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Some new notions of fractional Hermite-Hadamard type inequalities involving applications to the physical sciences

Hijaz Ahmad[a,b,c], Rahim Bux Khokhar[d], Muhammad Suleman[d], Muhammad Tariq[d], Sotiris K. Ntouyas[e], Jessada Tariboon[f,*]

[a]*Department of Mathematics, Faculty of Science, Islamic University of Madinah, Medina, 42210, Saudi Arabia.*

[b]*Near East University, Operational Research Center in Healthcare, TRNC Mersin 10, Nicosia, 99138, Turkey.*

[c]*Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon.*

[d]*Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan.*

[e]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.*

[f]*Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

Abstract

The term convexity in the frame of fractional calculus is a well-established concept that has assembled significant attention in mathematics and various scientific disciplines for over a century. It offers valuable insights and results in diverse fields, along with practical applications due to its geometric interpretation. Moreover, convexity provides researchers with powerful tools and numerical methods for addressing extensive interconnected problems. In the realm of applied mathematics, convexity, particularly in relation to fractional analysis, finds extensive and remarkable applications. In this manuscript, we construct new fractional identities for differentiable preinvex functions to strengthen the recently assigned approach even more. Then utilizing these identities, some generalizations of the Hermite-Hadamard type inequality involving generalized preinvexities in the frame of fractional integral operator, namely Riemann-Liouville (R-L) fractional integrals are explored. Finally, we examined some applications to the q-digamma and Bessel functions via the established results. We used fundamental methods to arrive at our conclusions. We anticipate the techniques and approaches addressed by this study will further pique and spark the researcher's interest.

Keywords: Convex function, invex sets, preinvex functions, Hölder's inequality, power mean inequality.

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1. Introduction

A lot of mathematicians have investigated the notion of "convexity" throughout the past century. Many researchers have provided this phrase with remarkable thought as they work to advance various

*Corresponding author

Email addresses: hijaz555@gmail.com (Hijaz Ahmad), rahimbux.khokhar@faculty.muet.edu.pk (Rahim Bux Khokhar), sulemansial02@gmail.com (Muhammad Suleman), captaintariq2187@gmail.com (Muhammad Tariq), sntouyas@uoi.gr (Sotiris K. Ntouyas), jessada.t@sci.kmutnb.ac.th (Jessada Tariboon)

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branches of the pure and applied sciences. It has assumed a significant role in these efforts. In the fields of statistics, financial, economics, optimization and functional analysis, the idea of convex analysis assumes a significant role. Numerous real-world applications of optimization of convex functions exist, including circuit design, controller design, modelling, etc. The term "convexity" has a wide range of applications and significance, making it a rich source of inspiration and a mesmerizing topic for scientists and mathematicians. For the literature regarding convexity, see the references [4, 8, 10, 12, 15, 16, 25–27, 32, 33, 36].

The concept of inequalities and the convexity property are crucial in today's mathematics research. Both terms have a tight relationship with one another. Inequalities play a significant role in an assortment of fields, including mechanics, probability, mathematical quadrature formulas, functional analysis, information technology, and statistical issues. In this way, it is possible to think of the hypothesis of inequality as a separate area of mathematical analysis. For the literature regarding inequalities, see the references [1, 7, 9, 17, 19, 20, 28, 30, 31].

Fractional calculus has developed into a prominent and viable topic for study in the varying field of applied mathematics over the past several years. Some researchers have explored and solved practical problems in numerous domains and sectors of applied sciences using freshly introduced fractional derivatives and integrals with multiple points of view and approaches.

The goal of this article is to prove some integral inequalities for derivable mapping whose absolute value are preinvex. Next we will review some concepts in invexity analysis that will be utilized throughout the paper (see [2, 21, 23, 29, 37] and references therein).

The idea of convexity is a strong and magnificent tool for dealing with a huge range of applied and pure science problems. Many researchers have recently devoted themselves to researching the properties and inequalities associated with topic of convexity in different areas (see [5, 6] and the references therein).

We constructed this manuscript in the following way: first, we explore some fundamental ideas and definitions in Section 2. In Section 3, we investigate and prove the new integral identities. In order to these integral identities, with the aid of Hölder and Power mean inequality, we will attain the estimations of H-H inequalities pertaining to fractional operator. In Section 4, we investigate some applications involving modified Bessel functions and q-digamma function. Lastly, in Section 5, future directions and conclusion of the newly discussed concept are elaborated.

2. Preliminaries

This section's main objective is to remember and discuss specific related ideas and concepts that are pertinent to our analysis in later sections of this paper.

Definition 2.1 ([13])**.** A mapping $\Lambda : \mathfrak{K} \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex if

$$\Lambda\left(\wp\varpi_1 + (1-\wp)\,\varpi_2\right) \leqslant \wp\Lambda\left(\varpi_1\right) + (1-\wp)\,\Lambda\left(\varpi_2\right),$$

for all $\varpi_1, \varpi_2 \in \mathfrak{K}$ and $\wp \in [0,1]$.

Definition 2.2 ([3])**.** The term invexity (η-connected set) is always defined on set $\mathfrak{K} \subset \mathbb{R}^n$ w.r.t. $\eta(\cdot,\cdot)$, if $\varpi_1, \varpi_2 \in \mathfrak{K}$ and $\wp \in [0,1]$,

$$\varpi_1 + \wp\eta(\varpi_2, \varpi_1) \in \mathfrak{K}.$$

It is self-evident that every convex set is invex in terms of $\eta(\varpi_2, \varpi_1) = \varpi_2 - \varpi_1$. However, there are invex sets that are not convex [2].

In the year 1988, Mond and Weir [35] explored the idea of invex set to introduced the idea of preinvexity.

Definition 2.3 ([35])**.** The following inequality

$$\Lambda\left(\varpi_1 + \wp\eta\left(\varpi_2, \varpi_1\right)\right) \leqslant (1-\wp)\,\Lambda\left(\varpi_1\right) + \wp\,\Lambda\left(\varpi_2\right), \quad \forall \varpi_1, \varpi_2 \in \mathfrak{K}, \quad \wp \in [0,1],$$

is said to be preinvex w.r.t. η.

