve the following problem:

$$\begin{cases} \min F(X), \\ C(X) \succeq 0, \end{cases} \tag{P1}$$

where

1. let S^n denotes the space of all $n \times n$ real symmetric matrices, S^n_+ and S^n_{++} are the cone of symmetric semidefinite and symmetric positive definite matrices, respectively;

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2. the objective function to be minimized is $F : \mathbb{S}_+^n \to \mathbb{R}$, must verify: must satisfy the conditions of convexity and be at least twice differentiable;
3. $C(X) \succeq 0$, where $C : \mathbb{S}_+^n \to \mathbb{S}_+^m$;
4. $\mathring{\mathbb{P}} = \{X \in \mathbb{S}_+^n : C(X) \succeq 0\}$ is a primal feasible set.

Beginning with an initial value of μ, the perturbed optimality conditions are approximately solved. The algorithm necessary for this solution employs the iterates generated by the perturbed algorithm as inner iterates.

The most robust quasi-Newton formula is the BFGS formula, proposed independently by Broyden [8], Fletcher [11], Goldfarb [13], Shanno [18] and others.

This updated formula approximates the Hessian of the Lagrangian. It mentions that, under the strong convexity assumption, the matrix being approximated is symmetric and positive definite. Additionally, it states that the authors or researchers "require that the matrix M they compute is symmetric and positive definite". This indicates that the authors have a specific requirement for the computed matrix M.

For the linear search, it consists of calculating a step such that the new iteration obtained is always strictly feasible. Additionally, the Armijo condition, as described in [4], must be satisfied. Any differentiable convex quadratic problem can be formulated into a monotone Linear Complementarity Problem (LCP in short), and vice versa. This makes the LCP one of the fundamental problems of mathematical programming, where LCP are used in the study of equilibrium problems in, e.g., economics, transportation planning and game theory in [9]. There have been successful generalizations of a number of IPMs for linear optimization (LO in short) to LCP [14].

The logarithmic barrier function is the foundation of the majority of IPMs. In [1], proposed a new class of non-Self-Regular kernel functions for LO and obtained $O(q\sqrt{n}\log(n^{1+\frac{1}{q}})\log\frac{n}{\varepsilon})$ with $q \geqslant 1$, iteration bounds for large update dual-primal MPIs. By introducing a parameter $q \geqslant 1$ into the kernel function of [5], they obtained the best known iteration bound for LO in large and small updates by choosing $q = O(\log(n+1))$. The first trigonometric kernel function that gives better results $O(\sqrt{n}\log n \log\frac{n}{\varepsilon})$ was proposed by [7]. [10], proposed a new barrier function $(m+1)t^2 - (m+2) + \frac{1}{t^m}$ with $m > 4$ and primal-dual interior point algorithms for problems LCP and an analysis of the kernel function based algorithm. They obtained $O(m^{\frac{3m+1}{2m}}\sqrt{n}\log\frac{(x^0)^t y^0)}{\varepsilon})$ for small update and $O(m^{\frac{3m+1}{2m}} n^{\frac{m+1}{2m}}\log\frac{(x^0)^t y^0)}{\varepsilon})$ for large-update methods, which are the best known iteration bounds for such methods. [19] introduced a primal-dual IPM specifically designed for SDO, achieving complexity results of $O(q^2\sqrt{n}\log\frac{n}{\varepsilon})$, and $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ for updates at both small and large scales, with $q > 1$. In [12] extended the IPM from LO to SDO and obtained similar iteration bounds.

Motivated by [2], who proposed a BFGS method with a primal-dual barrier function (IPMs) to solve convex nonlinear class problems, where the objective function and the constraints sets are in general convex, and with a reformulation equivalent to the central path, a solution to the linear complementarity problem was proposed in [10] and they were able to find a new search direction that targets a small neighbourhood of the central path.

In this context, the main objective of this study is to use an LCP and combine it with a new kernel function and the quasi-Newton BFGS to solve the nonlinear convex semi-definite problem (P1).

This paper introduces a new method for solving semi-definite convex problems using Newton and quasi-Newton algorithms to obtain new directions, along with Armijo test algorithms to determine a new step in Section 2. In Section 3, we examine a novel kernel function and its theoretical properties for SDO. In Section 4, we present the new complexity of the central path algorithm, and a conclusion is drawn in Section 5.

2. The resolution of problem (P1)

We consider the following non-empty sets of constraints:

1. $\mathbb{P} = \{X \in \mathbb{S}_+^n : C(X) \succeq 0\}$ is a primal feasible set;

2. $\mathring{\mathbb{P}} = \{X \in \mathbb{S}^n_+ : C(X) \succ 0\}$ is a primal strictly feasible set;
3. $\mathring{\mathbb{P}}_{\mathbb{D}} = \{(X, S) \in \mathbb{S}^n_+ \times \mathbb{S}^m_+ : (C(X), S) \succ 0\}$ is a primal-dual strictly feasible set.

The objective in the following is to solve the fundamental problem (P1), based on [3, 7, 10]. First of all, the optimality conditions will be specified, as shown below.

