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Existence and stability results for the integrable solution of a singular stochastic fractional-order integral equation with delay

A. M. A. El-Sayed[a], Mawaddah Abdurahman[b], Hoda A. Fouad[a,b,*]

[a] Faculty of Science, Alexandria University, Alexandria, Egypt.

[b] College of Science, Taibah University, Al-Madinah, Saudi Arabia.

Abstract

In this paper, we are concerning with the existence of the solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ of the singular stochastic fractional-order integral equation with delay $\rho(.)$,

$$\mathcal{V}(t) = B(t)t^{\alpha-1} + \lambda\, \mathcal{I}^{\beta} \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0,\tau],$$

where $B(t)$ is a given second order mean square stochastic process, λ is a parameter, $\rho(t) \leqslant t$, and $\mathcal{G}(t,\mathcal{V})$ is a measurable function in $t \in (0,\tau]$ and satisfies Lipschitz condition on the second argument. The Hyers-Ulam and generalized Hyers-Ulam-Rassias stability will be proved. Moreover, the continuous dependence of the solution on the process $B(t)$ and λ will be studied. As applications, some nonlocal, weighted and nonlocal-weighted integral problems of stochastic fractional-order differential equations will be studied.

Keywords: Stochastic fractional calculus, singular stochastic integral equation, stochastic fractional-order differential equations, existence of integrable solution, continuous dependence, Hyers-Ulam stability.

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1. Introduction

Stochastic differential equations are a powerful tool for describing systems affected by external noise. These equations utilize random numbers or functions as coefficients for independent or dependent variables. Recently, El-Sayed and Fouad [15–17] studied a specific category of problems dealing with stochastic differential equations with nonlocal conditions. Their research shows that using Schauder's fixed point theorem, there is always at least one solution for a functional nonlocal random integral equation within the space of all squared integrable stochastic processes with a finite second moment. Nonlocal and weighted conditions provide more precise measurements taken at multiple locations compared to local conditions. In stochastic differential equations (SDEs) with non-local conditions, the behavior of the

*Corresponding author

Email addresses: amasayed@alexu.edu.eg (A. M. A. El-Sayed), mawaddah.a.s.a@gmail.com (Mawaddah Abdurahman), hoda.fouad@alexu.edu.eg (Hoda A. Fouad)

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solution at a given point depends on the values of the solution at other points in the domain rather than just the local behavior near that point. This means that the solution of a non-local SDE is influenced by the global structure of the domain rather than just the local behavior around a point. Non-local conditions in SDEs arise when the stochastic process is affected by long-range interactions or non-local effects, such as non-local diffusion or fractional Brownian motion. These non-local effects can arise, for example, when the underlying phenomenon being modeled exhibits memory or long-range correlations [6, 26]. Overall, non-local conditions in SDEs can significantly affect the solutions of these equations and have important implications for the modeling and analysis of a wide range of phenomena in physics, biology, finance, and other fields [7, 27].

Let $(\Omega, \mathbb{F}, \rho)$ be a complete probability space where Ω is a sample space, $\mathbb{F}$ is a σ-algebra of events of Ω occurring during the time interval $[0,\tau]$, and ρ is a probability measure, let $\mathcal{W}(t)$, $t \geqslant 0$ be a standard Brownian motion on $(\Omega, \mathbb{F}, \rho)$. Let $\mathcal{V}(t;\omega) = \mathcal{V}(t)$, $t \in [0,\tau]$, $\omega \in \Omega$ be a second order stochastic process, i.e., $E(\mathcal{V}^2(t)) < +\infty$, $t \in [0,\tau]$. Let $C(I, L_2(\Omega))$ be the class of all continuous stochastic processes in mean square notion on $I = [0,\tau]$ with the norms ([33, 35, 36])

$$\| \mathcal{V} \|_C = \sup_{t\in I} \| \mathcal{V}(t) \|_2, \qquad \| \mathcal{V}(t) \|_2 = \sqrt{E(\mathcal{V}^2(t))}.$$

