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Spherical interval valued fuzzy ideals which coincide in semigroups

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Abstract

The concept of spherical fuzzy set was introduced by Gun et al. in 2018. It is generalization of Pythagorean fuzzy set. Our main paper, we give the concepts of spherical interval valued fuzzy ideals in semigroups, and properties of a spherical fuzzy ideal in semigroups with prove. Moreover, we investigate necessary and sufficient conditions of coincidences spherical interval valued fuzzy ideals in semigroups.

Keywords: Spherical fuzzy set, spherical interval valued fuzzy set, spherical interval valued fuzzy ideals.

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1. Introduction

There are numerous uncertain, imprecise, and incomplete problems in the real world. Zadeh's fuzzy set theory is a successful and effective tool to solve many above (similar) problems. The concept of fuzzy sets was published by Zadeh in 1965 [13]. In 1979, Kuroki [8] studied fuzzy subsemigroups and various kinds of fuzzy ideals in semigroups. Later in 1975 Zadeh [14] studied theory interval valued fuzzy set have been successfully applied to pattern recognition medical diagnosis[3], fuzzy logic, decision-making image processing [1] and decision making method [15] and so on. Biswas [2] used the ideal of interval valued fuzzy sets to interval valued subgroups in 1994. In 2006, Narayanan and Manikantan [10] were developed theory of an interval valued fuzzy set to interval valued fuzzy subsemigroups and types interval valued fuzzy ideals in semigroups. In 2012, Kim et al. [7] gave the concepts of interval valued fuzzy quasi-ideals on semigroups and they studied of its properties. As another extension of fuzzy set theory, Yager [12] proposed another class of nonstandard fuzzy sets, called Pythagorean fuzzy sets. The sets are represented by pairs of two values $\langle \eta(x), \vartheta(x) \rangle$, which satisfies $0 \leqslant (\eta(x))^2 + (\vartheta(x))^2 \leqslant 1$. Gun et al. [6] gave the concept of spherical fuzzy set and studied properties it. The spherical fuzzy set is a generalization of the picture fuzzy sets and Pythagorean fuzzy sets. In 2020 Veerappan and Venkatesan

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[11] studied propeties of spherical interval valued fuzzy bi-ideals of Γ near-rings. In 2022 Krailoet et al. [9] studued spherical fuzzy sets and rough sets in ternaray semigroups. Recently, Chinnadual et al. [4] introduced spherical fuzzy ideals and discussed properties spherical fuzzy ideals in semigroups.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts and results of semigroup, fuzzy sets, interval valued fuzzy sets, Pythagorean fuzzy sets and spherical fuzzy sets. Section 3, proposes the definition of types spherical interval valued fuzzy ideals and we investigate necessary and sufficient conditions of coincidences spherical interval valued fuzzy ideals in semigroups. In Section 4, we discuss some basic properties of spherical interval valued fuzzy ideals of a spherical fuzzy ideal in semigroups.

2. Preliminaries

To assemble this work self sufficient, we briefly introduce a few definitions engaged in the remaining work.

A *subsemigroup* of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}^2 \subseteq \mathfrak{K}$. A *left (right) ideal* of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{G}\mathfrak{K} \subseteq \mathfrak{K}$ ($\mathfrak{K}\mathfrak{G} \subseteq \mathfrak{K}$). By an *ideal* of a semigroup $\mathfrak{S}$, we mean a non-empty set of $\mathfrak{S}$ which is both a left and a right ideal of $\mathfrak{G}$. A *generalized bi-ideal* of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}\mathfrak{G}\mathfrak{K} \subseteq \mathfrak{K}$. A *qausi-ideal* of a semigroup $\mathfrak{G}$ is a non-empty set $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{K}\mathfrak{G} \cap \mathfrak{G}\mathfrak{K} \subseteq \mathfrak{K}$. A subsemigroup $\mathfrak{K}$ of a semigroup $\mathfrak{G}$ is called a *bi-ideal (interior ideal,* $(1,2)$*-ideal)* of $\mathfrak{S}$ if $\mathfrak{K}\mathfrak{G}\mathfrak{K} \subseteq \mathfrak{K}$ ($\mathfrak{G}\mathfrak{K}\mathfrak{G} \subseteq \mathfrak{K}$, $\mathfrak{K}\mathfrak{G}\mathfrak{K}^2 \subseteq \mathfrak{K}$). A semigroup $\mathfrak{G}$ is said to be *regular* if for each element $u \in \mathfrak{G}$, there exists an element $x \in \mathfrak{G}$ such that $u = uxu$. A semigroup $\mathfrak{G}$ is called *intra-regular* if for every $u \in \mathfrak{G}$ there exist $x, y \in \mathfrak{G}$ such that $u = xu^2y$. A semigroup $\mathfrak{G}$ is said to be *left* (right) regular if for each element $u \in \mathfrak{S}$, there exists an element $x \in \mathfrak{G}$ such that $u = xu^2 (u = u^2x)$. A semigroup $\mathfrak{G}$ is called *semisimple* if for every $u \in \mathfrak{G}$, there exist $x, y, z \in \mathfrak{G}$ such that $u = xuyuz$. A semigroup $\mathfrak{G}$ is called *weakly regular* if for every $u \in \mathfrak{G}$ there exist $x, y \in \mathfrak{G}$ such that $u = uxuy$. A semigroup $\mathfrak{G}$ is a *left (right) quasi-regular* if for every $u \in \mathfrak{G}$, there exist $x, y \in \mathfrak{G}$ such that $u = xuyu$ ($u = uxuy$).

For any $\eta_i \in [0,1]$, where $i \in \mathfrak{J}$, define

$$\bigvee_{i\in\mathfrak{J}} \eta_i := \sup_{i\in\mathfrak{J}}\{\eta_i\} \quad \text{and} \quad \bigwedge_{i\in\mathfrak{J}} \eta_i := \inf_{i\in\mathfrak{J}}\{\eta_i\}.$$

We see that for any $\eta_1, \eta_2 \in [0,1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \quad \text{and} \quad \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

Definition 2.1 ([13]). A fuzzy subset (fuzzy set) η of a non-empty set $\mathfrak{T}$ is a function from $\mathfrak{T}$ into the closed interval $[0,1]$, i.e., $\eta : \mathfrak{T} \to [0,1]$.

The set of all closed subintervals of $[0,1]$ is denoted by $\mathcal{C}$, that is,

$$\mathcal{C} = \{\overline{\eta} = [\eta^-, \eta^+] \mid 0 \leqslant \eta^- \leqslant \eta^+ \leqslant 1\}.$$

We note that $[\eta, \eta] = \{\eta\}$ for all $\eta \in [0,1]$. For $\eta = 0$ or 1 we shall denote $[0,0]$ by $\overline{0}$ and $[1,1]$ by $\overline{1}$. Let $\overline{\eta} = [\eta^-, \eta^+]$ and $\overline{\vartheta} = [\vartheta^-, \vartheta^+]$ in $\mathcal{C}$. Define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{\eta} \preceq \overline{\vartheta}$ if and only if $\eta^- \leqslant \vartheta^-$ and $\eta^+ \leqslant \vartheta^+$;
(2) $\overline{\eta} = \overline{\vartheta}$ if and only if $\eta^- = \vartheta^-$ and $\eta^+ = \vartheta^+$;
(3) $\overline{\eta} \curlywedge \overline{\vartheta} = [(\eta^- \wedge \vartheta^-), (\eta^+ \wedge \vartheta^+)]$;
(4) $\overline{\eta} \curlyvee \overline{\vartheta} = [(\eta^- \vee \vartheta^-), (\eta^+ \vee \vartheta^+)]$, if $\overline{\eta} \succeq \overline{\vartheta}$, we write $\overline{\vartheta} \preceq \overline{\eta}$.

Proposition 2.2 ([5]). *For any* $\overline{\eta}, \overline{\vartheta}, \overline{\varpi} \in \mathcal{C}$, *the following properties are true:*

(1) $\overline{\eta} \curlywedge \overline{\eta} = \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\eta} = \overline{\eta}$;

(2) $\overline{\eta} \curlywedge \overline{\vartheta} = \overline{\vartheta} \curlywedge \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\vartheta} = \overline{\vartheta} \curlyvee \overline{\eta}$*;*
(3) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlywedge \overline{\varpi} = \overline{\eta} \curlywedge (\overline{\vartheta} \curlywedge \overline{\varpi})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlyvee \overline{\varpi} = \overline{\eta} \curlyvee (\overline{\vartheta} \curlyvee \overline{\varpi})$*;*
(4) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlyvee \overline{\varpi} = (\overline{\eta} \curlyvee \overline{\varpi}) \curlywedge (\overline{\vartheta} \curlyvee \overline{\varpi})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlywedge \overline{\varpi} = (\overline{\eta} \curlywedge \overline{\varpi}) \curlyvee (\overline{\vartheta} \curlywedge \overline{\varpi})$*;*
(5) *if* $\overline{\eta} \preceq \overline{\vartheta}$*, then* $\overline{\eta} \curlywedge \overline{\varpi} \preceq \overline{\vartheta} \curlywedge \overline{\varpi}$ *and* $\overline{\eta} \curlyvee \overline{\varpi} \preceq \overline{\vartheta} \curlyvee \overline{\varpi}$*.*

Definition 2.3 ([14])**.** An interval valued fuzzy subset (shortly, IVF subset) of a non-empty set $\mathfrak{T}$ is a function $\overline{\eta} : \mathfrak{T} \to \mathcal{C}$.