2.1. The optimality conditions (K.K.T)

The Lagrangian $\mathbb{L} : \mathbb{S}^n_+ \times \mathbb{S}^m_+ \to \mathbb{R}$ is an important step towards solving the problem (P1):

$$\mathbb{L}(X, S) = F(X) + S^T C(X),$$

with

1. for all (X, S), the function $\mathbb{L}$ is convex and at least twice differentiable;
2. the gradient's formula from $\mathbb{L}$ to X is:
$$\nabla_X L(X, S) = \nabla F(X) - \nabla C(X)S;$$
3. the gradient's formula from $\mathbb{L}$ to S is:
$$\nabla_S \mathbb{L}(X, S) = C(X);$$
4. the Hessian of $\mathbb{L}$ to X is:
$$\nabla^2_{XX}\mathbb{L}(X, S) = \nabla^2 F(X) + \nabla^T C(X) S \nabla C(X).$$

The conditions of optimality are proposed by problem (P1) under constraint qualification requirements: there exists a vector of multipliers $S \in \mathbb{S}^m_+$ that verifies the following problem:

$$\begin{cases} \nabla F(X) - \nabla C(X)S = 0, \\ C(X)S = 0, \\ (C(X), S) \succeq 0, \end{cases} \tag{2.1}$$

Our primal-dual (IPs) approximation is standard and has been discussed in [3, 6]. The next step is to associate problem (2.1) with the penalized problem, as shown below:

$$\begin{cases} \nabla F(X) - \nabla C(X)S = 0, \\ C(X)S = \mu I, \\ (C(X), S) \succ 0, \end{cases} \tag{2.2}$$

with:

1. I is an identity matrix, where $I \in \mathbb{S}^m$;
2. the parameter penalization is $\mu > 0$ is fixed at each iteration.

The Newton's procedure iterative is used to solve a nonlinear equation.

2.2. Kernel function with Newton and BFGS quasi-Newton methods

2.2.1. New class of search directions

We are basing this on [10], where, the main concept is now to replace the perturbative complementarity equation in the problem (2.1); in other words, the term:

$$\frac{1}{\mu} D^{-1} C(X) S D = I, \tag{2.3}$$

is substitute by:

$$\psi\left(\frac{1}{\mu} D^{-1} C(X) S D\right) = \psi(I),$$

where ψ is considered a *barrier kernel function.*

Definition 2.1. All functions $\psi : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ twice differentiable and satisfy the conditions:

1. $\psi(1) = \psi'(1) = 0$;
2. for all $t > 0$, $\psi''(t) > 0$;
3. $\lim_{t\to 0}\psi(t) = \lim_{t\to +\infty}\psi(t) = +\infty$,

is a barrier kernel function.

Remark 2.2.

1. We will obtain strict convexity of ψ and its minimum is reached at $t = 1$ according to conditions (1) and (2) of the Definition 2.1.
2. Based on the last condition of the Definition 2.1, ψ yields the barrier property.

Thus, the problem (2.2) is expressed as follows:

$$\begin{cases} \nabla F(X) - \nabla C(X)S = 0, \\ \psi(\frac{1}{\mu}D^{-1}C(X)SD) - \psi(I) = 0, \\ (C(X), S) \succ 0. \end{cases} \tag{2.4}$$

Let

$$\mathbb{G}(X, S) = 0,$$

in a manner that:

$$\mathbb{G}(X, S) = \begin{bmatrix} \nabla F(X) - \nabla C(X)S \\ \\ \psi(\frac{1}{\mu}D^{-1}C(X)SD) - \psi(I) \end{bmatrix} \in \mathbb{S}^{n+m}. \tag{2.5}$$

2.3. Newton's step

We use the search direction for SDO proposed by Nestrov:

$$P = C(X)^{\frac{1}{2}}(C(X)^{\frac{1}{2}}SC(X)^{\frac{1}{2}})^{\frac{-1}{2}}C(X)^{\frac{1}{2}} = S^{\frac{-1}{2}}(S^{\frac{1}{2}}C(X)S^{\frac{1}{2}})^{\frac{1}{2}}S^{\frac{-1}{2}}.$$

To simplify the resolution process we posit $D = P^{-1}$, we will need to define new vectors as follows:

$$V = \frac{1}{\sqrt{\mu}}(D^{-1}C(X)SD)^{\frac{1}{2}} = \frac{1}{\sqrt{\mu}}D^{-1}C(X)D^{-1} = \frac{1}{\sqrt{\mu}}DSD, \tag{2.6}$$

where

$$A^{\frac{1}{2}} = (\sqrt{a_{ij}})_{1\leqslant i,j\leqslant n},$$

$$D_X = \frac{1}{\sqrt{\mu}}D^{-1}C^{-1}(X)\nabla C(X)V\Delta XD, \tag{2.7}$$

$$D_S = \frac{1}{\sqrt{\mu}}D^{-1}S^{-1}V\Delta SD. \tag{2.8}$$

$\mathbb{G}$ is a nonlinear function, so with $\mu > 0$ held constant, we apply Newton's method to solve (2.5) and obtain the solution, which is known as Newton's primal-dual direction $(\Delta X, \Delta S) \in \mathbb{S}^n \times \mathbb{S}^m$ of the subsequent system:

$$\begin{pmatrix} \mathbb{H} & \nabla C(X) \\ \frac{D^{-1}S\nabla C(X)^T D}{\mu}\nabla\psi(V^2) & \frac{D^{-1}C(X)D}{\mu}\nabla\psi(V^2) \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta S \end{pmatrix} = \begin{pmatrix} -\nabla_X \mathbb{L}(X, S) \\ -\psi(V^2) + \psi(I) \end{pmatrix}, \tag{2.9}$$

where

1. by design of the kernel function, we get $\psi(I) = 0$;
2. from (2.4) and (2.3), we obtain
$$\nabla_X \mathbb{L}(X) = \nabla F(X) - \mu \nabla C(X)(C(X))^{-1};$$
3. $\mathbb{H}$ is an approximation of $\nabla^2_{X,X}\mathbb{L}(X,S)$ as it is positive-definite, and we will get its formula from the quasi-Newton method with the BFGS technique; it is our goal in the sequel.