It is very important to mark that every convex is a preinvex function but the converse is not true [21]. For example $\Lambda(\wp) = -|\wp|, \ \forall \wp \in \mathbb{R}$, is preinvex but not convex w.r.t.

$$\eta\left(\varpi_2, \varpi_1\right) = \begin{cases} \varpi_2 - \varpi_1, & \text{if } \varpi_1\varpi_2 \geqslant 0, \\ \varpi_1 - \varpi_2, & \text{if } \varpi_1\varpi_2 < 0. \end{cases}$$

The folowing condition C first time explored and discussed by Mohan and Neogy [22].

Condition-$\mathcal{C}$: Assume that $\mathfrak{K} \subset \mathfrak{R}^n$ be an open invex subset w.r.t. $\eta : \mathfrak{K} \times \mathfrak{K} \to \mathbb{R}$. For any $\varpi_1, \varpi_2 \in \mathfrak{K}$ and $\wp \in [0, 1]$,

$$\eta\left(\varpi_2, \varpi_2 + \wp\, \eta\left(\varpi_1, \varpi_2\right)\right) = -\wp\, \eta\left(\varpi_1, \varpi_2\right), \qquad \eta\left(\varpi_1, \varpi_2 + \wp\, \eta\left(\varpi_1, \varpi_2\right)\right) = \left(1 - \wp\right)\, \eta\left(\varpi_1, \varpi_2\right). \tag{2.1}$$

For any $\varpi_1, \varpi_2 \in \mathfrak{K}$ and $\wp_1, \wp_2 \in [0, 1]$ from condition $\mathcal{C}$, we have

$$\eta\left(\varpi_2 + \wp_2\, \eta\left(\varpi_1, \varpi_2\right),\ \varpi_2 + \wp_1\, \eta\left(\varpi_1, \varpi_2\right)\right) = \left(\wp_2 - \wp_1\right) \eta\left(\varpi_1, \varpi_2\right).$$

If Λ is preinvex on $[\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)]$ and η satisfies condition $\mathcal{C}$, then for each $\wp \in [0, 1]$, from Eq. (2.1), it yields

$$\begin{aligned} \left|\Lambda\left(\varpi_1 + \wp\eta\left(\varpi_2, \varpi_1\right)\right)\right| &= \left|\Lambda\left(\varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right) + \left(1 - \wp\right) \eta\left(\varpi_1, \varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right)\right| \\ &\leqslant \wp\ \left|\Lambda\left(\varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right)\right| + \left(1 - \wp\right) \left|\Lambda\left(\varpi_1\right)\right|, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \left|\Lambda\left(\varpi_1 + \left(1 - \wp\right) \eta\left(\varpi_2, \varpi_1\right)\right)\right| &= \left|\Lambda\left(\varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right) + \wp\eta\left(\varpi_1, \varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right)\right| \\ &\leqslant \left(1 - \wp\right) \left|\Lambda\left(\varpi_1 + \eta\left(\varpi_2, \varpi_1\right)\right)\right| + \wp \left|\Lambda\left(\varpi_1\right)\right|. \end{aligned}$$

There are numerous vector functions that meet the condition $\mathcal{C}$ in [3], which trivial case $\eta\left(\varpi_1, \varpi_2\right) = \varpi_1 - \varpi_2$. For example suppose $\mathfrak{K} = \mathbb{R} \backslash \{0\}$ and

$$\eta\left(\varpi_2, \varpi_1\right) = \begin{cases} \varpi_2 - \varpi_1, & \text{if } \varpi_1 > 0, \varpi_2 > 0, \\ \varpi_2 - \varpi_1, & \text{if } \varpi_1 < 0, \varpi_2 < 0, \\ -\varpi_2, & \text{otherwise}. \end{cases}$$

The set $\mathfrak{K}$ is invex set and the condition $\mathcal{C}$ is satisfied by η.

Noor [24], demonstrated the following H-H type inequalities.

Theorem 2.4. *Assume that function* $\Lambda : \mathfrak{K} = [\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)] \to (0, \infty)$ *is a preinvex on* $\mathfrak{K}^\circ$ *with* $\eta(\varpi_2, \varpi_1) > 0$. *Then*

$$\Lambda\left(\frac{2c_1 + \eta\left(\varpi_2, \varpi_1\right)}{2}\right) \leqslant \frac{1}{\eta\left(\varpi_2, \varpi_1\right)} \int_{\varpi_1}^{\varpi_1 + \eta\left(\varpi_2, \varpi_1\right)} \Lambda\left(x\right) dx \leqslant \frac{\Lambda\left(\varpi_1\right) + \Lambda\left(\varpi_2\right)}{2}.$$

The history of fractional calculus can be traced back to the question of whether the significance of a derivative in integer order can be extended to a fractional order that is not an integer. Following this unique discussion between Leibniz and L'Hôpital, the idea of fractional calculus piqued the interest of several notable academics, including Euler, Laplace, Fourier, Lacroix, Abel, Riemann, and Liouville. As kernel structures have advanced, fractional operators have become more distinct in terms of singularity, locality, and having generic forms. In this regard, several new developments based on the fundamental ideas of analysis in the frame of fractional, R-L, and Caputo-Fabrizio have been effective. Fractional inequalities are useful for constructing the quantitative and qualitative properties of convex functions. There is a growing interest in such a field of study as dealing with the challenges of various applications of these versions.

Definition 2.5 ([11])**.** Suppose $\Lambda \in \mathcal{L}[\varpi_1, \varpi_2]$. The left-sided and right-sided R-L fractional integrals of order $\rho > 0$ ae respectively defined by

$$J^{\rho}_{\varpi_1}\Lambda(\tau) = \frac{1}{\Gamma(\rho)}\int_{\varpi_1}^{\tau}(\tau-\mu)^{\rho-1}\Lambda(\mu)d\mu, \qquad \varpi_1 < \tau,$$

and

$$J^{\rho}_{\varpi_2}\Lambda(\tau) = \frac{1}{\Gamma(\rho)}\int_{\tau}^{\varpi_2}(\mu-\tau)^{\rho-1}\Lambda(\mu)d\mu, \qquad \tau < \varpi_2.$$

Gamma function is defined as $\Gamma(\rho) = \int_0^{\infty} e^{-u}u^{\rho-1}du$. Note that $J^{0}_{\varpi_1}\Lambda(\tau) = J^{0}_{\varpi_2}\Lambda(\tau) = \Lambda(\tau)$.