Definition 2.4 ([10])**.** Let $\mathfrak{K}$ be a subset of a non-empty set $\mathfrak{T}$. An interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ of $\mathfrak{K}$ is defined to be a function $\overline{\chi}_{\mathfrak{K}} : \mathfrak{K} \to \mathcal{C}$ by

$$\overline{\chi}_{\mathfrak{K}}(u) = \begin{cases} \overline{1}, & \text{if } u \in \mathfrak{K}, \\ \overline{0}, & \text{if } u \notin \mathfrak{K}, \end{cases}$$

for all $u \in \mathfrak{K}$.

For two IVF subsets $\overline{\eta}$ and $\overline{\vartheta}$ of a non-empty set $\mathfrak{T}$, define

(1) $\overline{\eta} \sqsubseteq \overline{\vartheta} \Leftrightarrow \overline{\eta}(u) \preceq \overline{\vartheta}(u)$ for all $u \in \mathfrak{T}$;
(2) $\overline{\eta} = \overline{\vartheta} \Leftrightarrow \overline{\eta} \sqsubseteq \overline{\vartheta}$ and $\overline{\vartheta} \sqsubseteq \overline{\eta}$;
(3) $(\overline{\eta} \sqcap \overline{\vartheta})(u) = \overline{\eta}(u) \curlywedge \overline{\vartheta}(u)$ for all $u \in \mathfrak{T}$.

For two IVF subsets $\overline{\eta}$ and $\overline{\vartheta}$ of a semigroup $\mathfrak{G}$, define the product $\overline{\eta} \circ \overline{\vartheta}$ as follows: for all $u \in \mathfrak{G}$,

$$(\overline{\eta} \circ \overline{\vartheta})(u) = \begin{cases} \displaystyle\curlyvee_{(x,y) \in F_u} \{\overline{\eta}(x) \curlywedge \overline{\vartheta}(y)\}, & \text{if } F_u \neq \emptyset, \\ \overline{0}, & \text{if } F_u = \emptyset, \end{cases}$$

where $F_u := \{(x, y) \in \mathfrak{G} \times \mathfrak{G} \mid u = xy\}$.

Definition 2.5 ([10])**.** An IVF subset $\overline{\eta}$ of a semigroup $\mathfrak{G}$ is said to be

(1) an *IVF subsemigroup* of $\mathfrak{G}$ if $\overline{\eta}(uv) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$ for all $u, v \in \mathfrak{G}$;
(2) an *IVF left (right) ideal* of $\mathfrak{G}$ if $\overline{\eta}(uv) \succeq \overline{\eta}(v)$ $(\overline{\eta}(uv) \succeq \overline{\eta}(u))$ for all $u, v \in \mathfrak{G}$;
(3) an *IVF ideal* of $\mathfrak{G}$ if it is both an IVF left ideal and an IVF right ideal of $\mathfrak{G}$.

Definition 2.6 ([10])**.** An IVF subsemigroup $\overline{\eta}$ of a semigroup $\mathfrak{G}$ is said to be

(1) an *IVF bi-ideal* of $\mathfrak{G}$ if $\overline{\eta}(uvw) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w)$ for all $u, v, w \in \mathfrak{G}$;
(2) an *IVF interior ideal* of $\mathfrak{G}$ if $\overline{\eta}(uvw) \succeq \overline{\eta}(v)$ for all $u, v, w \in \mathfrak{G}$.

Definition 2.7 ([7])**.** An IVF subset $\overline{\eta}$ of a semigroup $\mathfrak{G}$ is said to be an *IVF quasi-ideal* of $\mathfrak{G}$ if $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\eta})(u) \curlywedge (\overline{\eta} \circ \overline{\mathfrak{G}})(u)$ for all $u \in \mathfrak{G}$.

Definition 2.8 ([12])**.** Let $\mathfrak{T}$ be a non-empty set. A *Pythagorean fuzzy set* (PFS) $P := \{u, \eta(u), \vartheta(u) \mid u \in \mathfrak{G}\}$, where $\eta : \mathfrak{T} \to [0, 1]$ and $\vartheta : \mathfrak{T} \to [0, 1]$ represent the degree of membership and non-membership of the object $z \in \mathfrak{T}$ to the set P subset to the condition $0 \leqslant (\eta(u))^2 + (\vartheta(u))^2 \leqslant 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity a PFS is denoted as $P = (\eta(u); \vartheta(u))$.

Definition 2.9 ([6])**.** Let $\mathfrak{T}$ be a non-empty set. A *spherical fuzzy set* (SFS) $SP := \{u, \eta(u), \vartheta(u), \omega(u) \mid u \in \mathfrak{G}\}$, where $\eta : \mathfrak{T} \to [0, 1]$, $\vartheta : \mathfrak{T} \to [0, 1]$ and $\omega : \mathfrak{T} \to [0, 1]$ represent the degree of membership, non-membership and hesitancy of the object $u \in \mathfrak{T}$ to the set SP subset to the condition $0 \leqslant (\eta(u))^2 + (\vartheta(u))^2 + (\omega(u))^2 \leqslant 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity a PFS is denoted as $SP = (\eta(u); \vartheta(u), \omega(u))$.

3. Spherical interval valued fuzzy ideal in semigroups

In this section, we will study concepts of spherical interval valued fuzzy set in a semigroup and we study properties of those.

Definition 3.1. Let $\mathfrak{T}$ be a non-empty set. A *spherical interval valued fuzzy set* (SIVF-set) $\overline{SP} := \{u, \overline{\eta}(u), \overline{\vartheta}(u), \overline{\varpi}(u) \mid u \in \mathfrak{G}\}$, where $\overline{\eta} : \mathfrak{T} \to \mathcal{C}$, $\overline{\vartheta} : \mathfrak{T} \to \mathcal{C}$ and $\overline{\varpi} : \mathfrak{T} \to \mathcal{C}$ represent the degree of membership, non-membership and hesitancy of the object $u \in \mathfrak{T}$ to the set $\overline{SP}$ subset to the condition $\overline{0} \leqslant (\overline{\eta}(u))^2 + (\overline{\vartheta}(u))^2 + (\overline{\varpi}(u))^2 \leqslant \overline{1}$ for all $u \in \mathfrak{T}$. For the sake of simplicity a PFS is denoted as $\overline{SP} := (\overline{\eta}, \overline{\vartheta}, \overline{\varpi})$.

Definition 3.2. Let $\mathfrak{K}$ be a subset of a non-empty set $\mathfrak{T}$. A spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ of $\mathfrak{K}$ is defined to be a function $\overline{\chi}_{\mathfrak{K}} : \mathfrak{K} \to \mathcal{C}$ by

$$\overline{\eta}_{\chi_{\mathfrak{K}}}(u) := \begin{cases} \overline{1}, & u \in \mathfrak{K}, \\ \overline{0}, & u \notin \mathfrak{K}, \end{cases} \quad \text{and} \quad \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) := \begin{cases} \overline{1}, & u \in \mathfrak{K}, \\ \overline{0}, & u \notin \mathfrak{K}, \end{cases} \quad \text{and} \quad \overline{\varpi}_{\chi_{\mathfrak{K}}}(u) := \begin{cases} \overline{0}, & u \in \mathfrak{K}, \\ \overline{1}, & u \notin \mathfrak{K}, \end{cases}$$

for all $u \in \mathfrak{K}$.

Remark 3.3. To simplify matters, we will employ the symbol $\overline{\chi}_{\mathfrak{K}} = (\overline{\eta}_{\chi_{\mathfrak{K}}}, \overline{\vartheta}_{\chi_{\mathfrak{K}}}, \overline{\varpi}_{\chi_{\mathfrak{K}}})$ for the IF set $\overline{\chi}_{\mathfrak{K}} := \{(u, \overline{\eta}_{\chi_{\mathfrak{K}}}(u), \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u), \overline{\varpi}_{\chi_{\mathfrak{K}}}(u),) \mid u \in \mathfrak{T}\}$.