2.3.1. Approximation of the Hessian matrix

At each iteration, the matrix $\mathbb{H}$ is calculated from the initial matrix $\mathbb{H}_0$ as an approximation of the Hessian matrix or its inverse. The BFGS updates the matrix $\mathbb{H}_k$, resulting in a revised approximation. The following formulas yield the approximation $\mathbb{H}_{k+1}$: using (2.6), (2.7), and (2.8) in (2.9), we derive a new problem, which is the following:

$$\begin{cases} \mathbb{H}D_X + \nabla C(X)C(X)^{-1}S\nabla C(X)^T D^{-1}D_S D^{-1} = \nabla_X \mathbb{L}(X)C(X)^{-1}\nabla C(X)D^{-1}VD^{-1}, \\ D_X + D_S = -\frac{\psi(V^2)}{V\nabla\psi(V^2)}, \end{cases} \tag{2.10}$$

where $\mathbb{H}$ is a positive definite matrix and $\nabla C(X)C(X)^{-1}S\nabla C(X)^T$ is a semi-definite matrix, therefore we find a unique solution called (D_X, D_S) of the system (2.10). Now, we will present a quasi-Newton formula for calculating $\mathbb{H}$.

$$\begin{cases} D_X = \mathbb{H}^{-1}(S\nabla C(X)^T D^{-1}D_S D^{-1} - \nabla_X \mathbb{L}(X)D^{-1}VD^{-1})C(X)^{-1}\nabla C(X), \\ D_S = -\frac{\psi(V^2)}{V\nabla\psi(V^2)} - D_X, \end{cases}$$

and $\mathbb{H}^{-1}$ is updated with the quasi-Newton formula so as to approximate of the inverse of the Lagrangian's Hessian:

$$(\mathbb{H}_k{}')^{-1} = \mathbb{H}_k^{-\frac{1}{2}}\left(I - \frac{\mathbb{H}_k^{-\frac{1}{2}}\bar{\lambda}_k\bar{\lambda}_k^T\mathbb{H}_k^{-\frac{1}{2}}}{\bar{\lambda}_k^T Y_k + Y_k^T \mathbb{H}_k^{-1} Y_k}\right)\mathbb{H}_k^{-\frac{1}{2}}, \quad \mathbb{H}'_k = \mathbb{H}_k + \frac{Y_k Y_k^T}{Y^T\bar{\lambda}_k} \text{ and } \mathbb{H}_{k+1} = \mathbb{H}'_k - \frac{\mathbb{H}'_k\mathbb{H}_k\bar{\lambda}_k\bar{\lambda}_k\mathbb{H}_k\mathbb{H}'_k}{\bar{\lambda}_k^T\mathbb{H}_k\bar{\lambda}_k},$$

where $\bar{\lambda}_k \in \mathbb{R}^n$ and $Y_k \in \mathbb{R}^n$ are defined by

$$\bar{\lambda}_k = \alpha\lambda_k, \quad \lambda_k = X_{k+1} - X_k \text{ and } Y = \nabla\mathbb{L}(X_{k+1}, S_{k+1}) - \nabla\mathbb{L}(X_k, S_{k+1}),$$

(X_{k+1}, S_{k+1}) is a new iteration derived from (X_k, S_k). How to ensure that $\mathbb{H}_k$ remains a positive definite matrix at each iteration, using the following theorem, is addressed in [2].

Theorem 2.3 ([2])**.** *$\mathbb{H}_k$ is well-defined and yields a definite positive matrix if these matrices $Y_k^T\bar{\lambda}_k$ are positive definite matrices.*

2.4. The central path algorithm and Armijo test

2.4.1. The central path algorithm for convex problems

The algorithm for the kernel function is: From the initial solution primal-dual (X^0, S^0), $\mu^0 > 0$, with the precision parameter $\varepsilon > 0$, and employing a new kernel proposed in the remainder of our work at each outer iteration, the penalty parameter decreases by a value of $(1-\theta)$, where $\theta \in (0, 1)$. To have a solution of the equation (2.9) where this solution is close to the central path, we will use Newton's directions, and a second problem will appear in our directions, which is the calculation of the Lagrange hessian matrix; for this type of question, we relied on the quasi-Newton method where we could have an approximation of the matrix. We repeat the procedure until we have $n\mu < \varepsilon$, at which point we can say that we have obtained ε-approximate solutions to our primary problem of the primal X and dual S.
In the following, we will have the problem of the step, that's why we will use the Armijo test. This is shown in the Algorithm 1.

2.4.2. Armijo test algorithm

The algorithm of Armijo [4] is: we have $\alpha \in]0, +\infty[$ and $0 < \beta < 1$, in general we choose the initial values: $\alpha_0 = 1$ and $\beta = 0.95$, where to verify the following conditions:

$$(X, S) \in \mathring{\mathfrak{F}},$$

where

$$\alpha_{S,\text{Armijo}} = \min\{\alpha : S + \alpha\beta\Delta S \succ 0\},$$

and

$$\alpha_{X,\text{Armijo}} = \min\{\alpha : C(X) + \alpha\beta\nabla C(X)^T \Delta X \succ 0\}.$$

In the next step, the so-called Armijo test is passed, defined as follows:

$$C(X_k + \alpha D_k) \preceq C(X_k) + \alpha\beta\nabla C(X_k)^T D_k.$$

1. If the condition is verified, we choose the largest integer $n_0 \geqslant 0$, such that we perform the following procedure (we will grow the step):
$$\alpha_1 = 2\alpha_0,$$
and we repeat the Armijo test for the new step value, if the test is verified, we continue and calculate the new iteration:
$$\alpha_2 = 2\alpha_1 = 2^2\alpha_0,$$
to the largest value n_0 that satisfies the Armijo test.
2. If α_0 doesn't check Armijo's tests, in the case we choose the smallest integer $m_0 \geqslant 0$ that will verify the test, such that we perform the following procedure (we will decrease the step value):
$$\alpha_1 = \frac{\alpha_0}{2},$$
and we repeat the Armijo test for the new step value, if the test is not verified, we continue and calculate the new iteration:
$$\alpha_2 = \frac{\alpha_1}{2} = \frac{\alpha_0}{2^2},$$
to the smallest value m_0 that satisfies the Armijo test.