Let $L_1([0,\tau], L_2(\Omega))$ be the class of all second order integrable stochastic processes in mean square notion on $[0,\tau]$. The norm of $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ is given by

$$\|\mathcal{V}(t)\|_1^* = \int_0^\tau \|\mathcal{V}(t)\|_2 dt.$$

Definition 1.1. Let $\{\mathcal{V}(t),\ t \in [0,\tau]\}$ be a second order continuous or Riemann integrable stochastic process in mean square notion and $\alpha, \beta \in (0,1]$. The fractional-order integral $\mathcal{I}^\beta \mathcal{V}(t)$ is defined by

$$\mathcal{I}^\beta \mathcal{V}(t) = \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)} \mathcal{V}(\xi) d\xi.$$

If $\{\mathcal{V}(t),\ t \in [0,\tau]\}$ is mean square differentiable and the derivative $\frac{d}{dt}\mathcal{V}(t)$ is continuous or Riemann integrable on $[0,\tau]$, then the fractional-order derivative is defined by

$$D^\alpha \mathcal{V}(t) = \mathcal{I}^{1-\alpha} \frac{d\mathcal{V}}{dt}.$$

For the properties of stochastic fractional calculus (see [12, 19, 22]).

Some stochastic and deterministic problem of fractional order integral and differential equations have be studied by authors (see [3, 4, 9–11, 13–18, 20, 21]). Let $\alpha,\ \beta \in (0,1]$. Consider t as

$$\mathcal{V}(t) = B(t)t^{\alpha-1} + \lambda \mathcal{I}^\beta \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0,\tau], \tag{1.1}$$

where $B(t)$ is a given mean square second order stochastic process and λ is a parameter. The existence of solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ will be studied. The Hyers-Ulam stability of the integral equation (1.1) will be proved in the class $L_1([0,\tau], L_2(\Omega))$. The continuous dependence of the solution on the second order process $B(t)$ and the parameter λ will be proved.

The following are examples of the second order process $B(t)$.

1. The Brownian motion with volatility σ and Drift σ ([28, 32]),

$$B(t) = \sigma t + \sigma \mathcal{W}(t), \quad t \in \mathbb{R}_+.$$

2. The Brownian bridge [30]

$$B(s) = l(1-s) + ms + (1-s)\int_0^s \frac{d\mathcal{W}(t)}{1-t}, \quad s \in [0,1), \quad l, m \in \mathbb{R}.$$

3. The Brownian motion started at A, $A \in L_2(\Omega)$ ([29]),

$$B(t) = A + \mathcal{W}(t),$$

where $\mathcal{W}(t)$ is a standard Brownian motion will be considered.

Finally, as applications, the nonlocal problem

$$\begin{cases} {}^R D^\beta \mathcal{V}(t) = \lambda \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0, \tau], \\ \mathcal{I}^{1-\beta} \mathcal{V}(t)|_{t=0} = \mathcal{V}_0, \end{cases} \tag{1.2}$$

the weighted problem

$$\begin{cases} {}^R D^\beta \mathcal{V}(t) = \lambda \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0, \tau], \\ t^{1-\beta} \mathcal{V}(t)|_{t=0} = \mathcal{V}_1, \end{cases} \tag{1.3}$$

and the nonlocal-weighted integral problem

$$\begin{cases} {}^R D^\beta \mathcal{V}(t) = \lambda \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0, \tau], \\ t^{1-\beta} \mathcal{V}(t)|_{t=0} = \mathcal{V}_1 + \int_0^\tau \mathcal{V}(t) dt, \end{cases} \tag{1.4}$$

where $\mathcal{V}_o$ and $\mathcal{V}_1$ are second order random variables, will be studied.

2. Existence of the solution

Under the following assumptions, we study the existence of solution of (1.1) .

(A1) $\rho : [0, \tau] \to [0, \tau]$ is increasing, $\rho(t) \leqslant t$ and $\rho'(t) \geqslant \rho > 1$.