Throughout the paper, we will consider that $\Gamma(.)$ is the gamma function and $\rho > 0$.

3. Main results

Lemma 3.1. *Let an open invex subset $\mathfrak{K} \subseteq \mathbb{R}$ w.r.t. $\eta : \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathbb{R}$ and $\varpi_1, \varpi_2 \in \mathfrak{K}$ with $\varpi_1 < \varpi_1 + \eta(\varpi_2, \varpi_1)$. Assume that $\Lambda : \mathfrak{K} \to \mathbb{R}$ is a differentiable function on $\mathfrak{K}$such that $\Lambda' \in \mathcal{L}([\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)])$. Then*

$$\begin{aligned}&\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1 + \frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\\ &\quad= \eta(\varpi_2,\varpi_1)\left[-\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho}\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \int_{\frac{\rho}{\rho+1}}^{1}(1-\wp^{\rho})\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp\right].\end{aligned} \tag{3.1}$$

Proof. By applying the partial integration to the right hand side of (3.1) we have

$$\begin{aligned}&\eta(\varpi_2,\varpi_1)\left[-\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho}\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \int_{\frac{\rho}{\rho+1}}^{1}(1-\wp^{\rho})\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp\right]\\ &\quad= \eta(\varpi_2,\varpi_1)\left[-\int_0^{1}\wp^{\rho}\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \int_{\frac{\rho}{\rho+1}}^{1}\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp\right]\\ &\quad= -\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) + \rho\int_0^1\wp^{\rho-1}\Lambda(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\\ &\qquad -\Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\\ &\quad= \frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right).\end{aligned}$$

This ends the proof. □

Lemma 3.2. *Let an open invex subset $\mathfrak{K} \subseteq \mathbb{R}$ w.r.t. $\eta : \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathbb{R}$ and $\varpi_1, \varpi_2 \in \mathfrak{K}$ with $\varpi_1 < \varpi_1 + \eta(\varpi_2, \varpi_1)$. Assume that $\Lambda : \mathfrak{K} \to \mathbb{R}$ is a twice differentiable function on $\mathfrak{K}$such that $\Lambda'' \in \mathcal{L}([\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)])$. Then*

$$\begin{aligned}&\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1) + J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\\ &\quad - \frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\\ &\quad= \int_0^{\frac{1}{2}}\wp^{\rho}\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \int_{\frac{1}{2}}^{1}(1-\wp)^{\rho}\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp.\end{aligned} \tag{3.2}$$

Proof. It suffices to write that

$$I=\int_0^{\frac{1}{2}}\wp^{\rho}\Lambda''\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp+\int_{\frac{1}{2}}^{1}\left(1-\wp\right)^{\rho}\Lambda''\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp=I_1+I_2, \tag{3.3}$$

where

$$\begin{aligned}I_1&=\int_0^{\frac{1}{2}}\wp^{\rho}\Lambda''\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp\\&=\frac{\wp^{\rho}\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)}{\eta\left(\varpi_2,\varpi_1\right)}\Bigg|_0^{\frac{1}{2}}-\frac{\rho}{\eta\left(\varpi_2,\varpi_1\right)}\int_0^{\frac{1}{2}}\wp^{\rho-1}\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp\\&=\frac{\Lambda'\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)}{2^{\rho}\eta\left(\varpi_2,\varpi_1\right)}-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)}{2^{\rho-1}\left[\eta\left(\varpi_2,\varpi_1\right)\right]^2}+\frac{\Gamma\left(\rho+1\right)}{\left[\eta\left(\varpi_2,\varpi_1\right)\right]^{\rho+1}}J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)^-}\Lambda\left(\varpi_1\right),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}I_2&=\int_{\frac{1}{2}}^{1}\left(1-\wp\right)^{\rho}\Lambda''\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp\\&=\frac{\left(1-\wp\right)^{\rho}\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)}{\eta\left(\varpi_2,\varpi_1\right)}\Bigg|_{\frac{1}{2}}^{1}+\frac{\rho}{\eta\left(\varpi_2,\varpi_1\right)}\int_{\frac{1}{2}}^{1}\left(1-\wp\right)^{\rho-1}\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp\\&=-\frac{\Lambda'\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)}{2^{\rho}\eta\left(\varpi_2,\varpi_1\right)}+\frac{\rho}{\eta\left(\varpi_2,\varpi_1\right)}\int_{\frac{1}{2}}^{1}\left(1-\wp\right)^{\rho-1}\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)d\wp\\&=-\frac{\Lambda'\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)}{2^{\rho}\eta\left(\varpi_2,\varpi_1\right)}-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)}{2^{\rho-1}\left[\eta\left(\varpi_2,\varpi_1\right)\right]^2}\\&\quad+\frac{\Gamma\left(\rho+1\right)}{\left[\eta\left(\varpi_2,\varpi_1\right)\right]^{\rho+1}}J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta\left(\varpi_2,\varpi_1\right)\right)^+}\Lambda\left(\varpi_1+\eta\left(\varpi_2,\varpi_1\right)\right).\end{aligned} \tag{3.5}$$

Combining Eqs. (3.4) and (3.5) with (3.3), we get (3.2). □

Theorem 3.3. *Assume all conditions in Lemma 3.1 are satisfied. If $|\Lambda'|$ is preinvex on $[\varpi_1,\varpi_1+\eta\left(\varpi_2,\varpi_1\right)]$, then, for fractional integrals, the following inequality with $\rho>0$ holds*

$$\begin{aligned}&\left|\frac{\Gamma\left(\rho+1\right)}{\left[\eta\left(\varpi_2,\varpi_1\right)\right]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda\left(\varpi_1+\eta\left(\varpi_2,\varpi_1\right)\right)-\Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta\left(\varpi_2,\varpi_1\right)\right)\right|\\&\leqslant\eta\left(\varpi_2,\varpi_1\right)\left[-\frac{\rho}{2\left(\rho+1\right)^2\left(\rho+2\right)}\left|\Lambda'\left(\varpi_1\right)\right|+\frac{\rho\left(-\rho^{\rho+1}+2\rho^{\rho}+\left(\rho+1\right)^{\rho}\right)}{2\left(\rho+1\right)^{\rho+2}\left(\rho+2\right)}\left|\Lambda'\left(\varpi_2\right)\right|\right].\end{aligned} \tag{3.6}$$