Definition 3.4. Let $\overline{SP}_1 = (\overline{\eta}, \overline{\vartheta}, \overline{\varpi})$ and $\overline{SP}_2 = (\overline{\tau}, \overline{\nu}, \overline{\alpha})$ be two SIVF-set of a semigroup $\mathfrak{G}$. Then the product $\overline{SP}_1 \overline{\circ} \overline{SP}_2$ is defined by

$$\overline{SP}_1 \overline{\circ} \overline{SP}_2 := \{u, \overline{\eta} \,\overline{\circ}\, \overline{\tau}(u), \overline{\vartheta} \,\overline{\circ}\, \overline{\nu}(u), \overline{\varpi} \,\overline{\circ}\, \overline{\alpha}(u) : u \in \mathfrak{G}\},$$

where

$$(\overline{\eta} \,\overline{\circ}\, \overline{\tau})(u) = \begin{cases} \underset{(x,y) \in F_u}{\curlyvee} \{\overline{\eta}(x) \curlywedge \overline{\tau}(y)\}, & \text{if } F_u \neq \emptyset, \\ \overline{0}, & \text{if } F_u = \emptyset, \end{cases}$$

$$(\overline{\vartheta} \,\overline{\circ}\, \overline{\nu})(u) = \begin{cases} \underset{(x,y) \in F_u}{\curlyvee} \{\overline{\vartheta}(x) \curlywedge \overline{\nu}(y)\}, & \text{if } F_u \neq \emptyset, \\ \overline{0}, & \text{if } F_u = \emptyset, \end{cases}$$

$$(\overline{\varpi} \,\overline{\circ}\, \overline{\alpha})(u) = \begin{cases} \underset{(x,y) \in F_u}{\curlywedge} \{\overline{\varpi}(x) \curlyvee \overline{\alpha}(y)\}, & \text{if } F_u \neq \emptyset, \\ \overline{1}, & \text{if } F_u = \emptyset, \end{cases}$$

for all $u \in \mathfrak{G}$.

Definition 3.5. A SIVF-set $\overline{SP}$ of a semigroup $\mathfrak{G}$ is called

(1) a *spherical interval valued fuzzy subsemigroup* (SIVFS) if $\overline{\eta}(uv) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)$, and $\overline{\varpi}(uv) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(v)$ for all $u, v \in \mathfrak{G}$;
(2) a *spherical interval valued fuzzy left* (SIVFL) if $\overline{\eta}(uv) \succeq \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(v)$, and $\overline{\varpi}(uv) \preceq \overline{\varpi}(v)$ for all $u, v \in \mathfrak{G}$;
(3) a *spherical interval valued fuzzy right* (SIVFR) if $\overline{\eta}(uv) \succeq \overline{\eta}(u)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u)$, and $\overline{\varpi}(uv) \preceq \overline{\varpi}(u)$ for all $u, v \in \mathfrak{G}$;
(4) a *spherical interval valued fuzzy ideal* (SIVFI) if it is both a SIVFL and SIVFR of $\mathfrak{G}$;
(5) a *spherical interval valued fuzzy quasi-ideal* (SIVFQ) if $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}} \,\overline{\circ}\, \overline{\eta})(u) \curlywedge (\overline{\eta} \,\overline{\circ}\, \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}(u) \succeq (\overline{\mathfrak{G}} \,\overline{\circ}\, \overline{\vartheta})(u) \curlywedge (\overline{\vartheta} \,\overline{\circ}\, \overline{\mathfrak{G}})(u)$, and $\overline{\varpi}(u) \preceq (\overline{\mathfrak{G}} \,\overline{\circ}\, \overline{\varpi})(u) \curlyvee (\overline{\varpi} \,\overline{\circ}\, \overline{\mathfrak{G}})(u)$ for all $u \in \mathfrak{G}$.

It is claerly every SIVFI of a semigroup $\mathfrak{G}$ is SIVFS of $\mathfrak{G}$.

Definition 3.6. A SIVFS $\overline{SP}$ of a semigroup $\mathfrak{G}$ is called

(1) a *spherical interval valued fuzzy bi-ideal* (SIVFB) if $\overline{\eta}(uvw) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w)$, $\overline{\vartheta}(uvw) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w)$, and $\overline{\varpi}(uvw) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(w)$ for all $u, v, w \in \mathfrak{G}$;

(2) a *spherical interval valued fuzzy interior ideal* (SIVFII) if $\overline{\eta}(uvw) \succeq \overline{\eta}(v)$, $\overline{\vartheta}(uvw) \succeq \overline{\vartheta}(v)$, and $\overline{\varpi}(uvw) \preceq \overline{\varpi}(v)$ for all $u, v, w \in \mathfrak{G}$;

(3) a *spherical interval valued fuzzy (1,2)-ideal* (SIVF (1,2)-ideal) if $\overline{\eta}(uv(wy) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w) \curlywedge \overline{\eta}(y)$, $\overline{\vartheta}(uv(wy) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w) \curlywedge \overline{\vartheta}(y)$, and $\overline{\varpi}(uv(wy) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(w) \curlyvee \overline{\varpi}(y)$ for all $u, v, w, y \in \mathfrak{G}$.

Example 3.7. Let $\mathfrak{G} = \{\Psi, \Omega, \Upsilon, \Pi\}$ be semigroup with the following Cayley table:

$\cdot$	Ψ	Ω	Υ	Π
Ψ	Ψ	Ω	Υ	Π
Ω	Ω	Ψ	Π	Υ
Υ	Υ	Π	Ω	Ψ
Π	Π	Υ	Ω	Ψ

Define SIVF-set $\overline{\eta} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\eta}(\Psi) = [0.2, 0.3]$, $\overline{\eta}(\Omega) = [0.3, 0.5]$, $\overline{\eta}(\Upsilon) = [0.7, 0.8]$, $\overline{\eta}(\Pi) = [0.5, 0.8]$; $\overline{\vartheta} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\vartheta}(\Psi) = [0.2, 0.4]$, $\overline{\vartheta}(\Omega) = [0.5, 0.6]$, $\overline{\vartheta}(\Upsilon) = [0.6, 0.7]$, $\overline{\vartheta}(\Pi) = [0.7, 0.9]$; and $\overline{\varpi} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\varpi}(\Psi) = [0.1, 0.3]$, $\overline{\varpi}(\Omega) = [0.4, 0.5]$, $\overline{\varpi}(\Upsilon) = [0.5, 0.7]$, $\overline{\varpi}(\Pi) = [0.5, 0.7]$. Then $\overline{SP}$ is a SIVFB of $\mathfrak{G}$.

Example 3.8. Let $\mathfrak{G} = \{\Psi, \Omega, \Upsilon, \Pi\}$ be semigroup with the following Cayley table:

$\cdot$	Ψ	Ω	Υ	Π
Ψ	Ψ	Ψ	Ψ	Ψ
Ω	Ψ	Ψ	Ψ	Ψ
Υ	Ψ	Ψ	Ω	Ψ
Π	Ψ	Ψ	Ψ	Ω

Define SIVF-set $\overline{\eta} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\eta}(\Psi) = [0.5, 0.6]$, $\overline{\eta}(\Omega) = [0.3, 0.4]$, $\overline{\eta}(\Upsilon) = [0.2, 0.3]$, $\overline{\eta}(\Pi) = [0.3, 0.4]$; $\overline{\vartheta} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\vartheta}(\Psi) = [0.4, 0.5]$, $\overline{\vartheta}(\Omega) = [0.2, 0.3]$, $\overline{\vartheta}(\Upsilon) = [0.1, 0.2]$, $\overline{\vartheta}(\Pi) = [0.2, 0.3]$; and $\overline{\varpi} : \mathfrak{G} \to \mathcal{C}$ by $\overline{\varpi}(\Psi) = [0.3, 0.4]$, $\overline{\varpi}(\Omega) = [0.4, 0.5]$, $\overline{\varpi}(\Upsilon) = [0.3, 0.4]$, $\overline{\varpi}(\Pi) = [0.6, 0.7]$. Then $\overline{SP}$ is a SIVFI of $\mathfrak{G}$.

The following lemma shows that every SIVFI is a SIVFB of a semigroups.

Lemma 3.9. *Every SIVFI of a semigroup $\mathfrak{G}$ is a SIVFB of $\mathfrak{G}$.*

Proof. Suppose that $\overline{SP}$ is a SIVFI of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\overline{SP}$ is a SIVFI of $\mathfrak{G}$, we have that $\overline{SP}$ is a SIVFR of $\mathfrak{G}$. Thus,

$$\overline{\eta}(uv) \succeq \overline{\eta}(u), \quad \overline{\vartheta}(uv) \succeq \overline{\vartheta}(u), \quad \text{and} \quad \omega(uv) \preceq \omega(u),$$

and so $\overline{\eta}(uv) \succeq \overline{\eta}(u) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)$ and $\omega(uv) \preceq \omega(u) \preceq \omega(u) \curlyvee \omega(v)$. Hence, $\overline{SP}$ is a SIVFS of $\mathfrak{G}$. Let $u, v, w \in \mathfrak{G}$. Since $\overline{SP}$ is a SIVFI of $\mathfrak{G}$, we have that $\overline{SP}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\eta}(uvw) = \overline{\eta}((uv)w) \succeq \overline{\eta}(w)$, $\overline{\vartheta}(uvw) = \overline{\vartheta}((uv)w) \succeq \overline{\vartheta}(w)$, and $\overline{\varpi}(uvw) = \overline{\varpi}((uv)w) \preceq \overline{\varpi}(w)$ and so $\overline{\eta}(uvw) \succeq \overline{\eta}(w) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w)$, $\overline{\vartheta}(uvw) \succeq \overline{\vartheta}(w) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w)$, and $\omega(uvw) \preceq \omega(w) \preceq \omega(u) \curlyvee \omega(w)$. Hence $\overline{SP}$ is a SIVFB of $\mathfrak{G}$. □

In order to consider the converse of Lemma 3.9, we need to strengthen the condition of a semigroup.