(A2) $\mathcal{G} : [0, \tau] \times L_2(\Omega) \to L_2(\Omega)$ is measurable in $t \in [0, \tau]$ and the Lipschitz condition is satisfied,

$$\|\mathcal{G}(t, \mathcal{V}(t)) - \mathcal{G}(t, \mathcal{U}(t))\|_2 \leqslant b\|\mathcal{V}(t) - \mathcal{U}(t)\|_2,$$

and $\mathcal{G}(t, 0) \in L_1([0, \tau], L_2(\Omega))$. From this assumption we can deduce that

$$\|\mathcal{G}(t, \mathcal{V}(t))\|_2 - \|\mathcal{V}(t, 0)\|_2 \leqslant \|\mathcal{V}(t, x(t)) - \mathcal{V}(t, 0)\|_2 \leqslant b\|\mathcal{V}(t)\|_2 + \|\mathcal{G}(t, 0)\|_2$$

and

$$\|\mathcal{G}(t, \mathcal{V}(t)\|_2 \leqslant \|\mathcal{G}(t, 0)\|_2 + b\|\mathcal{V}(t)\|_2.$$

(A3)

$$|\lambda| b \tau^* < 1, \quad \text{where } \tau^* = \max\{\frac{\tau^\alpha}{\alpha}, \frac{\tau^\beta}{\Gamma(\beta+1)}\}.$$

Theorem 2.1. *Let the assumptions* (A1)-(A3) *be satisfied, then the singular stochastic fractional-order integral equation* (1.1) *has a unique solution* $\mathcal{V} \in L_1([0, \tau], L_2(\Omega))$.

Proof. Define the operator F by

$$F\mathcal{V}(t) = B(t) t^{\alpha-1} + \lambda \mathcal{I}^\beta \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0, \tau]$$

and the set $\mathcal{Q} \subset L_1(I, L_2(\Omega))$ by

$$\mathcal{Q} = \{x \in L_1(I, L_2(\Omega)), \quad \|\mathcal{V}\|_1^* \leqslant r\}.$$

Let $\mathcal{V} \in \mathcal{Q}$, then we have

$$\|F\mathcal{V}(t)\|_2 \leqslant \|B(t) t^{\alpha-1}\|_2 + \|\lambda \mathcal{I}^\beta \mathcal{G}(t, \mathcal{V}(\rho(t)))\|_2$$

$$\leqslant \|B(t)\|_2 t^{\alpha-1} + |\lambda| \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)} \|\mathcal{G}(\xi, \mathcal{V}(\rho(\xi)))\|_2 d\xi$$
$$\leqslant \|B\|_C t^{\alpha-1} + |\lambda| \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)} [\|\mathcal{G}(\xi,0)\|_2 + b\|\mathcal{V}(\rho(\xi)))\|_2] d\xi.$$

But the integral

$$J = \int_0^\tau \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)} [\|\mathcal{G}(\xi,0)\|_2 + b\|\mathcal{V}(\rho(\xi)))\|_2] d\xi dt$$
$$= \int_0^\tau \left([\|\mathcal{G}(\xi,0)\|_2 + b\|\mathcal{V}(\rho(\xi))\|_2] \right) \int_\xi^\tau \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)} dt d\xi$$
$$\leqslant \frac{\tau^\beta}{\Gamma(1+\beta)} \int_0^\tau \left([\|\mathcal{G}(\xi,0)\|_2 + b\|\mathcal{V}(\rho(\xi))\|_2 \right) d\xi.$$

Let $\rho(\xi) = \theta$, then $d\xi = \frac{1}{\rho'} d\theta \leqslant \frac{1}{\rho} d\theta \leqslant d\theta$ and

$$J \leqslant \frac{\tau^\beta}{\Gamma(\beta+1)} (\|\mathcal{G}\|_1^* + \frac{b}{\rho} \int_0^\tau \|\mathcal{V}(\theta))\|_2 d\theta) \leqslant \frac{\tau^\beta}{\Gamma(1+\beta)} (\|\mathcal{G}\|_1^* + b\|\mathcal{V}\|_1^*) \leqslant \frac{\tau^\beta}{\Gamma(1+\beta)} (a+br), \ a = \|\mathcal{G}\|_1^*.$$