We repeat the same procedures mentioned above for S. And we get the following:

$$\alpha = \min(0.95 \min(\alpha_{X,\text{Armijo}}, \alpha_{S,\text{Armijo}}), \alpha).$$

This is shown in lines 7 through 28 of the Algorithm 1.

2.4.3. Initial data

- For the problem (P1), we must provide a strictly feasible iterate X^0 strictly feasible at the beginning of the algorithm, which means that for all $i = \overline{1, m}$, we have $C_i(X^0) \prec 0$.
- The initial step chosen as $\alpha^0 = 1$.
- For the initial solutions of the problem (2.2), noted S^0 and μ^0 are provided in the following way:
  1. $(S^0, \mu^0) \in \{(S, \mu) \in \mathbb{R}^m \times \mathbb{R} : (S, \mu) \succ 0\}$;
  2. we set the perturbation parameter μ^0 to 1, i.e., $\mu^0 = 1$;
  3. and deduce the corresponding multiplier using the perturbed complementarity condition, i.e.,
  $$S^0 = C^{-1}(X^0)I. \tag{2.11}$$
- The initial approximation of the Hessian matrix is $\mathbb{H}^0 = I$.

Algorithm 1 A path central algorithm with Armijo test.

Input: an accuracy parameter $\varepsilon > 0$, an update parameter $\tau \in (0,1)$, a threshold parameter $\theta \in (0,1)$,
Begin $X^0 = I, C(X) := C(X^0), S := C^{-1}(X^0)I, \mu^0 = 1, \alpha^0 = (\alpha_X^0, \alpha_S^0) = (1,1), \mathbb{M}^0 = D^0 = I, \beta = 0.95, V^0 = \frac{1}{\sqrt{\mu^0}}((D^0)^{-1}C(X^0)S^0(D^0))^{\frac{1}{2}}, \Psi(V^0) \leqslant \tau$
while $n\mu_{k+1} > \varepsilon$ **do (outer iteration)** (go to 2.4.1)
  $\mu_{k+1} = (1-\theta)\mu_k$
  while $\Psi(V_{k+1}) > \tau$ **do (inner iteration)**
    Armijo test: (go to 2.4.2)
    if $(S_{k+1} \preceq S_k + \beta\alpha_S^0 D_S)$ **then**
      $k' = 1$ and $\alpha_S^1 = 2\alpha_S^0$
      while $S_{k+1} \preceq S_k + \alpha_S^{k'}\beta D_S$ **do**
        $\alpha_S^{k'} = 2^{k'}\alpha_S^0$; $k' = k'+1$
      end while
    else
      k'=1 and $\alpha_S^1 = \frac{\alpha_S^0}{2}$
      while $S_{k+1} \succ S_k + \beta\alpha_S^{k'} D_S$ **do**
        $\alpha_S^{k'} = \frac{\alpha_S^0}{2^{k'}}$; $k' = k'+1$
      end while
    end if
    if $C(X_{k+1}) \preceq C(X_k) + \beta\alpha_X^0 \nabla C(X_k)^T D_X$ **then**
      $k' = 1$ and $\alpha_S^1 = 2\alpha_S^0$
      while $C(X_{k+1}) \preceq C(X_k) + \beta\alpha_x^{k'} \nabla C(X_k)^T D_X$ **do**
        $\alpha_X^{k'} = 2^{k'}\alpha_X^0$, $k' = k'+1$
      end while
    else
      while $C(X_{k+1}) \succ C(X_k) + \beta\alpha_X^{k'} \nabla C(X_k)^T D_S$ **do**
        $\alpha_X^{k'} = \frac{\alpha_X^0}{2^{k'}}$; $k' = k'+1$
      end while
    end if
    $\alpha = \min(0.95\min(\alpha_X, \alpha_S), \alpha)$
    $X_{k+1} := X_k + \alpha\Delta X_k$
    $S_{k+1} := S_k + \alpha\Delta S_k$
    $V_{k+1} := \frac{1}{\sqrt{\mu}}((D_k)^{-1}C(X_{k+1})S_{k+1}D_k)^{\frac{1}{2}}$
  end while (inner iteration)
end while (outer iteration)

3. Theoretical properties of the new kernel function

We suggest a new simple kernel function with a number of key properties, including a new Newton direction for complexity analysis. One of these functions is described by:

$$\begin{array}{llll} \psi: & \mathbb{R}_+^* & \to & \mathbb{R} \\ & t & \mapsto & \psi(t) = t^2 + \frac{2}{t} - 3. \end{array} \tag{3.1}$$

Throughout this work, we will use three derivatives of ψ with respect to t, for $t > 0$, we have:

$$\begin{cases} \psi'(t) = 2t - \frac{2}{t^2}, \\ \psi''(t) = 2 + \frac{4}{t^3}, \\ \psi'''(t) = -\frac{12}{t^4}. \end{cases} \tag{3.2}$$

Lemma 3.1. *The function ψ defined by (3.1) has the following properties:*

1. $\psi''(t) > 2, \forall t > 0$;
2. $t\psi''(t) + \psi'(t) > 0, \forall t > 0$;
3. $\psi'''(t) < 0, \forall t > 0$;
4. $t\psi''(t) - \psi'(t) > 0, \forall t > 0$;
5. $2\psi''(t)^2 - \psi'''(t) > 0, \forall t > 0$;
6. $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \forall t > 0, \forall \beta > 1$;
7. $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \forall t < 1$.

Remark 3.2. It should be noted that all conditions in Lemma 3.1 are satisfied.