Theorem 3.10. *In a regular semigroup $\mathfrak{G}$, the SIVFBs and the SIVFIs coincide.*

Proof. Suppose that $\overline{SP}$ is a SIVFB of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is regular, we have $uv \in (u\mathfrak{G}u)\mathfrak{G} \subseteq u\mathfrak{G}$, which implies that $uv = uku$ for some $k \in \mathfrak{G}$. Thus, $\overline{\eta}(uv) = \overline{\eta}(uku) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(u) = \overline{\eta}(u)$, $\overline{\vartheta}(uv) = \overline{\vartheta}(uku) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(u) = \overline{\vartheta}(u)$ and $\overline{\varpi}(uv) = \overline{\varpi}(uku) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(u) = \overline{\varpi}(u)$. Hence $\overline{SP}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{SP}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{SP}$ is a SIVFI of $\mathfrak{G}$. □

The following lemma shows that every SIVFB is a SIVF $(1, 2)$-ideal on a semigroup.

Lemma 3.11. *Every SIVFB of a semigroup $\mathfrak{S}$ is a SIVF $(1,2)$-ideal of $\mathfrak{S}$.*

Proof. Suppose that $\overline{SP}$ is a SIVFB of $\mathfrak{S}$ and let $a, u, v, w \in \mathfrak{S}$. Then

$$\begin{aligned}
\overline{\eta}(ua(vw)) &= \overline{\eta}((uav)w) \succeq (\overline{\eta}(uav) \curlywedge \overline{\eta}(w)) \succeq ((\overline{\eta}(u) \curlywedge \overline{\eta}(v)) \curlywedge \overline{\eta}(w)) = \overline{\eta}(u) \curlywedge \overline{\eta}(v) \curlywedge \overline{\eta}(w),\\
\overline{\vartheta}(ua(vw)) &= \overline{\vartheta}((uav)w) \succeq (\overline{\vartheta}(uav) \curlywedge \overline{\vartheta}(w)) \succeq ((\overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)) \curlywedge \overline{\vartheta}(w)) = \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v) \curlywedge \overline{\vartheta}(w),\\
\overline{\varpi}(ua(vw)) &= \overline{\varpi}((uav)w) \preceq (\overline{\varpi}(uav) \curlyvee \overline{\varpi}(w)) \preceq ((\overline{\varpi}(u) \curlyvee \overline{\varpi}(v)) \curlyvee \overline{\varpi}(w)) = \overline{\varpi}(u) \curlyvee \overline{\varpi}(v) \curlyvee \overline{\varpi}(w).
\end{aligned}$$

Hence, $\overline{SP}$ is a SIVF $(1,2)$-ideal of $\mathfrak{S}$. □

In order to consider the converse of Lemma 3.11, we need to strengthen the condition of a semigroup.

Theorem 3.12. *In a regular semigroup $\mathfrak{S}$, the SIVF $(1,2)$-ideals and the SIVFBs coincide.*

Proof. Assume that $\overline{SP}$ is a SIVF $(1,2)$-ideal of $\mathfrak{S}$ and let $u, v, w \in \mathfrak{S}$. Since $\mathfrak{S}$ is regular, we have $uv \in (u\mathfrak{S}u)\mathfrak{S} \subseteq u\mathfrak{S}u$, which implies that $uv = usu$ for some $s \in \mathfrak{S}$. Thus,

$$\begin{aligned}
\overline{\eta}(uvw) &= \overline{\eta}((usu)w) = \overline{\eta}(us(uw)) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(u) \curlywedge \overline{\eta}(w) = \overline{\eta}(u) \curlywedge \overline{\eta}(w),\\
\overline{\vartheta}(uvw) &= \overline{\vartheta}((usu)w) = \overline{\vartheta}(us(uw)) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w) = \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w),\\
\overline{\varpi}(uvw) &= \overline{\varpi}((usu)w) = \overline{\varpi}(us(uw)) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(u) \curlyvee \overline{\varpi}(w) = \overline{\varpi}(u) \curlyvee \overline{\varpi}(w).
\end{aligned}$$

Hence, $\overline{SP}$ is a SIVFB of $\mathfrak{S}$. □

The following theorem shows that every SIVFI is a SIVF $(1,2)$-ideal on a semigroup.

Theorem 3.13. *Every SIVFI of a semigroup $\mathfrak{S}$ is a SIVF $(1,2)$-ideal of $\mathfrak{S}$.*

Proof. Suppose that $\overline{SP}$ is a SIVFI of $\mathfrak{S}$ and let $u, v \in \mathfrak{S}$. Since $\overline{SP}$ is a SIVFI of $\mathfrak{S}$, we have that $\overline{SP}$ is a SIVFR of $\mathfrak{S}$. Thus,

$$\overline{\eta}(uv) \succeq \overline{\eta}(u), \quad \overline{\vartheta}(uv) \succeq \overline{\vartheta}(u), \quad \text{and} \quad \omega(uv) \preceq \omega(u)$$

and so $\overline{\eta}(uv) \succeq \overline{\eta}(u) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)$ and $\omega(uv) \preceq \omega(u) \preceq \omega(u) \curlyvee \omega(v)$. Hence, $\overline{SP}$ is a SIVFS of $\mathfrak{S}$.
Let $a, u, v, w \in \mathfrak{S}$. Since $\overline{SP}$ is a SIVFI of $\mathfrak{S}$, we have that $\overline{SP}$ is a SIVFL of $\mathfrak{S}$. Thus,

$$\overline{\eta}(uvw) = \overline{\eta}((uv)w) \succeq \overline{\eta}(w), \quad \overline{\vartheta}(uvw) = \overline{\vartheta}((uv)w) \succeq \overline{\vartheta}(w), \quad \text{and} \quad \overline{\varpi}(uvw) = \overline{\varpi}((uv)w) \preceq \overline{\varpi}(w),$$

and so

$$\begin{aligned}
\overline{\eta}(ua(vw)) &\succeq \overline{\eta}(w) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v) \curlywedge \overline{\eta}(w),\\
\overline{\vartheta}(ua(vw)) &\succeq \overline{\vartheta}(w) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v) \curlywedge \overline{\vartheta}(w),\\
\overline{\varpi}(ua(vw)) &\preceq \overline{\varpi}(w) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(v) \curlyvee \overline{\varpi}(w).
\end{aligned}$$

Hence, $\overline{SP}$ is a SIVF $(1,2)$-ideal of $\mathfrak{S}$. □

The following lemma shows that every SIVFI is a SIVFII on a semigroup.

Lemma 3.14. *Every SIVFI of a semigroup $\mathfrak{S}$ is a SIVFII of $\mathfrak{S}$.*

Proof. Suppose that $\overline{SP}$ is a SIVFI of $\mathfrak{S}$ and let $u, v \in \mathfrak{S}$. Since $\overline{SP}$ is a SIVFI of $\mathfrak{S}$, we have that $\overline{SP}$ is a SIVFR of $\mathfrak{S}$. Thus,

$$\overline{\eta}(uv) \succeq \overline{\eta}(u), \quad \overline{\vartheta}(uv) \succeq \overline{\vartheta}(u), \quad \text{and} \quad \omega(uv) \preceq \omega(u),$$

and so $\overline{\eta}(uv) \succeq \overline{\eta}(u) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)$, and $\omega(uv) \preceq \omega(u) \subseteq \omega(u) \curlyvee \omega(v)$. Hence, $\overline{SP}$ is a SIVFS of $\mathfrak{S}$. Let $a, u, v \in \mathfrak{S}$. Then, $\overline{\eta}(uav) = \overline{\eta}(u(av)) \succeq \overline{\eta}(av) \succeq \overline{\eta}(a)$, $\overline{\vartheta}(uav) = \overline{\vartheta}(u(av)) \succeq \overline{\vartheta}(av) \succeq \overline{\vartheta}(a)$, and $\overline{\varpi}(uav) = \overline{\varpi}(u(av)) \preceq \overline{\varpi}(av) \preceq \overline{\varpi}(a)$. Thus, $\overline{\eta}(uav) \succeq \overline{\eta}(a)$, $\overline{\vartheta}(uav) \succeq \overline{\vartheta}(a)$, and $\overline{\varpi}(uav) \preceq \overline{\varpi}(a)$. Hence, $\overline{SP}$ is a SIVFII of $\mathfrak{S}$. □

In order to consider the converse of Lemma 3.14, we need to strengthen the condition of a semigroup $\mathfrak{G}$.