Then

$$\|F\mathcal{V}\|_1^* = \int_0^\tau \|F\mathcal{V}(t)\|_2 dt \leqslant \frac{\|B\|_C \tau^\alpha}{\alpha} + |\lambda| \frac{\tau^\beta}{\Gamma(1+\beta)} (a+br) = \tau^*(\|B\|_C + a|\lambda| + rb|\lambda|) = r,$$

where

$$r = \frac{\tau^*(\|B\|_C + a|\lambda|)}{1 - |\lambda| b\tau^*}.$$

This proves that $F\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$. Let $\mathcal{V}, \mathcal{U} \in \mathcal{Q}$, then we have

$$\|F\mathcal{V}(t) - F\mathcal{U}(t)\|_2 = \|\lambda \mathcal{I}^\beta [\mathcal{G}(t, \mathcal{V}(\rho(t))) - \mathcal{G}(t, \mathcal{U}(\rho(t)))]\|_2 \leqslant |\lambda| b \mathcal{I}^\beta \|\mathcal{V}(\rho(t))) - \mathcal{U}(\rho(t)))\|_2$$

and

$$\|F\mathcal{V} - F\mathcal{U}\|_1^* \leqslant |\lambda| \tau^* \frac{b}{\rho} \|\mathcal{V} - \mathcal{U}\|_1^* \leqslant |\lambda| \tau^* b \|\mathcal{V} - \mathcal{U}\|_1^*,$$

which proves that F is contraction on $\mathcal{Q}$ [8] and the singular fractional stochastic integral equation (1.1) has a unique solution $\mathcal{V} \in \mathcal{Q} \subset L_1([0,\tau], L_2(\Omega))$. □

3. Hyers-Ulam stability

For $I = [0,\tau], \varepsilon > 0, \psi \in C(I, \mathbb{R}_+)$, we consider the integral equation (1.1) and the following two inequalities (see [2, 24, 25]):

$$\|\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^\beta \mathcal{G}(t, \bar{\mathcal{V}}(\rho(t)))\|_2 \leqslant \varepsilon, \quad t \in I, \tag{3.1}$$
$$\|\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^\beta \mathcal{G}(t, \bar{\mathcal{V}}(\rho(t)))\|_2 \leqslant \psi(t), \quad t \in I. \tag{3.2}$$

Definition 3.1 ([23])**.** Equation (1.1) is the Hyers-Ulam stable if there exists a real number $c > 0$ such that for $\varepsilon > 0$ and for each solution $\bar{\mathcal{V}} \in L_1(I, L_2(\Omega))$ to (3.1) there exists a solution $\mathcal{V} \in L_1(I, L_2(\Omega))$ to (1.1) with $\|\bar{\mathcal{V}}(t) - \mathcal{V}(t)\|_2 \leqslant c\varepsilon, \ \ t \in I$.

Definition 3.2 ([31])**.** Equation (1.1) is generalized Hyers-Ulam-Rassias stable with respect to ψ if there exists a real number $c_\psi > 0$ such that for each solution $\bar{\mathcal{V}} \in L_1(I, L_2(\Omega))$ to (3.2) there exists a solution $\mathcal{V} \in L_1(I, L_2(\Omega))$ to (1.1) with $\|\bar{\mathcal{V}} - \mathcal{V}\|_2 \leqslant c_\psi \psi(t), \ \ t \in I$.

Theorem 3.3. *Let the assumptions of Theorem 2.1 be satisfied. Then the equation (1.1) is Hyers-Ulam stable.*

Proof. Let $\varepsilon > 0$ be given such that (3.1) holds and $\mathcal{V}$ be the solution of (1.1). Then

$$\begin{aligned}||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2 &= ||\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t)))||_2 \\ &= ||\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \bar{\mathcal{V}}(\rho(t))) + \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \bar{\mathcal{V}}(\rho(t))) - \lambda I^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))))||_2 \\ &\leqslant \varepsilon + |\lambda| b \mathcal{I}^{\beta}||\mathcal{V}(\rho(t)) - \mathcal{V}_s(\rho(t))||_2 \leqslant \varepsilon + |\lambda| b \frac{\tau^{\beta}}{\Gamma(1+\beta)}||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2,\end{aligned}$$

then

$$||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2 \leqslant \frac{\varepsilon}{1 - |\lambda| b \tau^*}.$$