The barrier function Ψ, is determined by:

$$\Psi(V) = \mathrm{tr}(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)).$$

Definition 3.3. In the performance analysis of the algorithm, we also employ the normalized proximity measure $\delta(V)$ given by:

$$\delta(V) = \frac{1}{2}\|\nabla\Psi(V)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{i=n}(\psi(\lambda_i(V)))^2} = \frac{1}{2}\|D_X + D_S\|. \tag{3.3}$$

The properties of ψ defined above imply that if $\psi(t)$ is twice differentiable, then it is completely determined by its second derivative:

$$\psi(t) = \int_1^t \int_1^\zeta \psi''(x)dxd\zeta = \int_1^t \int_1^\zeta (2 + \frac{4}{x^3})dxd\zeta.$$

Lemma 3.4 ([16]). *The function ψ is exponentially convex, that is to say:*

$$\textit{for all } V_1,\ V_2 \in \mathbb{S}^n_{++};\ \ \Psi\left((V_1^{\frac{1}{2}}V_2V_1^{\frac{1}{2}})^{\frac{1}{2}}\right) \leqslant \frac{1}{2}\left[\Psi(V_1) + \Psi(V_2)\right].$$

Lemma 3.5 ([16]). *Let ψ be a twice differentiable function for $t > 0$. The properties below are all equivalent.*

1. *For all $t > 0$, ψ is an exponentially convex.*
2. *$\psi(e^\zeta)$ is convex.*
3. *$\psi(\sqrt{\zeta})$ is convex.*
4. $t\psi''(t) + \psi'(t) > 0, \quad t > 0$.
5. $t\psi''(t) - \psi'(t) > 0, \quad t > 0$.

Lemma 3.6. *For all $t > 1$ and $\beta > 1$, $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0$.*

Proof. Let's set $f(\beta) = \psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t)$. Note that $f(1) = 0$. We have

$$f'(\beta) = t\psi''(t)\psi''(\beta t) - \psi'(t)\psi''(\beta t) - \beta t\psi'(t)\psi'''(\beta t) = \psi''(\beta t)[t\psi''(t) - \psi'(t)] - \beta t\psi'(t)\psi'''(\beta t).$$

According to Lemma 3.5, we have $t\psi''(t) + \psi'(t) > 0$, for all $t > 0$. By definition of the kernel function, we have:

1. $\psi''(\beta t) = 2 + \frac{4}{(\beta t)^3} > 2$, for all $t > 0$;
2. for all $t > 1$, $\psi'(t) = 2t - \frac{2}{t^2} > 0$;

3. for all $t, \beta > 0$, $\psi'''(\beta t) = -\frac{12}{(\beta t)^4} < 0$,

then, we obtain the result. □

Lemma 3.7 ([16]). *We have $\psi'''(t) = -\frac{12}{t^4} < 0$, for all $t > 0$, thence ψ satisfies the next mentioned properties.*

Case1: *For all $t < 1$, we have:*

$$3(t-1)^2 < \psi(t) < (t - \frac{1}{t^2})(t-1) < (1 + \frac{2}{t^3})(t-1)^2.$$

Case2: *For all $t > 1$, we have:*

$$3(t-1)^2 > \psi(t) > (t - \frac{1}{t^2})(t-1) > (1 + \frac{2}{t^3})(t-1)^2.$$

Lemma 3.8. *If we have the following condition satisfied: $\psi(t_1) = \psi(t_2)$, then one has the satisfaction of the following properties.*

1. *If $t_1 \leqslant \beta \leqslant t_2$ and $\beta \geqslant 1$, then $\psi(\beta t_1) \leqslant \psi(\beta t_2)$.*
2. *If $t_1 \leqslant 1 \leqslant t_2$, then $\psi'(t_1) \leqslant 0$ and $\psi'(t_2) \geqslant 0$ and $\psi'(t_2) < -\psi'(t_1)$.*
3. *If $t_1 \geqslant \beta \leqslant t_2$ and $\beta \geqslant 1$, then $-\psi'(\beta t_1) \leqslant \psi'(\beta t_2)$.*

Proof.

1. We consider $h(\beta) = \psi(\beta t_2) - \psi(\beta t_1)$, hence $h(1) = 0$, and

$$h'(\beta) = t_2\psi'(\beta t_2) - t_1\psi'(\beta t_1).$$

By definition of the new kernel function, we have $\psi''(t) \geqslant 2$, for all $t > 0$, which gives us the increase of the function ψ', we obtain:

$$h'(\beta) = t_2\psi'(\beta t_2) - t_1\psi'(\beta t_1) \geqslant (t_2 - t_1)\psi'(\beta t_2) - \psi'(\beta t_1) \geqslant 0,$$

per conclusion, h is an increasing function,

$$h(\beta) \geqslant h(1) = 0,$$

and we obtain the result.

2. We know that $\psi''(t) \geqslant 0$, for all $t > 0$, and $\psi'(1) = 0$, then, ψ' is an increasing function, therefore

$$\psi'(t_1) \leqslant \psi'(1) = 0 \leqslant \psi'(t_2).$$

In the second part of the proof, we have by assumption and Lemma 3.7 that

$$3(t_1 - 1)^2 < \psi(t_1) = \psi(t_2) < 3(t_2 - 1)^2,$$

thus $1 - t_1 < t_2 - 1$. On the other hand, let us assume that $-\psi'(t_1) < \psi'(t_2)$ and from Lemma 3.7, we obtain

$$\psi(t_2) > (t_2 - \frac{1}{t_2^2})(t_2 - 1) > (t_2 - \frac{1}{t_2^2})(1 - t_1) > -(t_1 - \frac{1}{t_1^2})(1 - t_1) = (t_1 - \frac{1}{t_1^2})(t_1 - 1) > \psi(t_1)$$

and this is a contradiction with our assumptions.