Proof. From (3.1) and the preinvexity of $|\Lambda'|$, we have

$$\begin{aligned}&\left|\frac{\Gamma\left(\rho+1\right)}{\left[\eta\left(\varpi_2,\varpi_1\right)\right]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda\left(\varpi_1+\eta\left(\varpi_2,\varpi_1\right)\right)-\Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta\left(\varpi_2,\varpi_1\right)\right)\right|\\&\leqslant\eta\left(\varpi_2,\varpi_1\right)\left[\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho}\left|\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)\right|d\wp+\int_{\frac{\rho}{\rho+1}}^{1}\left(1-\wp^{\rho}\right)\left|\Lambda'\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)\right|d\wp\right]\\&\leqslant\eta(\varpi_2,\varpi_1)\left[\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho}\left\{(1-\wp)\left|\Lambda'(\varpi_1)\right|+\wp\left|\Lambda'(\varpi_2)\right|\right\}d\wp\right.\\&\quad\left.+\int_{\frac{\rho}{\rho+1}}^{1}\left(1-\wp^{\rho}\right)\left\{(1-\wp)\left|\Lambda'(\varpi_1)\right|+\wp\left|\Lambda'(\varpi_2)\right|\right\}d\wp\right]\end{aligned}$$

$$\leqslant \eta(\varpi_2,\varpi_1)\left[|\Lambda'(\varpi_1)|\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho}(1-\wp)d\wp+|\Lambda'(\varpi_2)|\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho+1}d\wp\right.$$
$$\left.+|\Lambda'(\varpi_1)|\int_{\frac{\rho}{\rho+1}}^1(1-\wp^{\rho})(1-\wp)d\wp+|\Lambda'(\varpi_2)|\int_{\frac{\rho}{\rho+1}}^1(1-\wp^{\rho})\wp d\wp\right]$$
$$=\eta(\varpi_2,\varpi_1)\left[\frac{\rho}{2(\rho+1)^2(\rho+2)}|\Lambda'(\varpi_1)|+\frac{\rho\left(-\rho^{\rho+1}+2\rho^{\rho}+(\rho+1)^{\rho}\right)}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_2)|\right],$$

where

$$\int_0^{\frac{\rho}{\rho+1}}\wp^{\rho+1}d\wp=\frac{\rho^{\rho+2}}{(\rho+1)^{\rho+2}(\rho+2)},$$
$$\int_0^{\frac{\rho}{\rho+1}}\left(\wp^{\rho}-\wp^{\rho+1}\right)d\wp=\frac{2\rho^{\rho+1}}{(\rho+1)^{\rho+2}(\rho+2)},$$
$$\int_{\frac{\rho}{\rho+1}}^1\left(\wp-\wp^{\rho+1}\right)d\wp=\frac{2\rho^{\rho+1}+\rho(\rho+1)^{\rho}}{2(\rho+1)^{\rho+2}(\rho+2)},$$
$$\int_{\frac{\rho}{\rho+1}}^1(1-\wp^{\rho})(1-\wp)d\wp=\frac{4\rho^{\rho+1}-\rho(\rho+1)^{\rho}}{2(\rho+1)^{\rho+2}(\rho+2)}.$$

This ends the proof. □

Remark 3.4.

1. In (3.6), if we take $\eta(\varpi_2,\varpi_1)=\varpi_2-\varpi_1$ and $\rho=1$, then we get the inequality proved in [18, Theorem 2.2].
2. If η satisfies the condition $\mathcal{C}$, then by definition of the preinvexity of $|\Lambda'|$, we get

$$\begin{aligned}\left|\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|&=\left|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))+(1-\wp)\eta(\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1))\right|\\&\leqslant\wp\left|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))\right|+(1-\wp)\left|\Lambda'(\varpi_1)\right|.\end{aligned}\tag{3.7}$$

Using inequality (3.7) in proof of Theorem 3.3, the inequality (3.6) becomes the following

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))-\Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right|\\&\leqslant\eta(\varpi_2,\varpi_1)\left[\frac{\rho}{2(\rho+1)^2(\rho+2)}|\Lambda'(\varpi_1)|+\frac{\rho\left(-\rho^{\rho+1}+2\rho^{\rho}+(\rho+1)^{\rho}\right)}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))|\right].\end{aligned}\tag{3.8}$$

We observe that by employing the preinvexity of $|\Lambda'|$,

$$\left|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))\right|\leqslant\left|\Lambda'(\varpi_2)\right|.$$

Therefore, inequality (3.8) is better than the inequality (3.6).

Theorem 3.5. *Assume all conditions in Lemma 3.1 are satisfied. If* $|\Lambda'|^q$ *is preinvex on* $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ *for* $q\geqslant 1$, *then, for fractional integrals, the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho}}J^{\rho}_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))-\Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right|\\&\leqslant\eta(\varpi_2,\varpi_1)\left(\frac{\rho^{\rho+1}}{(\rho+1)^{\rho+2}}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2\rho^{\rho+1}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q-\frac{\rho^{\rho+2}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{4\rho^{\rho+1}-\rho(\rho+1)^{\rho}}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q+\frac{2\rho^{\rho+1}+\rho(\rho+1)^{\rho}}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{3.9}$$

where $p^{-1}=1-q^{-1}$.

Proof. From (3.1), utilizing power-mean inequality and definition of preinvexity of $|\Lambda'|^q$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^\rho} J^\rho_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right| \\
&\leqslant \eta(\varpi_2,\varpi_1)\left[\int_0^{\frac{\rho}{\rho+1}} \wp^\rho \Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp + \int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\,d\wp\right] \\
&\leqslant \eta(\varpi_2,\varpi_1)\left\{\left(\int_0^{\frac{\rho}{\rho+1}} \wp^\rho\,d\wp\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\rho}{\rho+1}} \wp^\rho|\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))|\,d\wp\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)\,d\wp\right)^{1-\frac{1}{q}}\left(\int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)|'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))|\,d\wp\right)^{\frac{1}{q}}\right\} \\
&\leqslant \eta(\varpi_2,\varpi_1)\left(\frac{\rho^{\rho+1}}{(\rho+1)^{\rho+2}}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^{\frac{\rho}{\rho+1}} \wp^\rho((1-\wp)|\Lambda'(\varpi_1)|^q+\wp|\Lambda'(\varpi_2)|^q)\,d\wp\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)((1-\wp)|\Lambda'(\varpi_1)|^q+\wp|\Lambda'(\varpi_2)|^q)\,d\wp\right\} \\
&= \eta(\varpi_2,\varpi_1)\left(\frac{\rho^{\rho+1}}{(\rho+1)^{\rho+2}}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2\rho^{\rho+1}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q - \frac{\rho^{\rho+2}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{4\rho^{\rho+1}-\rho(\rho+1)^\rho}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q + \frac{2\rho^{\rho+1}+\rho(\rho+1)^\rho}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This ends the proof. □

Remark 3.6.