□

Theorem 3.4. *Let the assumptions of Theorem 2.1 be satisfied and the function $\psi \in C(I, \mathbb{R}_+)$. Then equation (1.1) is a generalized Hyers-Ulam-Rassias stable with respect to ψ.*

Proof. Let $\mathcal{V}$ be a solution of (1.1). Then

$$\begin{aligned}||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2 &\leqslant ||\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \bar{\mathcal{V}}(\rho(t))) + \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \bar{\mathcal{V}}(\rho(t))) - \lambda I^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))))||_2 \\ &\leqslant ||\bar{\mathcal{V}}(t) - B(t)t^{\alpha-1} - \lambda \mathcal{I}^{\beta}\mathcal{G}(t, \bar{\mathcal{V}}(\rho(t)))||_2 + |\lambda| b \mathcal{I}^{\beta}||\mathcal{V}(\rho(t)) - \mathcal{V}_s(\rho(t))||_2 \\ &\leqslant \psi(t) + |\lambda| b \frac{\tau^{\beta}}{\Gamma(1+\beta)}||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2,\end{aligned}$$

then

$$||\bar{\mathcal{V}}(t) - \mathcal{V}(t)||_2 \leqslant \frac{\psi(t)}{1 - |\lambda| b \tau^*}.$$

which completes the proof. □

4. Continuous dependence of solutions

The concept of continuous dependence solution is presented in the following definition.

Definition 4.1. The solution $\mathcal{V} \in L_1([0, \tau], L_2(\Omega))$ of the singular fractional stochastic integral equation (1.1) depends continuously on the second order stochastic process $B(t)$ and the parameter λ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $\max\{||B(t) - B^*(t)||_2, |\lambda - \lambda^*|\} \leqslant \delta$ implies that $||\mathcal{V} - \mathcal{V}^*||_1 \leqslant \epsilon$.

Theorem 4.2. *Let the assumptions of Theorem 2.1 be satisfied, then the solution $\mathcal{V} \in L_1([0, \tau], L_2(\Omega))$ of the integral equation (1.1) depends continuously on the second order stochastic process $B(t)$ and the parameter λ.*

Proof. Let $\mathcal{V}^*$ be the solution of the equation

$$\mathcal{V}^*(t) = B^*(t)t^{\alpha-1} + \lambda^* \mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t))), \quad t \in (0, \tau].$$

It follows that,

$$\mathcal{V}(t) - \mathcal{V}^*(t) = (B(t) - B^*(t))t^{\alpha-1} + (\lambda - \lambda^*)\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t))) + \lambda(\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))) - \mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t))))$$

and

$$\begin{aligned}||\mathcal{V}(t) - \mathcal{V}^*(t)||_2 &\leqslant ||B(t) - B^*(t)||_2 t^{\alpha-1} + |\lambda - \lambda^*|||\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t)))||_2 \\ &\quad + |\lambda|||\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))) - \mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t)))||_2) \\ &\leqslant \delta t^{\alpha-1} + \delta||\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}^*(\rho(t)))||_2 + b|\lambda|\mathcal{I}^{\beta}||\mathcal{V}(\rho(t))) - \mathcal{V}^*(\rho(t)))||_2,\end{aligned}$$

so that,

$$\|\mathcal{V}-\mathcal{V}^*\|_1^* \leqslant \delta\frac{\tau^\alpha}{\alpha}+\delta\frac{\tau^\beta}{\Gamma(1+\beta)}(a+br)+\frac{|\lambda|\tau^\beta b}{\Gamma(\beta+1)}\|\mathcal{V}-\mathcal{V}^*\|_1,$$

then we can obtain that

$$\|\mathcal{V}-\mathcal{V}^*\|_1^* \leqslant \frac{\delta\tau^*(1+a+br)}{1-|\lambda|b\tau^*}=\epsilon,$$

which completes the proof. □

4.1. Examples

Example 4.3. Let $B(t) = \sigma t + \sigma\mathcal{W}(t)$ be the Brownian motion with drift, $B^*(t) = \sigma^* t + \sigma^*\mathcal{W}(t)$. Then $\forall\ \epsilon > 0$, there exists $\delta > 0$ such that

$$\max\{|\sigma-\sigma^*|, |\sigma-\sigma^*|\} \leqslant \delta_1,$$

we obtain

$$||B(t)-B^*(t)||_2 = t|\sigma-\sigma^*| + ||\mathcal{W}(t)||_2|\sigma-\sigma^*| \leqslant \delta_1(\tau+\sqrt{\tau}) = \delta.$$

Then our results of Theorem 4.2 are satisfied.