Lemma 3.15. *In a regular semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is regular, there exists $x \in \mathfrak{G}$ such that $u = uxu$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}((uxu)v) = \overline{\eta}((ux)uv) \succeq \overline{\eta}(u),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}((uxu)v) = \overline{\vartheta}((ux)uv) \succeq \overline{\vartheta}(u),\\ \overline{\varpi}(uv) &= \overline{\varpi}((uxu)v) = \overline{\varpi}((ux)uv) \preceq \overline{\varpi}(u).\end{aligned}$$

Hence, $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{\mathrm{SP}}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\mathrm{SP}}$ is a SIVFI of $\mathfrak{G}$. □

Lemma 3.16. *In a left (right) regular semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is left regular, there exists $k \in \mathfrak{G}$ such that $u = ku^2$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}((ku^2)v) = \overline{\eta}(kuuv) = \overline{\eta}((ku)uv) \succeq \overline{\eta}(u),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}((ku^2)v) = \overline{\vartheta}(kuuv) = \overline{\vartheta}((ku)uv) \succeq \overline{\vartheta}(u),\\ \overline{\varpi}(uv) &= \overline{\varpi}((ku^2)v) = \overline{\varpi}(kuuv) = \overline{\varpi}((ku)uv) \preceq \overline{\varpi}(u).\end{aligned}$$

Hence $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{\mathrm{SP}}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\mathrm{SP}}$ is a SIVFI of $\mathfrak{G}$. □

Lemma 3.17. *In an intra-regular semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is intra-regular, there exist $x, y \in \mathfrak{G}$ such that $u = xu^2y$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}((xu^2y)v) = \overline{\eta}((xuuy)v) = \overline{\eta}((xu)uv) \succeq \overline{\eta}(u),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}((xu^2y)v) = \overline{\vartheta}((xuuy)v) = \overline{\vartheta}((xu)uv) \succeq \overline{\vartheta}(u),\\ \overline{\varpi}(uv) &= \overline{\varpi}((xu^2)v) = \overline{\varpi}((xuuy)v) = \overline{\varpi}((xu)uv) \preceq \overline{\varpi}(u).\end{aligned}$$

Hence, $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{\mathrm{SP}}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\mathrm{SP}}$ is a SIVFI of $\mathfrak{G}$. □

Lemma 3.18. *In a semisiple semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is semisimple, there exist $x, y, z \in \mathfrak{G}$ such that $u = xuyuz$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}((xuyuz)v) = \overline{\eta}((xuy)u(zv)) \succeq \overline{\eta}(u),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}((xuyuz)v) = \overline{\vartheta}((xuy)u(zv)) \succeq \overline{\vartheta}(u),\\ \overline{\varpi}(uv) &= \overline{\varpi}((xuyuz)v) = \overline{\varpi}((xuy)u(zv)) \preceq \overline{\varpi}(u).\end{aligned}$$

Hence, $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{\mathrm{SP}}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\mathrm{SP}}$ is a SIVFI of $\mathfrak{G}$. □

Lemma 3.19. *In a let (right) quasi-regular semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is left quasi-regular, there exist $x, y \in \mathfrak{G}$ such that $v = xvyv$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}(u(xvyv)) = \overline{\eta}((ux)v(yv)) \succeq \overline{\eta}(v),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}(u(xvyv)) = \overline{\vartheta}((ux)v(yv)) \succeq \overline{\vartheta}(v),\\ \overline{\varpi}(uv) &= \overline{\varpi}(u(xvyv)) = \overline{\varpi}((ux)v(yv)) \preceq \overline{\varpi}(v).\end{aligned}$$

Hence, $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{\mathrm{SP}}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{\mathrm{SP}}$ is a SIVFI of $\mathfrak{G}$. □

Lemma 3.20. *In a weakly regular semigroup $\mathfrak{G}$, the SIVFIIs and the SIVFIs coincide.*

Proof. Suppose that $\overline{SP}$ is a SIVFII of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exist $p, q \in \mathfrak{G}$ such that $u = upuq$. Thus,

$$\begin{aligned}\overline{\eta}(uv) &= \overline{\eta}((upuq)v) = \overline{\eta}((up)u(qv)) \succeq \overline{\eta}(u),\\ \overline{\vartheta}(uv) &= \overline{\vartheta}((upuq)v) = \overline{\vartheta}((up)u(qv)) \succeq \overline{\vartheta}(u),\\ \overline{\varpi}(uv) &= \overline{\varpi}((upuq)v) = \overline{\varpi}((up)u(qv)) \preceq \overline{\varpi}(u).\end{aligned}$$

Hence, $\overline{SP}$ is a SIVFR of $\mathfrak{G}$. Similarly, we can show that $\overline{SP}$ is a SIVFL of $\mathfrak{G}$. Thus, $\overline{SP}$ is a SIVFI of $\mathfrak{G}$. □

By Lemmas 3.15, 3.16, 3.17, 3.18, 3.19, and 3.20, we have Theorem 3.21.

Theorem 3.21. *Let $\mathfrak{G}$ be a semigroup. If $\mathfrak{G}$ is regular, left (right) regular, intra-regular, semisimple, left (right) quasi-regular or weakly regular, then SIVFIIs and SIVFIs coincide.*

The following theorem shows that every SIVFI is a SIVFQ of a semigroup.

Theorem 3.22. *Every SIVFL (SIVFR) ideal of a semigroup $\mathfrak{G}$ is a SIVFQ of $\mathfrak{G}$.*

Proof. Suppose that $\overline{SP}$ is a SIVFL of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. If $F_u = \emptyset$, then it is easy to verify that $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\eta})(u) \curlywedge (\overline{\eta} \circ \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta})(u) \curlywedge (\overline{\vartheta} \circ \overline{\mathfrak{G}})(u)$ and $\overline{\varpi}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\varpi})(u) \curlyvee (\overline{\varpi} \circ \overline{\mathfrak{G}})(u)$. If $F_u \neq \emptyset$, then

$$\begin{aligned}(\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\eta})(u) &= \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\mathfrak{G}}(i) \curlywedge \overline{\eta}(j)\} = \underset{(i,j)\in F_u}{\curlyvee}\{\overline{1} \curlywedge \overline{\eta}(j)\} \preceq \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\eta}(j)\} \preceq \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\eta}(ij)\} \preceq \overline{\eta}(u),\\ (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\vartheta})(u) &= \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\mathfrak{G}}(i) \curlywedge \overline{\vartheta}(j)\} = \underset{(i,j)\in F_u}{\curlyvee}\{\overline{1} \curlywedge \overline{\vartheta}(j)\} \preceq \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\vartheta}(j)\} \preceq \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\vartheta}(ij)\} \preceq \overline{\eta}(u),\\ (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\varpi})(u) &= \underset{(i,j)\in F_u}{\curlywedge}\{\overline{\mathfrak{G}}(i) \curlyvee \overline{\varpi}(j)\} = \underset{(i,j)\in F_u}{\curlywedge}\{\overline{0} \curlyvee \overline{\varpi}(j)\} \succeq \underset{(i,j)\in F_u}{\curlywedge}\{\overline{\varpi}(j)\} \succeq \underset{(i,j)\in F_u}{\curlywedge}\{\overline{\varpi}(ij)\} \succeq \overline{\varpi}(u).\end{aligned}$$

Thus, $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\eta})(u)$, $\overline{\vartheta}(u) \succeq (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\vartheta})(u)$ and $(\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\varpi})(u) \preceq \overline{\varpi}(u)$ and so $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\eta})(u) \curlywedge (\overline{\eta}\,\overline{\circ}\,\overline{\mathfrak{G}})(u)$, $\overline{\vartheta}(u) \succeq (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\vartheta})(u) \curlywedge (\overline{\vartheta}\,\overline{\circ}\,\overline{\mathfrak{G}})(u)$ and $\overline{\varpi}(u) \preceq (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\varpi})(u) \curlyvee (\overline{\varpi}\,\overline{\circ}\,\overline{\mathfrak{G}})(u)$. Hence, $\overline{SP}$ is a SIVFQ of $\mathfrak{G}$. Similarly, if $\overline{SP}$ is a SIVFR of $\mathfrak{G}$, then $\overline{SP}$ is a SIVFQ of $\mathfrak{G}$. □

The following theorem shows that every SIVFQ is a SIVFS on a semigroup.

Theorem 3.23. *Every SIVFQ of a semigroup $\mathfrak{G}$ is a SIVFS of $\mathfrak{G}$.*

Proof. Assume that $\overline{SP}$ is a SIVFQ of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Then