1. In (3.9), if we set $\eta(\varpi_2,\varpi_1)=\varpi_2-\varpi_1$ and $\rho=1$, then we get the following midpoint type inequality

$$\begin{aligned}
&\left|\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\Lambda(x)\,dx - \Lambda\left(\frac{\varpi_1+\varpi_2}{2}\right)\right| \\
&\leqslant \frac{\varpi_2-\varpi_1}{8}\left\{\left(\frac{|\Lambda'(\varpi_1)|^q+2\left|\Lambda'(\varpi_2)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{2|\Lambda'(\varpi_1)|^q+|\Lambda'(\varpi_2)|^q}{3}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.10}$$

2. In (3.9), considering that η meets the condition $\mathcal{C}$ and using inequality (2.2), we get

$$\begin{aligned}
&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^\rho} J^\rho_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right| \\
&\leqslant \eta(\varpi_2,\varpi_1)\left(\frac{\rho^{\rho+1}}{(\rho+1)^{\rho+2}}\right)^{1-\frac{1}{q}} \\
&\quad \times\left\{\left(\frac{2\rho^{\rho+1}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q - \frac{\rho^{\rho+2}}{(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{4\rho^{\rho+1}-\rho(\rho+1)^\rho}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1)|^q + \frac{2\rho^{\rho+1}+\rho(\rho+1)^\rho}{2(\rho+1)^{\rho+2}(\rho+2)}|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 3.7. *Assume all conditions in Lemma 3.1 are satisfied. If* $|\Lambda'|^q$ *is preinvex on* $[\varpi_1, \varpi_1 + \eta(\varpi_2, \varpi_1)]$ *for* $q > 1$, *then, for fractional integrals, the following inequality is satisfied:*

$$\begin{aligned} &\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^\rho} J^\rho_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right| \\ &\leqslant \eta(\varpi_2,\varpi_1)\left\{\Big(M(\rho,p)\Big)^{\frac{1}{p}}\left(\frac{\rho^2+2\rho}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q - \frac{\rho^2}{2(\rho+1)^2}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \Big(N(\rho,p)\Big)^{\frac{1}{p}}\left(\frac{1}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q + \frac{2\rho+1}{2(\rho+1)^2}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right\}, \end{aligned} \tag{3.11}$$

where

$$M(\rho,p) = \int_0^{\frac{\rho}{\rho+1}} \wp^{\rho p} d\wp \quad \textit{and} \quad N(\rho,p) = \int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)^p \, d\wp,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From (3.1), Hölder integral inequality, and preinvexity of $|\Lambda'|^q$, we have

$$\begin{aligned} &\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^\rho} J^\rho_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right| \\ &\leqslant \eta(\varpi_2,\varpi_1)\left[\int_0^{\frac{\rho}{\rho+1}} \wp^\rho \left|\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right| d\wp + \int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)\left|\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right| d\wp\right] \\ &\leqslant \eta(\varpi_2,\varpi_1)\left\{\left(\int_0^{\frac{\rho}{\rho+1}} \wp^{\rho p} d\wp\right)^{\frac{1}{p}}\left(\int_0^{\frac{\rho}{\rho+1}} |\Lambda'(\varpi_1+t\eta(\varpi_2,\varpi_1))|^q d\wp\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)^p d\wp\right)^{\frac{1}{p}}\left(\int_{\frac{\rho}{\rho+1}}^1 |\Lambda'(\varpi_1+\wp\eta(\varpi_2,\varpi_1))|^q d\wp\right)^{\frac{1}{q}}\right\} \\ &\leqslant \eta(\varpi_2,\varpi_1)\left\{\left(\int_0^{\frac{\rho}{\rho+1}} \wp^{\rho p} d\wp\right)^{\frac{1}{p}}\left(\int_0^{\frac{\rho}{\rho+1}} \{(1-\wp)|\Lambda'(\varpi_1)|^q + t|\Lambda'(\varpi_2)|^q\} d\wp\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\int_{\frac{\rho}{\rho+1}}^1 (1-\wp^\rho)^p d\wp\right)^{\frac{1}{p}}\left(\int_{\frac{\rho}{\rho+1}}^1 \{(1-\wp)|\Lambda'(\varpi_1)|^q + \wp|\Lambda'(\varpi_2)|^q\} d\wp\right)^{\frac{1}{q}}\right\} \\ &= \eta(\varpi_2,\varpi_1)\left\{\Big(M(\rho,p)\Big)^{\frac{1}{p}}\left(-\frac{\rho^2+2\rho}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q \frac{\rho^2}{2(\rho+1)^2}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \Big(N(\rho,p)\Big)^{\frac{1}{p}}\left(\frac{1}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q + \frac{2\rho+1}{2(\rho+1)^2}|\Lambda'(\varpi_2)|^q\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

This ends the proof. □

Remark 3.8.