3. Let's put $g(\beta) = \psi'(\beta t_2) + \psi'(\beta t_1)$. We have $\psi''(t) \geqslant 0$, for all $t > 0$, then

$$g'(\beta) = t_2\psi''(\beta t_2) + t_1\psi''(\beta t_1) \geqslant 0,$$

therefore, g is an increasing function and from the second property of Lemma 3.8 we get:

$$g(\beta) \geqslant g(1), \qquad \psi'(\beta t_2) + \psi'(\beta t_1) \geqslant \psi'(t_2) + \psi'(t_1) \geqslant 0.$$

These will give us an increasing function and we will then get the result. □

Lemma 3.9. *Let $\psi(t)$ be a kernel function, we have $\psi(t) \leqslant 3(t-1)^2$, for all $t > 0$.*

Proof. Using Taylor's development, we obtain

$$\begin{aligned}\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\zeta)(\zeta-1)^3 \\ &\leqslant \frac{1}{2}\psi''(1)(t-1)^2 = 3(t-1)^2, \text{ for all } \zeta \in]1, t[.\end{aligned}$$

□

Proposition 3.10.

1. *Let $\rho : [0, +\infty[\to [1, +\infty[$ is an inverse function of the kernel function ψ.*
2. *The inverse function of $-\frac{1}{2}\psi'$ is noted by $\eta : [1, +\infty[\to]0, 1[$.*

Lemma 3.11. *Let be ρ the inverse function of the kernel function ψ, then $1 + \sqrt{\frac{s}{3}} \leqslant \rho(s) \leqslant 1 + \sqrt{s}$, $s \geqslant 0$.*

Proof. Let $s = \psi(t)$, $t \geqslant 1$, i.e., $\rho(s) = t$, $t \geqslant 1$, therefore: $s = t^2 + \frac{2}{t} - 3$, $t > 0$ and we have also

$$\psi''(t) \geqslant 2 \Leftrightarrow \int_1^t \int_1^x \psi''(y)dydx > \int_1^x 2dydx,$$

then

$$s = \psi(t) > (t-1)^2,$$

so

$$\rho(s) = t < \sqrt{s} + 1. \tag{3.4}$$

And on the other hand, applying Lemma 3.7 (Case 2), we find

$$s = \psi(t) < 3(t-1)^2 \Leftrightarrow t - 1 > \sqrt{\frac{s}{3}} \Leftrightarrow t > 1 + \sqrt{\frac{s}{3}}. \tag{3.5}$$

From (3.4) and (3.5) we get the result. □

Lemma 3.12. *Let $\delta(V)$ defined in (3.3), we have*

1. $\delta(V) \geqslant \sqrt{\frac{\Psi(V)}{2}}$;
2. $\|V\| \leqslant n + \sqrt{\Psi(V)} \leqslant n + \sqrt{2}\delta(V)$.

Proof.

1. We use the second inequality of the Lemma 3.9,

$$\Psi(V) = \sum_{i=1}^{m} \psi(\lambda_i(V)) \leqslant \sum_{i=1}^{m} \frac{\psi'(\lambda_i(V))^2}{2} = \frac{1}{2}\|\Psi(V)\|^2 = 2\delta(V)^2.$$

And we get the outcome of lemma.

2. From Lemma 3.9, we have:

$$\Psi(V)=\sum_{i=1}^{n}\psi(\lambda_i(V))\geqslant\sum_{i=1}^{n}(\lambda_i(V)-1)^2\geqslant \mathrm{tr}(V)^2-n^2,$$

so

$$\|V\|\leqslant\sqrt{n^2+\Psi(V)}\leqslant n+\sqrt{\Psi(V)}\leqslant n+\sqrt{2}\delta(V).$$

□

Lemma 3.13. *Let $\beta\geqslant 1$. Then, we have $\psi(\beta t)\leqslant\psi(t)+(\beta^2-1)t^2$.*

Proof. Let us define the kernel function, we have

$$\psi(\beta t)-\psi(t)=t^2(\beta^2-1)+\frac{2}{t}(\frac{1}{\beta}-1),$$

we have $\beta\geqslant 0$, so $\beta-1>0$ and $\frac{1}{\beta}-1<0$. Then, we have $\psi(\beta t)-\psi(t)\leqslant t^2(\beta^2-1)$. □

Lemma 3.14. *Let $0<\theta<1$ and $V_+=\frac{V}{\sqrt{1-\theta}}$, then*

$$\Psi(V_+)\leqslant\Psi(V)+\frac{\theta(n+2\sqrt{n\Psi(V)}+\Psi(V))}{1-\theta},$$

and:

$$\Psi_0\leqslant\tau+\frac{\theta(n+2\sqrt{n\tau}+\tau)}{1-\theta}.$$

Proof. From Lemma 3.13, with $\beta=\frac{1}{\sqrt{(1-\theta)}}$, we obtain

$$\Psi(V_+)\leqslant\Psi(V)+\sum_{i=1}^{n}(\beta^2-1)(\lambda_i(V))^2=\Psi(V)+\frac{\theta(\mathrm{tr}(V))^2}{1-\theta}$$

and from Lemma 3.12, we get the result.

□

Theorem 3.15 ([16])**.** *Let ρ be the inverse function of $\psi(t)$, for all $t\geqslant 1$, we have*

$$\Psi(\beta V)\leqslant n\psi(\beta\rho(\frac{\Psi(V)}{n})),\ V\in S^n_{++},\ \beta\geqslant 1.$$

Lemma 3.16. *Let $0\leqslant\theta<1$, $V_+=\frac{V}{\sqrt{1-\theta}}$, if $\Psi(V)\leqslant\tau$, thus, we have*

$$\Psi(V_+)\leqslant\frac{3}{1-\theta}(\theta\sqrt{n}+\sqrt{\tau})^2=\Psi_0.$$

Such that, Ψ_0 is an upper bound for Ψ throughout our algorithm.