Example 4.4. Let

$$B(s) = l(1-s)+ms+(1-s)\int_0^s \frac{d\mathcal{W}(t)}{1-t}, \quad s\in[0,1), \quad l,m\in\mathbb{R}$$

and

$$B^*(s) = l^*(1-s)+m^*s+(1-s)\int_0^s \frac{d\mathcal{W}(t)}{1-t}, \quad s\in[0,1), \quad l^*,m^*\in\mathbb{R},$$

where

$$\max\{l-l^*,\ m-m^*\} \leqslant \delta.$$

So, we can get

$$||B(s)-B^*(s)||_2 = |(l-l^*)(1-t)+(m-m^*)t| \leqslant \delta|(1-t)+t| = \delta.$$

Then our results of Theorem 4.2 are satisfied.

Example 4.5. Finally, let A be a second order random variable, $A\in L_2(\Omega)$, and $B(t) = A + \mathcal{W}(t)$. Let

$$B^*(t) = A^* + \mathcal{W}(t), ||A-A^*||_2 \leqslant \delta,$$

then we can get

$$||B(t)-B^*(t)||_2 = ||A-A^*||_2 \leqslant \delta.$$

Then our results of Theorem 4.2 are satisfied.

5. Applications

The fractional calculus and fractional-order differential and fractional-order integral equations are important for the modeling of many important real deterministic and stochastic problems (see, for example [1, 3, 5, 9–11, 13–18, 20, 21, 28, 34]). Here, we apply our results to prove the existence of integrable solutions $\mathcal{V}\in L_([0,\tau], L_2(\Omega))$ for the problems (1.2), (1.3), and (1.4).

(I). Consider nonlocal problem of the stochastic fractional-order differential equation [4]:

$$\begin{cases} {}^RD^\beta\mathcal{V}(t) = \lambda\mathcal{G}(t,\mathcal{V}(\rho(t))),\ t\in(0,\tau], \\ \mathcal{I}^{1-\beta}\mathcal{V}(t)|_{t=0} = \mathcal{V}_0. \end{cases}$$

Theorem 5.1. *Let the assumptions of Theorem 2.1 be satisfied. Then the nonlocal problem (1.2) is equivalent to the singular integral equation (1.1) with $\alpha = \beta$. Consequently, the problem (1.2) has a unique solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$, which depends continuously on $\mathcal{V}_0$ and λ.*

Proof. Let $B(t) = A$, where A is a second order random variable. Let $\mathcal{V}$ satisfies (1.2). Integrating (1.2) we obtain

$$\mathcal{I}^{1-\beta}\mathcal{V}(t) = \mathcal{I}^{1-\beta}\mathcal{V}(t)|_{t=0} + \lambda\mathcal{I}\mathcal{G}(t, \mathcal{V}(\rho(t))) = \mathcal{V}_0 + \lambda\mathcal{I}\mathcal{G}(t, \mathcal{V}(\rho(t))).$$

Operating with $\mathcal{I}^{\beta}$, then we have

$$\mathcal{I}\mathcal{V}(t) = \frac{\mathcal{V}_0 t^{\beta}}{\Gamma(1+\beta)} + \lambda\mathcal{I}^{1+\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))).$$

Differentiate both sides, we get (1.1),

$$\mathcal{V}(t) = \frac{\mathcal{V}_0 t^{\beta-1}}{\Gamma(\beta)} + \lambda\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))), \quad A = \frac{\mathcal{V}_0}{\Gamma(\beta)}. \tag{5.1}$$