$$\begin{aligned}\overline{\eta}(uv) &\succeq (\overline{\eta}\,\overline{\circ}\,\overline{\mathfrak{G}})(uv) \curlywedge (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\eta})(uv)\\ &= \underset{(p,q)\in F_{uv}}{\curlyvee}\{\overline{\eta}(p) \curlywedge \overline{\mathfrak{G}}(q)\} \curlywedge \underset{(a,b)\in F_{uv}}{\curlyvee}\{\overline{\mathfrak{G}}(a) \curlywedge \overline{\eta}(b)\}\\ &\succeq (\overline{\eta}(u) \curlywedge \overline{\mathfrak{G}}(v)) \curlywedge (\overline{\mathfrak{G}}(u)) \curlywedge \overline{\eta}(v) = (\overline{\eta}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{\eta}(v)) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v),\\ \overline{\vartheta}(uv) &\succeq (\overline{\vartheta}\,\overline{\circ}\,\overline{\mathfrak{G}})(uv) \curlywedge (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\vartheta})(uv)\\ &= \underset{(p,q)\in F_{uv}}{\curlyvee}\{\overline{\vartheta}(p) \curlywedge \overline{\mathfrak{G}}(q)\} \curlywedge \underset{(a,b)\in F_{uv}}{\curlyvee}\{\overline{\mathfrak{G}}(a) \curlywedge \overline{\vartheta}(b)\}\\ &\succeq (\overline{\vartheta}(u) \curlywedge \overline{\mathfrak{G}}(v)) \curlywedge (\overline{\mathfrak{G}}(u)) \curlywedge \overline{\vartheta}(v) = (\overline{\vartheta}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{\vartheta}(v)) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v),\\ \overline{\varpi}(uv) &\preceq (\overline{\varpi}\,\overline{\circ}\,\overline{\mathfrak{G}})(uv) \curlyvee (\overline{\mathfrak{G}}\,\overline{\circ}\,\overline{\varpi})(uv)\\ &= \underset{(p,q)\in F_{uv}}{\curlywedge}\{\overline{\vartheta}(p) \curlyvee \overline{\mathfrak{G}}(q)\} \curlyvee \underset{(a,b)\in F_{uv}}{\curlywedge}\{\overline{\mathfrak{G}}(a) \curlyvee \overline{\vartheta}(b)\}\\ &\preceq (\overline{\vartheta}(u) \curlyvee \overline{\mathfrak{G}}(v)) \curlyvee (\overline{\mathfrak{G}}(u)) \curlyvee \overline{\vartheta}(v) = (\overline{\vartheta}(u) \curlyvee \overline{0}) \curlyvee (\overline{0} \curlyvee \overline{\vartheta}(v)) \preceq \overline{\vartheta}(u) \curlyvee \overline{\vartheta}(v).\end{aligned}$$

Thus, $\overline{\eta}(uv) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v)$, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v)$, and $\overline{\varpi}(uv) \preceq \overline{\vartheta}(u) \curlyvee \overline{\vartheta}(v)$. Hence, $\overline{SP}$ is a SIVFS of $\mathfrak{G}$. □

The following theorem shows that every SIVFQ is a SIVFB on a semigroup.

Theorem 3.24. *Every SIVFQ of a semigroup $\mathfrak{G}$ is a SIVFB of $\mathfrak{G}$.*

Proof. Assume that $\overline{SP}$ is a SIVFQ of $\mathfrak{G}$ and let $u, v \in \mathfrak{G}$. Then by Theorem 3.23, $\overline{SP}$ is a SIVFS of $\mathfrak{G}$. Let u, v, w. Then

$$\begin{aligned}
\overline{\eta}(uvw) &\succeq (\overline{\eta} \circ \overline{\mathfrak{G}})(uvw) \curlywedge (\overline{\mathfrak{G}} \circ \overline{\eta})(uvw)\\
&= \curlyvee_{(p,q)\in F_{uvw}} \{\overline{\eta}(p) \curlywedge \overline{\mathfrak{G}}(q)\} \curlywedge \curlyvee_{(a,b)\in F_{uvw}} \{\overline{\mathfrak{G}}(a) \curlywedge \overline{\eta}(b)\}\\
&\succeq (\overline{\eta}(u) \curlywedge \overline{\mathfrak{G}}(vw)) \curlywedge (\overline{\mathfrak{G}}(uvw)) \curlywedge \overline{\eta}(w) = (\overline{\eta}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{\eta}(v)) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w),\\
\overline{\vartheta}(uvw) &\succeq (\overline{\vartheta} \circ \overline{\mathfrak{G}})(uvw) \curlywedge (\overline{\mathfrak{G}} \circ \overline{\vartheta})(uvw)\\
&= \curlyvee_{(p,q)\in F_{uvw}} \{\overline{\vartheta}(p) \curlywedge \overline{\mathfrak{G}}(q)\} \curlywedge \curlyvee_{(a,b)\in F_{uvw}} \{\overline{\mathfrak{G}}(a) \curlywedge \overline{\vartheta}(b)\}\\
&\succeq (\overline{\vartheta}(u) \curlywedge \overline{\mathfrak{G}}(vw)) \curlywedge (\overline{\mathfrak{G}}(uv)) \curlywedge \overline{\vartheta}(w) = (\overline{\vartheta}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{\vartheta}(w)) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w),\\
\overline{\varpi}(uvw) &\preceq (\overline{\varpi} \circ \overline{\mathfrak{G}})(uvw) \curlyvee (\overline{\mathfrak{G}} \circ \overline{\varpi})(uvw)\\
&= \curlywedge_{(p,q)\in F_{uvw}} \{\overline{\vartheta}(p) \curlyvee \overline{\mathfrak{G}}(q)\} \curlyvee \curlywedge_{(a,b)\in F_{uvw}} \{\overline{\mathfrak{G}}(a) \curlyvee \overline{\vartheta}(b)\}\\
&\preceq (\overline{\vartheta}(u) \curlyvee \overline{\mathfrak{G}}(vw)) \curlyvee (\overline{\mathfrak{G}}(uv)) \curlyvee \overline{\vartheta}(w) = (\overline{\vartheta}(u) \curlyvee \overline{0}) \curlyvee (\overline{0} \curlyvee \overline{\vartheta}(w)) \preceq \overline{\vartheta}(u) \curlyvee \overline{\vartheta}(w).
\end{aligned}$$

Thus, $\overline{\eta}(uvw) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w)$, $\overline{\vartheta}(uvw) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w)$ and $\overline{\varpi}(uvw) \preceq \overline{\vartheta}(u) \curlyvee \overline{\vartheta}(w)$. Hence, $\overline{SP}$ is a SIVFB of $\mathfrak{G}$. ☐

The following result is an immediate consequence of Theorem 3.24 and Lemma 3.11.

Corollary 3.25. *Every SIVFQ of a semigroup $\mathfrak{G}$ is a SIVF $(1,2)$-ideal of $\mathfrak{G}$.*

4. Some basic properties of spherical interval valued fuzzy ideals in semigroups

In this section, we prove properties of spherical interval valued fuzzy ideals in semigroups.

Theorem 4.1. *Let $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ be a family of SIVFSs of a semigroup $\mathfrak{G}$. Then $\curlywedge_{i\in\mathfrak{J}} \overline{SP}_i$ is a SIVFS of $\mathfrak{G}$.*

Proof. Let $u, v \in \mathfrak{G}$. Then,

$$\begin{aligned}
&\curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(uv) \succeq \curlywedge_{i\in\mathfrak{J}} (\overline{\eta}_i(u) \curlywedge \overline{\eta}_i(v)) = \curlywedge(\overline{\eta}_i(u) \curlywedge \overline{\eta}_i(v)) = (\curlywedge\overline{\eta}_i(u)) \curlywedge (\curlywedge\overline{\eta}_i(v)) = \curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(u) \curlywedge \curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(v),\\
&\curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(uv) \succeq \curlywedge_{i\in\mathfrak{J}} (\overline{\vartheta}_i(u) \curlywedge \overline{\vartheta}_i(v)) = \curlywedge(\overline{\vartheta}_i(u) \curlywedge \overline{\vartheta}_i(v)) = (\curlywedge\overline{\vartheta}_i(u)) \curlywedge (\curlywedge\overline{\vartheta}_i(v)) = \curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(u) \curlywedge \curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(v),\\
&\curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(uv) \preceq \curlyvee_{i\in\mathfrak{J}} (\overline{\varpi}_i(u) \curlyvee \overline{\varpi}_i(v)) = \curlyvee(\overline{\varpi}_i(u) \curlyvee \overline{\varpi}_i(v)) = (\curlyvee\overline{\varpi}_i(u)) \curlyvee (\curlyvee\overline{\varpi}_i(v)) = \curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(u) \curlyvee \curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(v).
\end{aligned}$$

Thus, $\curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i$ is a SIVFS of $\mathfrak{G}$. ☐

Theorem 4.2. *Let $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ be a family of SIVFLs (SIVFRs) of a semigroup $\mathfrak{G}$. Then $\curlywedge_{i\in\mathfrak{J}} \overline{SP}_i$ is a SIVFI (SIVFR) of $\mathfrak{G}$.*

Proof. Let $u, v \in \mathfrak{G}$. Then,

$$\begin{aligned}
\curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(uv) &\succeq \curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(v) = \curlywedge(\curlywedge\overline{\eta}_i(v)) = \curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i(v),\\
\curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(uv) &\succeq \curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(v) = \curlywedge(\curlywedge\overline{\vartheta}_i(v)) = \curlywedge_{i\in\mathfrak{J}} \overline{\vartheta}_i(v),\\
\curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(uv) &\preceq \curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(v) = \curlywedge(\curlywedge\overline{\varpi}_i(v)) = \curlyvee_{i\in\mathfrak{J}} \overline{\varpi}_i(v).
\end{aligned}$$

Thus, $\curlywedge_{i\in\mathfrak{J}} \overline{\eta}_i$ is a SIVFL of $\mathfrak{G}$. ☐

Theorem 4.3. *Let $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ be a family of SIVFBs (SIVFIIs, SIVF (1,2)-ideals) of a semigroup $\mathfrak{G}$. Then $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{SP}_i$ is a SIVFB (SIVFII, SIVF (1,2)-ideal) of $\mathfrak{G}$.*