1. In (3.11), if we take $\eta(\varpi_2, \varpi_1) = \varpi_2 - \varpi_1$ and $\rho = 1$, then we get the inequality proved in [18, Theorem 2.3].
2. In (3.11), considering that η meets the condition $\mathcal{C}$ and using inequality (2.2), we get

$$\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^\rho} J^\rho_{\varpi_1^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1)) - \Lambda\left(\varpi_1+\frac{\rho}{\rho+1}\eta(\varpi_2,\varpi_1)\right)\right|$$

$$\leqslant \eta(\varpi_2,\varpi_1)\left\{\left(M(\rho,p)\right)^{\frac{1}{p}}\left(\frac{\rho^2+2\rho}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q-\frac{\rho^2}{2(\rho+1)^2}|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(N(\rho,p)\right)^{\frac{1}{p}}\left(\frac{1}{2(\rho+1)^2}|\Lambda'(\varpi_1)|^q+\frac{2\rho+1}{2(\rho+1)^2}|\Lambda'(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\right)^{\frac{1}{q}}\right\}.$$

Theorem 3.9. *Assume all conditions in Lemma 3.2 are satisfied. If $|\Lambda''|^q$ is preinvex on $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ for $q>1$, then, for fractional integrals, the following inequality is satisfied:*

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1))^-}\Lambda(\varpi_1)+J^{\rho-1}_{(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1))^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.\\&\qquad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|\\&\leqslant\left(\frac{2^{-\rho-1}}{\rho+1}\right)^{\frac{1}{p}}\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{(\rho+3)2^{-\rho-2}}{(\rho+1)(\rho+2)}\right)+|\Lambda''(\varpi_2)|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)\right]^{\frac{1}{q}}\right.\\&\qquad\left.+\left[|\Lambda''(\varpi_1)|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)+|\Lambda''(\varpi_2)|^q\left(\frac{4-(\rho+3)2^{-\rho}}{4(\rho+1)(\rho+2)}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.12}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (3.2) and the famous Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1))^-}\Lambda(\varpi_1)+J^{\rho-1}_{(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1))^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.\\&\qquad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|\\&\leqslant\int_0^{\frac{1}{2}}\wp^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|d\wp+\int_{\frac{1}{2}}^1(1-\wp)^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|d\wp\\&\leqslant\left(\int_0^{\frac{1}{2}}\wp^\rho d\wp\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\wp^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|^q d\wp\right)^{\frac{1}{q}}\\&\qquad+\left(\int_0^{\frac{1}{2}}(1-\wp)^\rho d\wp\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(1-\wp)^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|^q d\wp\right)^{\frac{1}{q}}.\end{aligned}\tag{3.13}$$

Since $|\Lambda''|^q$ is preinvex function on $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$, we have

$$\begin{aligned}\int_0^{\frac{1}{2}}\wp^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|^q d\wp&\leqslant\int_0^{\frac{1}{2}}\wp^\rho\left\{(1-\wp)\left|\Lambda''(\varpi_1)\right|^q+\wp\left|\Lambda''(\varpi_2)\right|^q\right\}d\wp\\&\leqslant\left|\Lambda''(\varpi_1)\right|^q\left(\frac{(\rho+3)2^{-\rho-2}}{(\rho+1)(\rho+2)}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^1(1-\wp)^\rho\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|^q d\wp&\leqslant\int_{\frac{1}{2}}^1(1-\wp)^\rho\left\{(1-\wp)\left|\Lambda''(\varpi_1)\right|^q+\wp\left|\Lambda''(\varpi_2)\right|^q\right\}d\wp\\&\leqslant\left|\Lambda''(\varpi_1)\right|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{4-(\rho+3)2^{-\rho}}{4(\rho+1)(\rho+2)}\right).\end{aligned}\tag{3.15}$$

Using equations (3.14) and (3.15) in (3.13), we get the result of (3.12), which completes the proof. □

Remark 3.10. In (3.12), considering that η meets the condition $\mathcal{C}$ and using inequality (2.2), we get

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.\\&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|\\&\leqslant\left(\frac{2^{-\rho-1}}{\rho+1}\right)^{\frac{1}{p}}\times\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{(\rho+3)2^{-\rho-2}}{(\rho+1)(\rho+2)}\right)+|\Lambda''(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)\right]^{\frac{1}{q}}\right.\\&\quad\left.+\left[|\Lambda''(\varpi_1)|^q\left(\frac{2^{-\rho-2}}{\rho+2}\right)+|\Lambda''(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\left(\frac{4-(\rho+3)2^{-\rho}}{4(\rho+1)(\rho+2)}\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Theorem 3.11. *Assume all conditions in Lemma 3.2 are satisfied. If $|\Lambda''|^q$ is preinvex function on $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$ for $q>1$, $q\geqslant r$, $s\geqslant 0$, then, for fractional integrals, the following inequality is satisfied:*

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.\\&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|\\&\leqslant\left(\frac{1}{2^{\rho\left(\frac{q-r}{q-1}\right)+1}.\rho\left(\frac{q-r}{q-1}\right)+1}\right)^{1-\frac{1}{q}}\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{(\rho r+3)2^{-\rho r-2}}{(\rho r+1)(\rho r+2)}\right)+|\Lambda''(\varpi_2)|^q\left(\frac{2^{-\rho r-2}}{\rho r+2}\right)\right]^{\frac{1}{q}}\right\}\\&\quad+\left(\frac{1}{2^{\rho\left(\frac{q-s}{q-1}\right)+1}.\rho\left(\frac{q-s}{q-1}\right)+1}\right)^{1-\frac{1}{q}}\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{2^{-\rho s-2}}{\rho s+2}\right)+|\Lambda''(\varpi_2)|^q\left(\frac{(\rho s+3)e^{-\ln(2)\rho s}}{4(\rho s+1)(\rho s+2)}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.16}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (3.2) and Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.\\&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|\\&\leqslant\left\{\left(\int_0^{\frac{1}{2}}\wp^{\rho\left(\frac{q-r}{q-1}\right)}d\wp\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\rho^{\rho r}\left|\Lambda''(\varpi_1+\rho\eta(\varpi_2,\varpi_1))\right|^q d\wp\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_{\frac{1}{2}}^1(1-\wp)^{\rho\left(\frac{q-s}{q-1}\right)}d\wp\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-\rho)^{\rho s}\left|\Lambda''(\varpi_1+\rho\eta(\varpi_2,\varpi_1))\right|^q d\wp\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.17}$$