Let $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ be the solution of (5.1). Operating with $\mathcal{I}^{1-\beta}$ we obtain

$$\mathcal{I}^{1-\beta}\mathcal{V}(t) = \mathcal{V}_0 + \lambda\int_0^t \mathcal{G}(\xi, \mathcal{V}(\rho(\xi)))d\xi, \qquad \mathcal{I}^{1-\beta}\mathcal{V}(t)|_{t=0} = \mathcal{V}_o,$$

and

$$\frac{d}{dt}\mathcal{I}^{1-\beta}\mathcal{V}(t) = {}^R D^{\beta}\mathcal{V}(t) = \lambda\mathcal{G}(t, \mathcal{V}(\rho(t))),$$

then (1.2) is equivalent to the nonlocal problem of (1.1). Therefore, Theorems 2.1 and 4.2 are satisfied and problem (1.2) has a unique solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$, which depends continuously on $\mathcal{V}_0$ and λ. □

Now, the following corollary can be proved.

Corollary 5.2. *Let the assumptions of Theorems 2.1, 3.3, and 3.4 are satisfied. Let $\alpha = \beta$ in (1.1), then the nonlocal problem (1.2) is Hyers-Ulam and generalized Hyers-Ulam-Rassias stable.*

(II). Consider the problem with weighted condition (1.3),

$$\begin{cases} {}^R D^{\beta}\mathcal{V}(t) = \lambda\mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0,\tau], \\ t^{1-\beta}\mathcal{V}(t)|_{t=0} = \mathcal{V}_1. \end{cases}$$

Theorem 5.3. *The weighted problem (1.3) is equivalent to the integral equation (1.1) with $\alpha = \beta$. Consequently, the problem (1.2) has a unique solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$, which depends continuously on $\mathcal{V}_1$ and λ.*

Proof. Let $B(t) = A$, where A is a second order random variable. Let $\mathcal{V}$ satisfies (1.3). Integrating (1.3) we obtain

$$\mathcal{I}^{1-\beta}\mathcal{V}(t) = c + \lambda\mathcal{I}\mathcal{G}(t, \mathcal{V}(\rho(t))).$$

Operating by $\mathcal{I}^{\beta}$ we get

$$\mathcal{I}\mathcal{V}(t) = \frac{ct^{\beta}}{\Gamma(1+\beta)} + \lambda\mathcal{I}^{1+\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))).$$

Differentiating both sides, we obtain

$$\mathcal{V}(t) = \frac{ct^{\beta-1}}{\Gamma(\beta)} + \lambda\mathcal{I}^{\beta}\mathcal{G}(t, \mathcal{V}(\rho(t))).$$

Multiplying by $t^{1-\beta}$, then

$$t^{1-\beta}\mathcal{V}(t)|_{t=0} = \frac{c}{\Gamma(\beta)} + \lambda t^{1-\beta}\mathcal{I}^{\beta}\mathcal{G}(t,\mathcal{V}(\rho(t)))|_{t=0}$$

and

$$\mathcal{V}_1 = \frac{c}{\Gamma(\beta)},$$

which implies (1.1),

$$\mathcal{V}(t) = \mathcal{V}_1 t^{\beta-1} + \lambda\mathcal{I}^{\beta}\mathcal{G}(t,\mathcal{V}(\rho(t))), \quad A = \mathcal{V}_1. \tag{5.2}$$

Let $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ be the solution of (5.2). Operating with $\mathcal{I}^{1-\beta}$ and $\frac{d}{dt}$, respectively, on (5.2) we obtain (1.1). This proves that (1.3) is equivalent to (1.1). Therefore, Theorems 2.1 and 4.2 are satisfied and problem (1.2) has a unique solution $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$, which depends continuously on $\mathcal{V}_1$ and λ. □

Now, the following corollary can be proved.