Proof. Let $u, v \in \mathfrak{G}$. Since $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ is a family of SIVFBs of $\mathfrak{G}$, then, $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ is a family of SIVFBS of $\mathfrak{G}$. Thus, by Theorem 4.1, $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i$ is a SIVF of $\mathfrak{G}$. Let $u, v, w \in \mathfrak{G}$. Then,

$$\begin{aligned}
&\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i(uvw) \succeq \underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\eta}_i(u) \curlywedge \overline{\eta}_i(w)) = \curlywedge(\overline{\eta}_i(u) \curlywedge \overline{\eta}_i(w)) = (\curlywedge\overline{\eta}_i(u)) \curlywedge (\curlywedge\overline{\eta}_i(w)) = \underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i(u) \curlywedge \underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i(w),\\
&\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\vartheta}_i(uvw) \succeq \underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\vartheta}_i(u) \curlywedge \overline{\vartheta}_i(w)) = \curlywedge(\overline{\vartheta}_i(u) \curlywedge \overline{\vartheta}_i(w)) = (\curlywedge\overline{\vartheta}_i(u)) \curlywedge (\curlywedge\overline{\vartheta}_i(w)) = \underset{i\in\mathfrak{J}}{\curlywedge}\overline{\vartheta}_i(u) \curlywedge \underset{i\in\mathfrak{J}}{\curlywedge}\overline{\vartheta}_i(w),\\
&\underset{i\in\mathfrak{J}}{\curlyvee}\overline{\varpi}_i(uvw) \preceq \underset{i\in\mathfrak{J}}{\curlyvee}(\overline{\varpi}_i(u) \curlyvee \overline{\varpi}_i(w)) = \curlyvee(\overline{\varpi}_i(u) \curlyvee \overline{\varpi}_i(w)) = (\curlyvee\overline{\varpi}_i(u)) \curlyvee (\curlyvee\overline{\varpi}_i(w)) = \underset{i\in\mathfrak{J}}{\curlyvee}\overline{\varpi}_i(u) \curlyvee \underset{i\in\mathfrak{J}}{\curlyvee}\overline{\varpi}_i(w).
\end{aligned}$$

Thus, $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i$ is a SIVFB of $\mathfrak{G}$. Similarly, we can show that $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i$ is a SIVFII (SIVF (1,2)-ideal) of $\mathfrak{G}$. □

Theorem 4.4. *Let $\{\overline{SP}_i \mid i \in \mathfrak{J}\}$ be a family of SIVFQs of a semigroup $\mathfrak{G}$. Then $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{SP}_i$ is a SIVFQ of $\mathfrak{G}$.*

Proof. Let $u \in \mathfrak{G}$. Then,

$$\begin{aligned}
\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i(u) &\succeq \underset{i\in\mathfrak{J}}{\curlywedge}((\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u) \curlywedge (\overline{\eta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u) = (\curlywedge(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u)) \curlywedge (\curlywedge\overline{\eta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u)) = \underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u)\curlywedge\underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\eta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u),\\
\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\vartheta}_i(u) &\succeq \underset{i\in\mathfrak{J}}{\curlywedge}((\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u) \curlywedge (\overline{\vartheta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u) = (\curlywedge(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u)) \curlywedge (\curlywedge\overline{\vartheta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u)) = \underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u)\curlywedge\underset{i\in\mathfrak{J}}{\curlywedge}(\overline{\vartheta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u),\\
\underset{i\in\mathfrak{J}}{\curlyvee}\overline{\varpi}_i(u) &\preceq \underset{i\in\mathfrak{J}}{\curlyvee}((\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u) \curlyvee (\overline{\varpi}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u)\\
&= (\curlyvee(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u)) \curlyvee (\curlyvee\overline{\eta}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u)) = \underset{i\in\mathfrak{J}}{\curlyvee}(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u)\curlyvee\underset{i\in\mathfrak{J}}{\curlyvee}(\overline{\varpi}\,\bar{\circ}\,\overline{\mathfrak{G}}))(u).
\end{aligned}$$

Thus, $\underset{i\in\mathfrak{J}}{\curlywedge}\overline{\eta}_i$ is a SIVFQ of $\mathfrak{G}$. □

The follows theorems are study the spherical interval valued characteristic function of types of subsemigroups on semigroups.

Theorem 4.5. *Let $\mathfrak{K}$ is a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is a subsemigroup of $\mathfrak{G}$ if and only if the spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{K}$ is a subsemigroup $\mathfrak{G}$ and let $u, v\mathfrak{G}$. Then we have the following cases.

Case 1. If $u, v \in \mathfrak{K}$, then $uv \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(u) = \overline{1} = \overline{\eta}_{\chi_{\mathfrak{K}}}(v) = \overline{\eta}_{\chi_{\mathfrak{K}}}(uv)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) = \overline{1} = \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v) = \overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(u) = \overline{0} = \overline{\varpi}_{\chi_{\mathfrak{K}}}(v) = \overline{\varpi}_{\chi_{\mathfrak{K}}}(uv)$. Hence $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$.

Case 2. If $u, v \notin \mathfrak{K}$, then $uv \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$. Therefore, $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$.

Conversely suppose that $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$ and $u, v \in \mathfrak{K}$. If $uv \notin \mathfrak{K}$, then $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$. Since $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$ we have $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$, which is a contradiction. Thus, $uv \in \mathfrak{K}$. Hence, $\mathfrak{K}$ is a subsemigroup of $\mathfrak{G}$. □

Theorem 4.6. *Let $\mathfrak{K}$ is a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is a left (right) ideal of $\mathfrak{G}$ if and only if the spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ is a SIVFL (SIVFR) of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{K}$ is a left ideal $\mathfrak{G}$ and let $u, v\mathfrak{G}$. Then we have the following cases.

Case 1. If $v \in \mathfrak{K}$, then $uv \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(v) = \overline{1} = \overline{\eta}_{\chi_{\mathfrak{K}}}(uv)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(v) = \overline{1} = \overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(v) = \overline{0} = \overline{\varpi}_{\chi_{\mathfrak{K}}}(uv)$. Hence $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\curlywedge\vartheta}_{\chi_{\mathfrak{K}}}(v)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \curlyvee\overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$.

Case 2. If $v \notin \mathfrak{K}$, then $uv \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$. Therefore, $\overline{\chi}_{\mathfrak{K}}$ is a SIVFL of $\mathfrak{G}$.

Conversely suppose that $\overline{\chi}_{\mathfrak{K}}$ is a SIVFL of $\mathfrak{G}$ and $v \in \mathfrak{K}$. If $uv \notin \mathfrak{K}$, then $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$. Since $\overline{\chi}_{\mathfrak{K}}$ is a SIVFL of $\mathfrak{G}$ we have $\overline{\eta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(v)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(v)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \preceq \overline{\varpi}_{\chi_{\mathfrak{K}}}(v)$, which is a contradiction. Thus, $uv \in \mathfrak{K}$. Hence, $\mathfrak{K}$ is a left ideal of $\mathfrak{G}$. □

Corollary 4.7. *Let $\mathfrak{K}$ is a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is an ideal of $\mathfrak{G}$ if and only if the spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ is a SIVFI of $\mathfrak{G}$.*

Theorem 4.8. *Let $\mathfrak{K}$ is a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is a bi-ideal (interior ideal, (1,2)-ideal) of $\mathfrak{G}$ if and only if the spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ is a SIVFB (SIVFII, SIVF(1, 2)-ideal) of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{K}$ is a bi-ideal $\mathfrak{G}$. Then $\mathfrak{K}$ is a subsemigroup $\mathfrak{G}$. Thus, by Theorem 4.5, $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$. Let $u, v, w \in \mathfrak{G}$. Then we have the following cases.

Case 1. If $u, w \in \mathfrak{K}$, then $uvw \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(u) = \overline{1} = \overline{\eta}_{\chi_{\mathfrak{K}}}(w) = \overline{\eta}_{\chi_{\mathfrak{K}}}(uvw)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) = \overline{1} = \overline{\vartheta}_{\chi_{\mathfrak{K}}}(w) = \overline{\vartheta}_{\chi_{\mathfrak{K}}}(uvw)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(u) = \overline{0} = \overline{\varpi}_{\chi_{\mathfrak{K}}}(w) = \overline{\varpi}_{\chi_{\mathfrak{K}}}(uvw)$. Hence $\overline{\eta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(w)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(w)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uvw) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(w)$.

Case 2. If $u, w \notin \mathfrak{K}$, then $uvw \in \mathfrak{K}$. Thus, $\overline{\eta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(w)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(w)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uvw) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(w)$. Therefore $\overline{\chi}_{\mathfrak{K}}$ is a SIVFB of $\mathfrak{G}$. Similarly, we can show that $\overline{\chi}_{\mathfrak{K}}$ is a SIVFII (SIVF(1, 2)-ideal) of $\mathfrak{G}$.