Proof. We have $\frac{1}{\sqrt{1-\theta}}\geqslant 1$ and $\rho(\frac{\Psi(V)}{n})\geqslant 1$, so $\frac{\rho(\frac{\Psi(V)}{n})}{\sqrt{1-\theta}}\geqslant 1$. From Theorem 3.15, we find

$$\Psi(V_+)\leqslant n\psi(\frac{1}{\sqrt{1-\theta}}\rho(\frac{\Psi(V)}{n})).$$

Using the last inequality of Lemma 3.9, we obtain

$$\Psi(V_+)\leqslant 3n(\frac{1}{\sqrt{1-\theta}}\rho(\frac{\Psi(V)}{n})-1)^2,$$

and from the first inequality of Lemma 3.11, we have

$$\Psi(V_+)\leqslant 3n(\frac{1}{\sqrt{1-\theta}}(1+\sqrt{\frac{\Psi(V)}{n}})-1)^2=\frac{3n}{1-\theta}(1+\sqrt{\frac{\Psi(V)}{n}}-\sqrt{1-\theta})^2\leqslant\frac{3}{1-\theta}(\theta\sqrt{n}+\sqrt{\tau})^2=\Psi_0.$$

□

4. Analysis of complexity

For each external iteration, we will compute in this section the number of inner iterations for the convex problem, using the proximity function defined by the new kernel function proposed in this paper, following the proximity function approach in [20].

4.1. *The step size* α

We calculate an approximation of the value of a step α and the resulting decrease of the barrier function. We have the new solution, which is in the following form:

$$X := X + \alpha\beta\Delta X, \quad S := S + \alpha\beta\Delta S.$$

We choose α as the largest step, so that iterating (X, S) is strictly feasible, i.e., $(X, S) \in \mathring{\mathfrak{F}}$, where

$$\alpha_S = \min\{\alpha : S + \alpha\beta\Delta S \succ 0\},$$

and

$$\alpha_X = \min\{\alpha : C(X) + \alpha\beta\nabla C(X)^T \Delta X \prec 0\},$$

then

$$\alpha = \min(0.95 \min(\alpha_X, \alpha_S), \alpha).$$

We have

$$S^+ = S + \alpha\beta\Delta S = D^{-1}SV^{-1}(V + \alpha\beta D_S)D^{-1}$$

and

$$C(X^+) = C(X) + \alpha\beta\nabla C(X)^T \Delta X = D(V + \alpha\beta D_X)V^{-1}C(X)D.$$

From (2.6), we get:

$$\mu V^2 = D^{-1}C(X)SD.$$

So

$$V_+^2 = \frac{D^{-1}C(X)SD}{\mu} = (V + \alpha\beta D_X)(V + \alpha\beta D_S),$$

and subsequently

$$V_+ = \sqrt{(V + \alpha\beta D_X)(V + \alpha\beta D_S)}.$$

Forall $\alpha > 0$, we put

$$\mathbb{F}(\alpha) = \Psi(V_+) - \Psi(V).$$

$\mathbb{F}(\alpha)$ is taken as the difference of the proximity between the new and the old iterations. The exponential convexity of the function Ψ gives us

$$\Psi(V_+) \leqslant \frac{\Psi(V + \alpha\beta D_X) + \Psi(V + \alpha\beta D_S)}{2},$$

and, if we posit

$$\mathbb{F}_1(\alpha) = \frac{\Psi(V + \alpha\beta D_X) + \Psi(V + \alpha\beta D_S)}{2} - \Psi(V), \tag{4.1}$$

we remark that

$$\mathbb{F}_1(\alpha) \geqslant \mathbb{F}(\alpha) \ \text{ and } \ \mathbb{F}_1(0) = \mathbb{F}(0) = 0.$$

We have

$$\mathbb{F}_1'(\alpha) = \frac{1}{2}\mathrm{tr}[\nabla\Psi(V_X)D_X + \nabla\Psi(V_X)D_S],$$

such as

$$V_X = V + \alpha\beta D_X \ \text{ and } \ V_S = V + \alpha\beta D_S$$

and
$$\mathbb{F}_1''(\alpha) = \frac{1}{2}\mathrm{tr}[\Delta\Psi(V_X)D_X^2 + \Delta\Psi(V_S)D_S^2].$$
We get
$$\mathbb{F}_1(0) = -2\delta^2(V).$$
To simplify all that follows we will put $\delta(V) = \delta$ and $V_{\min} = \min \lambda_i(V)$. The following lemmas are based on references [5, 7]:

Lemma 4.1. *Let $\mathbb{F}_1(\alpha)$ is defined in (4.1), then we have*
$$\mathbb{F}_1''(\alpha) \leqslant 2\delta^2\Psi''(V_{\min} - 2\alpha\delta).$$

Lemma 4.2. $F_1'(\alpha) \leqslant 0$ *is valid if α satisfies the inequality*
$$-\psi'(V_{\min} - 2\alpha\delta) + \psi'(V_{\min}) \leqslant 2\delta. \tag{4.2}$$

Lemma 4.3. *The largest step size α that satisfies the inequality (4.2) is given by:*
$$\alpha_1 := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

Lemma 4.4. *Let α_1, be as defined in Lemma 4.3, then, we obtain*
$$\alpha_1 \geqslant \frac{1}{\psi''(\rho(2\delta))} = \alpha_2.$$

Lemma 4.5. *If the step size α is such that $\alpha \leqslant \alpha_1$, so*
$$\mathbb{F}(\alpha) \leqslant -\alpha\delta^2. \tag{4.3}$$

4.2. Theoretical number of iterations in an algorithm

4.2.1. Outer iterations

About the outer iterations, it can be established from the following results.