Corollary 5.4. *Let the assumptions of Theorems 2.1, 3.3, and 3.4 are satisfied. Let $\alpha = \beta$ in (1.1), then the nonlocal problem (1.3) is Hyers-Ulam and generalized Hyers-Ulam-Rassias stable.*

(III). Consider the weighted-nonlocal-integral problem (1.4),

$$\begin{cases} {}^{R}D^{\beta}\mathcal{V}(t) = \lambda\mathcal{G}(t,\mathcal{V}(\rho(t))), \ t \in (0,\tau], \\ t^{1-\beta}\mathcal{V}(t)|_{t=0} = \mathcal{V}_1 + \int_0^{\tau}\mathcal{V}(t)dt. \end{cases}$$

Theorem 5.5. *The weighted-nonlocal problem (1.4) is equivalent to the s integral equation (1.1) with $\alpha = \beta$.*

Proof. Let $B(t) = A$, where A is a second order random variable. Let $\mathcal{V}$ be a solution of (1.4). Integrating equation (1.4) we obtain

$$\mathcal{I}^{1-\beta}\mathcal{V}(t) = c + \lambda\mathcal{I}\mathcal{G}(t,\mathcal{V}(\rho(t))).$$

Operating by $\mathcal{I}^{\beta}$ we obtain

$$\mathcal{I}\mathcal{V}(t) = \frac{ct^{\beta}}{\Gamma(1+\beta)} + \lambda\mathcal{I}^{1+\beta}\mathcal{G}(t,\mathcal{V}(\rho(t))).$$

Differentiating both sides we can get

$$\mathcal{V}(t) = \frac{ct^{\beta-1}}{\Gamma(\beta)} + \lambda\mathcal{I}^{\beta}\mathcal{G}(t,\mathcal{V}(\rho(t))).$$

Multiplying by $t^{1-\beta}$, then

$$t^{1-\beta}\mathcal{V}(t)|_{t=0} = \frac{c}{\Gamma(\beta)} + \lambda t^{1-\beta}\mathcal{I}^{\beta}\mathcal{G}(t,\mathcal{V}(\rho(t)))|_{t=0}$$

and

$$\mathcal{V}_1 + \int_0^{\tau}\mathcal{V}(t)dt = \frac{c}{\Gamma(\beta)},$$

which implies (1.1),

$$\mathcal{V}(t) = t^{\beta-1}(\mathcal{V}_1 + \int_0^{\tau}\mathcal{V}(t)dt) + \lambda\mathcal{I}^{\beta}\mathcal{G}(t,\mathcal{V}(\rho(t))), \quad A = (\mathcal{V}_1 + \int_0^{\tau}\mathcal{V}(t)dt). \tag{5.3}$$

Let $\mathcal{V}$ be a solution of (1.4). Multiplying (5.3) by $t^{1-\beta}$ we obtain

$$t^{1-\beta}\mathcal{V}(t)|_{t=0} = (\mathcal{V}_1 + \int_0^{\tau}\mathcal{V}(t)dt).$$

Operating with $\mathcal{I}^{1-\beta}$ and $\frac{d}{dt}$, respectively, on (5.3) we obtain (1.1). This proves that (1.4) is equivalent to (1.1). □

Corollary 5.6. *Let the assumptions of Theorems 2.1, 3.3, and 3.4 are satisfied. Let $\alpha = \beta$ in (1.1), then the nonlocal problem (1.4) is Hyers-Ulam and generalized Hyers-Ulam-Rassias stable.*

6. Conclusions

Let $B(t)$ be a given mean square second order stochastic process, $0 < \lambda < 1$ is a parameter, and $\rho(t) \leqslant t$. Here, we proved the existence of integrable solution $\mathcal{V} \in L_(\,[0,\tau], L_2(\Omega))$ of the singular stochastic fractional-order integral equation with delay (1.1),

$$\mathcal{V}(t) = B(t)t^{\alpha-1} + \lambda \mathcal{I}^{\beta} \mathcal{G}(t, \mathcal{V}(\rho(t))), \quad t \in (0, \tau].$$

The continuous dependence of this solution on $B(t)$ and λ have been proved and some examples of the mean square second order stochastic process $B(t)$ have been considered.

The Hyers-Ulam and generalized Hyers-Ulam-Rassias stability of (1.1) have been proved in the class $L_1([0,\tau], L_2(\Omega))$.

As application we proved the the equivalence of (1.1) and the problems of fractional order differential equations (1.2)-(1.4) and deduced the existence of solutions $\mathcal{V} \in L_1([0,\tau], L_2(\Omega))$ of these problems.

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