Since $|\Lambda''|^q$ is a preinvex function on $[\varpi_1,\varpi_1+\eta(\varpi_2,\varpi_1)]$, we have

$$\begin{aligned}\int_0^{\frac{1}{2}}\wp^{\rho r}\left|\Lambda''(\varpi_1+\wp\eta(\varpi_2,\varpi_1))\right|^q d\wp&\leqslant\int_0^{\frac{1}{2}}\wp^{\rho r}\left\{(1-\rho)\left|\Lambda''(\varpi_1)\right|^q+\wp\left|\Lambda''(\varpi_2)\right|^q\right\}d\wp\\&\leqslant\left|\Lambda''(\varpi_1)\right|^q\left(\frac{(\rho r+3)2^{-\rho r-2}}{(\rho r+1)(\rho r+2)}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{2^{-\rho r-2}}{\rho r+2}\right)\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-\wp)^{\rho s}\left|\Lambda''\left(\varpi_1+\wp\eta\left(\varpi_2,\varpi_1\right)\right)\right|^q d\wp \\
&\leqslant \int_{\frac{1}{2}}^{1}(1-\wp)^{\rho s}\left\{(1-\wp)\left|\Lambda''\left(\varpi_1\right)\right|^q+\wp\left|\Lambda''\left(\varpi_2\right)\right|^q\right\} d\wp \\
&\leqslant \left|\Lambda''\left(\varpi_1\right)\right|^q\left(\frac{2^{-\rho s-2}}{\rho s+2}\right)+\left|\Lambda''\left(\varpi_2\right)\right|^q\left(\frac{(\rho s+3)\, e^{-\ln(2)\rho s}}{4(\rho s+1)(\rho s+2)}\right).
\end{aligned} \tag{3.19}$$

Using equations (3.18) and (3.19) in (3.17), we get (3.16). This ends the proof. □

Remark 3.12. In (3.16), considering that η meets the condition $\mathcal{C}$ and using inequality (2.2), we get

$$\begin{aligned}
&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda\left(\varpi_1+\eta(\varpi_2,\varpi_1)\right)\right\}\right. \\
&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right| \\
&\leqslant\left(\frac{1}{2^{\rho\left(\frac{q-r}{q-1}\right)+1}.\rho\left(\frac{q-r}{q-1}\right)+1}\right)^{1-\frac{1}{q}}\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{(\rho r+3)2^{-\rho r-2}}{(\rho r+1)(\rho r+2)}\right)+|\Lambda''(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\left(\frac{2^{-\rho r-2}}{\rho r+2}\right)\right]^{\frac{1}{q}}\right\} \\
&\quad+\left(\frac{1}{2^{\rho\left(\frac{q-s}{q-1}\right)+1}.\rho\left(\frac{q-s}{q-1}\right)+1}\right)^{1-\frac{1}{q}} \\
&\quad\times\left\{\left[|\Lambda''(\varpi_1)|^q\left(\frac{2^{-\rho s-2}}{\rho s+2}\right)+|\Lambda''(\varpi_1+\eta(\varpi_2,\varpi_1))|^q\left(\frac{(\rho s+3)e^{-\ln(2)\rho s}}{4(\rho s+1)(\rho s+2)}\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 3.13. *Under the same conditions of* (3.16), *when* $r=s$, *we have*

$$\begin{aligned}
&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda\left(\varpi_1+\eta(\varpi_2,\varpi_1)\right)\right\}\right. \\
&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right| \\
&\leqslant\left(\frac{1}{2^{\rho\left(\frac{q-r}{q-1}\right)+1}.\rho\left(\frac{q-r}{q-1}\right)+1}\right)^{1-\frac{1}{q}}\left\{\left\{\left[\left|\Lambda''(\varpi_1)\right|^q\left(\frac{(\rho r+3)\,2^{-\rho r-2}}{(\rho r+1)(\rho r+2)}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{2^{-\rho r-2}}{\rho r+2}\right)\right]^{\frac{1}{q}}\right\}\right. \\
&\quad\left.+\left\{\left[\left|\Lambda''(\varpi_1)\right|^q\left(\frac{2^{-\rho r-2}}{\rho r+2}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{(\rho r+3)\, e^{-\ln(2)\rho r}}{4(\rho r+1)(\rho r+2)}\right)\right]^{\frac{1}{q}}\right\}\right\}.
\end{aligned}$$

Corollary 3.14. *Under the same conditions of* (3.16), *when* $r=0=s$, *we have*

$$\begin{aligned}
&\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(\varpi_1)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda\left(\varpi_1+\eta(\varpi_2,\varpi_1)\right)\right\}\right. \\
&\quad\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right| \\
&\leqslant\left(\frac{1}{2^{\rho\left(\frac{q}{q-1}\right)+1}.\ \rho\left(\frac{q}{q-1}\right)+1}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left\{\left(\left|\Lambda''(\varpi_1)\right|^q\frac{3}{8}+\left|\Lambda''(\varpi_2)\right|^q\frac{1}{8}\right)^{\frac{1}{q}}+\left(\left|\Lambda''(\varpi_1)\right|^q\frac{1}{8}+\left|\Lambda''(\varpi_2)\right|^q\frac{3}{8}\right)^{\frac{1}{q}}\right\}.$$

Corollary 3.15. *Under the same conditions of* (3.16), *when* $r = s = q$, *we have*

$$\left|\frac{\Gamma(\rho+1)}{[\eta(\varpi_2,\varpi_1)]^{\rho+1}}\left\{J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^-}\Lambda(a)+J^{\rho-1}_{\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)^+}\Lambda(\varpi_1+\eta(\varpi_2,\varpi_1))\right\}\right.$$
$$\left.-\frac{\rho\Lambda\left(\varpi_1+\frac{1}{2}\eta(\varpi_2,\varpi_1)\right)}{2^{\rho-2}[\eta(\varpi_2,\varpi_1)]^2}\right|$$
$$\leqslant\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\left|\Lambda''(\varpi_1)\right|^q\left(\frac{(\rho q+3)2^{-\rho q-2}}{(\rho q+1)(\rho q+2)}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{2^{-\rho q-2}}{(\rho q+2)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\Lambda''(\varpi_1)\right|^q\left(\frac{2^{-\rho q-2}}{(\rho q+2)}\right)+\left|\Lambda''(\varpi_2)\right|^q\left(\frac{(\rho q+3)e^{-\ln(2)\rho q}}{4(\rho q+1)(\rho q+2)}\right)\right)^{\frac{1}{q}}\right\}.$$

4. Applications to some special functions

4.1. *q-digamma function*

For $0 < q < 1$, the mathematically q-digamma function φ_q (see [14, 34]) is given as:

$$\varphi_q = -\ln(1-q)+\ln q\sum_{k=0}^{\infty}\frac{q^{k+\zeta}}{1-q^{k+\zeta}} = -\ln(1-q)+\ln q\sum_{k=0}^{\infty}\frac{q^{k\zeta}}{1-q^{k\zeta}}.$$