Lemma 4.6 ([16])**.** *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attains its (global) minimum at $t > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, one has*
$$h(t) = \frac{th'(0)}{2}.$$

Theorem 4.7. *If $\Psi \geqslant 1$, we have*
$$\mathbb{F}(\alpha_2) \leqslant -\frac{\Psi^{\frac{1}{4}}}{97}.$$

Proof. From the second inequality of Lemma 4.4 and the definition of the inverse of $-\frac{1}{2}\psi'$, whence
$$-(2t - \frac{2}{t^2}) = 2s,$$
we obtain
$$t = \eta(s) \geqslant \frac{1}{\sqrt{s+1}}. \tag{4.4}$$
From (3.2) and Lemma 4.4, it follows that
$$\alpha_2 = \frac{1}{\psi''(\eta(2\delta))} = \frac{1}{2 + \frac{4}{(\eta(2\delta))^3}},$$

from (4.4), we get

$$\alpha_2 \geqslant \frac{1}{2+4(2\delta+1)^{\frac{3}{2}}} = \widetilde{\alpha}.$$

From Lemma 4.5, hence

$$\mathbb{F}(\widetilde{\alpha}) \leqslant -\widetilde{\alpha}\delta^2 = -\frac{\delta^2}{2+4(2\delta+1)^{\frac{3}{2}}} \leqslant -\frac{\delta^2}{4\delta+4(4\delta)^{\frac{3}{2}}} \leqslant -\frac{\delta^{\frac{1}{2}}}{68}.$$

With the first property of Lemma 3.12 we obtain the following result:

$$\mathbb{F}(\widetilde{\alpha}) \leqslant -\frac{\Psi^{\frac{1}{4}}}{97} \leqslant \Theta(\Psi^{\frac{1}{4}}).$$

□

Lemma 4.8. *If* $\alpha \in [0,1]$, *then*

$$(1-t)^{\alpha} \leqslant 1-\alpha t, \quad \forall t \geqslant 0.$$

Proof. Suggest

$$\mathbb{F}(t) = (1-t)^{\alpha} - (1-\alpha t).$$

We achieve the following

1. $\mathbb{F}(0) = 0$;
2. $\mathbb{F}'(t) = -\alpha(1-t)^{\alpha-1} + \alpha$;
3. $\mathbb{F}'(0) = 0$;
4. $\mathbb{F}''(t) = \alpha(\alpha-1)(1-t)^{\alpha-1}$.

By definition, we have

$$\mathbb{F}''(t) < 0, \quad \forall t \in [0,1],$$

so, $\mathbb{F}'$ is a deceasing function, with $\mathbb{F}'(t) < \mathbb{F}'(0) = 0$. And we conclude the same for the function f, which will give us the result. □

Lemma 4.9. *We note by* Ψ_0 *the first update of* $\Psi(\nu)$ *and* Ψ_j, $j = 1, 2, \dots, K$ *sequence of values of* $\Psi(\nu)$ *in the inner iterations,*

$$K \leqslant \left[\frac{4\Psi_0^{\frac{1}{4}}}{3\beta}\right],$$

where

$$\Psi_0 = \frac{3}{1-\theta}(\theta\sqrt{n}+\sqrt{\tau})^2.$$

Proof. Let $t_0, t_1, \dots, t_K$ be a sequence of positive numbers that verifies

$$t_{j+1} \leqslant t_j - \beta t_j^{1-\gamma}, \quad j = 0, 1, \dots, K-1. \tag{4.5}$$

According to (4.5) and Lemma 4.8, we get

$$0 < t_{j+1}^{\gamma} \leqslant (t_j - \beta t_j^{1-\gamma})^{\gamma} = t_j^{\gamma}(1-\beta t_j^{-\gamma})^{\gamma} \leqslant t_j^{\gamma}(1-\beta\gamma t_j^{-\gamma}) = t_j^{\gamma} - \beta\gamma.$$

By recurrence we obtain

$$0 < t_K^{\gamma} \leqslant t_0^{\gamma} - K\beta\gamma$$

and we have

$$K \leqslant \left[\frac{t_0^{\gamma}}{\beta\gamma}\right]. \tag{4.6}$$

Furthermore, according to Theorem 4.7,

$$\Psi_{j+1} - \Psi_j \leqslant f(\alpha_2) \leqslant -\frac{\Psi_j^{\frac{1}{4}}}{97},$$

then it is assumed that

$$0 \leqslant \Psi_{j+1} \leqslant \Psi_j - \frac{\Psi_j^{\frac{1}{4}}}{97}.$$

Using (4.5), we get

$$0 \leqslant \Psi_{j+1}^{\frac{1}{4}} \leqslant \Psi_j^{\frac{1}{4}} - \frac{1}{384}.$$

We use the inequality (4.6) and obtain the number of outer iterations is given as:

$$K \leqslant \left[384\Psi_0^{\frac{1}{4}}\right] = \Theta(n^{\frac{1}{4}}).$$

□

4.2.2. Inner iterations

Lemma 4.10. *With a given accuracy* $\varepsilon > 0$, *we obtain*

$$k \geqslant \frac{1}{\theta} \log \frac{n}{\varepsilon} = \Theta\left(\log \frac{n}{\varepsilon}\right).$$

Proof. The central path parameter is $\mu^0 = 1$, $\mu^k = (1-\theta)^k \mu^0$ and $n\mu \leqslant \varepsilon$, so, we have:

$$n(1-\theta)^k \leqslant \varepsilon,$$

and with the following property:

$$-\log(1-\theta) \geqslant \theta.$$

Thus, we conclude the number of inner iterations in the following form:

$$\log \frac{n}{\varepsilon}.$$

□

Remark 4.11. The iterations bound becomes:

$$\Theta(n^{\frac{1}{4}} \log \frac{n}{\varepsilon}).$$

5. Conclusion

In this paper, the best polynomial theoretical complexity $\Theta(n^{\frac{1}{4}} \log \frac{n}{\varepsilon})$ has been found until now to solve convex nonlinear semi-definite problems based on the following methods. The first one concerns the quasi-Newton BFGS method where the following difficulties have been solved; the second derivatives where they are not available or difficult to compute and the algorithmic properties with the memory space is reduced, subsequently we combine it with the second technique called kernel function where the properties are needed to get a new direction's Newton, and to guarantee at each iteration, a strictly feasible solution, we have performed the Armijo test to have iterative steps. For future research we could focus on infeasible problems.

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