For $q > 1$ and $\zeta > 0$, q−digamma function φ_q can be given as:

$$\varphi_q = -\ln(q-1)+\ln q\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{q^{-(k+\zeta)}}{1-q^{-(k+\zeta)}}\right] = -\ln(q-1)+\ln q\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{q^{-k\zeta}}{1-q^{-k\zeta}}\right].$$

Proposition 4.1. *Assume that* $\varpi_1,\varpi_2\in\mathbb{R}$ *such that* $0<\varpi_1<\varpi_2$ *and* $0<q<1$. *Then*

$$\left|\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\varphi_q(\varepsilon)\,d\varepsilon-\varphi_q\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\leqslant\left(\frac{\varpi_2-\varpi_1}{8}\right)\left\{\left(\frac{\left|\varphi_q^{(1)}(\varpi_1)\right|^q+2\left|\varphi_q^{(1)}(\varpi_2)\right|^q}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2\left|\varphi_q^{(1)}(\varpi_1)\right|^q+\left|\varphi_q^{(1)}(\varpi_2)\right|^q}{3}\right)^{\frac{1}{q}}\right\}.$$

Proof. The assertion can be obtained immediately from equation (3.10), when $\Lambda(\varepsilon) = \varphi_q(\varepsilon)$ and $\varepsilon > 0$, since $\Lambda'(\varepsilon) = \varphi_q'(\varepsilon)$ is convex on $(0, +\infty)$. □

4.2. *Modified Bessel function*

First we add the mathematical form of modified Bessel function $\mathfrak{I}_m$, in a first sense, which is given by (see [34, p.77])

$$\mathfrak{I}_m(\zeta) = \Sigma_{n\geqslant 0}\frac{\left(\frac{\zeta}{2}\right)^{m+2n}}{n!\Gamma(m+n+1)},$$

where $\zeta\in\mathbb{R}$ and $m > -1$, while the mathematical form of modified Bessel function $\mathfrak{K}_m$ in second sense (see [34, p.78]) is usually explored as

$$\mathfrak{K}_m(\zeta) = \frac{\pi}{2}\frac{\mathfrak{I}_{-m}(\zeta)-\mathfrak{I}_m(\zeta)}{\sin m\pi}.$$

Consider the function $\Omega_{m}(\zeta) : \mathfrak{R} \to [1, \infty)$ defined by

$$\Omega_{m}(\zeta) = 2^{m}\Gamma(m+1)\zeta^{-m}\mathfrak{I}_{m}(\zeta),$$

where Γ is the gamma function. The 1st order derivative formula of $\Omega_{m}(\zeta)$ is given by [34]:

$$\Omega'_{m}(\zeta) = \frac{\zeta}{2(m+1)}\Omega_{m+1}(\zeta) \tag{4.1}$$

and the 2nd derivative can be attained easily from (4.1) to be

$$\Omega''_{m}(\zeta) = \frac{\zeta^{2}\Omega_{m+2}(\zeta)}{4(m+1)(m+2)} + \frac{\Omega_{m+1}(\zeta)}{2(m+1)}. \tag{4.2}$$

Proposition 4.2. *Suppose that* $m > -1$ *and* $0 < \varpi_1 < \varpi_2$. *Then we have*

$$\begin{aligned}
&\left|\frac{\varpi_1+\varpi_2}{4(m+1)}\Omega_{m+1}\left(\frac{\varpi_1+\varpi_2}{8}\right) - \frac{\Omega_{m}(\varpi_2)-\Omega_{m}(\varpi_1)}{\varpi_2-\varpi_1}\right| \\
&\leqslant \frac{\varpi_2-\varpi_1}{8}\left[\frac{1}{3}\left\{\left(\frac{\varpi_1^{2}\Omega_{m+2}(\varpi_1)}{4(m+1)(m+2)} + \frac{\Omega_{m+1}(\varpi_1)}{2(m+1)}\right)^{q} + 2\left(\frac{\varpi_2^{2}\Omega_{m+2}(\varpi_2)}{4(m+1)(m+2)} + \frac{\Omega_{m+1}(\varpi_2)}{2(m+1)}\right)^{q}\right\}^{\frac{1}{q}}\right. \\
&\quad\left. + \frac{1}{3}\left\{2\left(\frac{\varpi_1^{2}\Omega_{m+2}(\varpi_1)}{4(m+1)(m+2)} + \frac{\Omega_{m+1}(\varpi_1)}{2(m+1)}\right)^{q} + \left(\frac{\varpi_2^{2}\Omega_{m+2}(\varpi_2)}{4(m+1)(m+2)} + \frac{\Omega_{m+1}(\varpi_2)}{2(m+1)}\right)^{q}\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Applying the inequality in (3.10) to the mapping $\Lambda(\zeta) = \Omega'_{m}(\zeta)$, $\zeta > 0$ (note that all assumptions are satisfied) and the identities (4.1) and (4.2), we get the result. □

5. Conclusion

The literature on integral inequalities pertaining to fractional operators has grown to be a rich source of interest for many scholars in numerous domains as a result of the possible applications that fractional calculus holds. When compared to convex functions, improvements and extensions accomplished with preinvex functions result in better and sharper bounds. Last but not least, the novel idea of the R-L operator for the preinvexity has multiple potential applications and is significant in the field of applied sciences. In this paper, first we constructed few fractional identities. Using these identities and the new notation, we deduce some parameterized inequalities of the Hermite–Hadamard type relevant to the R-L fractional integrals. Additionally, several instances are given to show that the results are accurate. With the assistance of Hölder and Power mean inequality, we acquired the modifications of Hermite-Hadamard inequality that gave the work more aesthetic appeal. Our outcomes offer advancements and extensions of prior research, which encourage forward-looking study. Some new concepts such as interval-valued LR convexities, fuzzy interval convexities, and CR convexities can be used to establish further generalizations. Interested researchers can also utilize quantum calculus, coordinated interval-valued functions, fractional calculus, etc to see the behavior of these inequalities as well. It will be quite interesting to see how different types of new convex functions can be applied to investigate the fractional versions of inequalities.

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