Conversely suppose that $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$. Then, $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$. Thus, by Theorem 4.5, $\mathfrak{K}$ is a subsemigroup of $\mathfrak{G}$. Let $u, w \in \mathfrak{K}$. If $uvw \notin \mathfrak{K}$, then $\overline{\eta}_{\chi_{\mathfrak{K}}}(uvw) \preceq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(w)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uvw) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(w)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(w)$. Since $\overline{\chi}_{\mathfrak{K}}$ is a SIVFB of $\mathfrak{G}$ we have $\overline{\eta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\eta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\eta}_{\chi_{\mathfrak{K}}}(w)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uvw) \succeq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlywedge \overline{\vartheta}_{\chi_{\mathfrak{K}}}(w)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uvw) \preceq \overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \curlyvee \overline{\varpi}_{\chi_{\mathfrak{K}}}(w)$, which is a contradiction. Thus, $uvw \in \mathfrak{K}$. Hence, $\mathfrak{K}$ is a bi-ideal of $\mathfrak{G}$. □

Theorem 4.9. *Let $\mathfrak{K}$ is a nonempty subset of a semigroup $\mathfrak{G}$. Then $\mathfrak{K}$ is a quasi-ideal of $\mathfrak{G}$ if and only if the spherical interval valued characteristic function $\overline{\chi}_{\mathfrak{K}}$ is a SIVFQ of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{K}$ is a quasi-ideal $\mathfrak{G}$ and let $u \in \mathfrak{G}$. Then we have the following cases.

Case 1. If $u \in \mathfrak{K}$, then $\overline{\eta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\eta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\eta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\varpi}_{\chi_{\mathfrak{K}}})(u) \curlyvee (\overline{\varpi}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$.

Case 2. If $u \notin \mathfrak{K}$, then u is expressible $u = yz$ or not. Suppose u is expressible as $u = yz$. Since $u \notin \mathfrak{K}$ either $y \in \mathfrak{K}$ or $z \notin \mathfrak{K}$. If $y \in \mathfrak{K}$ and $z \notin \mathfrak{K}$, then there cannot be another exprssion of the form $x = ab$, where $a \in \mathfrak{K}$ and $b \in \mathfrak{K}$. Then $x \in \mathfrak{G}\mathfrak{K} \cap \mathfrak{K}\mathfrak{G} \subseteq \mathfrak{K}$. Thus $u \in \mathfrak{K}$. Thus, $uv \in \mathfrak{K}$. Thus, $(\overline{\mathfrak{G}} \circ \overline{\eta}_{\chi_{\mathfrak{K}}})(u) = \overline{0}$ or $(\overline{\eta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u) = \overline{0}$, $(\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) = \overline{0}$ or $(\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u) = \overline{0}$ and $(\overline{\mathfrak{G}} \circ \overline{\varpi}_{\chi_{\mathfrak{K}}})(u) = \overline{1}$ or $(\overline{\varpi}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u) = \overline{1}$. Hence, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\varpi}_{\chi_{\mathfrak{K}}})(u) \curlyvee (\overline{\varpi}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$. Therefore, $\overline{\chi}_{\mathfrak{K}}$ is a SIVFQ of $\mathfrak{G}$.

Conversely suppose that $\overline{\chi}_{\mathfrak{K}}$ is a SIVFQ of $\mathfrak{G}$ and $u \in \mathfrak{K}\mathfrak{G} \cap \mathfrak{G}\mathfrak{K}$. If $u \notin \mathfrak{K}$, then $\overline{\eta}_{\chi_{\mathfrak{K}}}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\eta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\eta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(uv) \preceq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(uv) \succeq (\overline{\mathfrak{G}} \circ \overline{\varpi}_{\chi_{\mathfrak{K}}})(u) \curlyvee (\overline{\varpi}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$. Since $\overline{\chi}_{\mathfrak{K}}$ is a SIVFS of $\mathfrak{G}$. Then, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, $\overline{\vartheta}_{\chi_{\mathfrak{K}}}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}_{\chi_{\mathfrak{K}}})(u) \curlywedge (\overline{\vartheta}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$ and $\overline{\varpi}_{\chi_{\mathfrak{K}}}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\varpi}_{\chi_{\mathfrak{K}}})(u) \curlyvee (\overline{\varpi}_{\chi_{\mathfrak{K}}} \circ \overline{\mathfrak{G}})(u)$, which is a contradiction. Thus, $u \in \mathfrak{K}$. Hence, $\mathfrak{K}$ is a quasi-ideal of $\mathfrak{G}$. □

Next, we give the definition of a $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut. And we prove the set $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut are subsemigoups of semigroups.

Definition 4.10. Let $\overline{SP}$ be a SIVF-set of a semigroup $\mathfrak{G}$ and $\overline{\beta} \in \mathcal{C}$. Then the set $\overline{\eta}_{\overline{\beta}} = \{u \in \mathfrak{G} : \overline{\eta}(u) \succeq \overline{\beta}\}$, $\overline{\vartheta}_{\overline{\beta}} = \{u \in \mathfrak{G} : \overline{\vartheta}(u) \succeq \overline{\beta}\}$, and $\overline{\varpi}_{\overline{\beta}} = \{u \in \mathfrak{G} : \overline{\varpi}(u) \preceq \overline{\beta}\}$ are called a $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\mathfrak{G}$, respectively.

Theorem 4.11. *Let $\overline{SP}$ be a SIVFS of a semigroup $\mathfrak{G}$. Then the $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\overline{SP}$ are subsemigroups of $\mathfrak{G}$, for every $\overline{\beta} \in Im(\overline{\eta}) \cap Im(\overline{\vartheta}) \cap Im(\overline{\varpi}) \subseteq \mathcal{C}$.*

Proof. Let $\overline{\beta} \in \mathrm{Im}(\overline{\eta}) \cap \mathrm{Im}(\overline{\vartheta}) \cap \mathrm{Im}(\overline{\varpi}) \subseteq \mathcal{C}$ and $u, v \in \overline{SP}$. If $u, v \in \overline{\eta}_{\overline{\beta}}$, then $\overline{\eta}(u) \succeq \overline{\beta}$ and $\overline{\eta}(v) \succeq \overline{\beta}$. Thus, $\overline{\eta}(uv) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v) \succeq \overline{\beta}$. Hence, $uv \in \overline{\eta}_{\overline{\beta}}$. If $u, v \in \overline{\vartheta}_{\overline{\beta}}$, then $\overline{\vartheta}(u) \succeq \overline{\beta}$ and $\overline{\vartheta}(v) \succeq \overline{\beta}$. Thus, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v) \succeq \overline{\beta}$. Hence, $uv \in \overline{\vartheta}_{\overline{\beta}}$. If $u, v \in \overline{\varpi}_{\overline{\beta}}$, then $\overline{\varpi}(u) \preceq \overline{\beta}$ and $\overline{\varpi}(v) \preceq \overline{\beta}$. Thus, $\overline{\varpi}(uv) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(v) \preceq \overline{\beta}$. Hence, $uv \in \overline{\varpi}_{\overline{\beta}}$. Therefore, $\overline{\eta}_{\overline{\beta}}$, $\overline{\vartheta}_{\overline{\beta}}$, and $\overline{\varpi}_{\overline{\beta}}$ are subsemigroups of $\mathfrak{G}$. □

Theorem 4.12. *Let $\overline{SP}$ be a SIVFL (SIVFR) of a semigroup $\mathfrak{G}$. Then the $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\overline{SP}$ are left (right) ideals of $\mathfrak{G}$, for every $\overline{\beta} \in Im(\overline{\eta}) \cap Im(\overline{\vartheta}) \cap Im(\overline{\varpi}) \subseteq \mathcal{C}$.*

Proof. Let $\overline{\beta} \in \mathrm{Im}(\overline{\eta}) \cap \mathrm{Im}(\overline{\vartheta}) \cap \mathrm{Im}(\overline{\varpi}) \subseteq \mathcal{C}$ and $u, v \in \overline{SP}$. If $v \in \overline{\eta}_{\overline{\beta}}$, then $\overline{\eta}(v) \succeq \overline{\beta}$. Thus, $\overline{\eta}(uv) \succeq \overline{\eta}(v) \succeq \overline{\beta}$. Hence, $uv \in \overline{\eta}_{\overline{\beta}}$. If $v \in \overline{\vartheta}_{\overline{\beta}}$, then $\overline{\vartheta}(v) \succeq \overline{\beta}$. Thus, $\overline{\vartheta}(uv) \succeq \overline{\vartheta}(v) \succeq \overline{\beta}$. Hence, $uv \in \overline{\vartheta}_{\overline{\beta}}$. If $v \in \overline{\varpi}_{\overline{\beta}}$, then $\overline{\varpi}(v) \preceq \overline{\beta}$. Thus, $\overline{\varpi}(uv) \preceq \overline{\varpi}(v) \preceq \overline{\beta}$. Hence, $uv \in \overline{\varpi}_{\overline{\beta}}$. Therefore, $\overline{\eta}_{\overline{\beta}}$, $\overline{\vartheta}_{\overline{\beta}}$, and $\overline{\varpi}_{\overline{\beta}}$ are left ideals of $\mathfrak{G}$. □

Corollary 4.13. *Let $\overline{SP}$ be a SIVFI of a semigroup $\mathfrak{G}$. Then the $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\overline{SP}$ are ideals of $\mathfrak{G}$, for every $\overline{\beta} \in Im(\overline{\eta}) \cap Im(\overline{\vartheta}) \cap Im(\overline{\varpi}) \subseteq \mathcal{C}$.*

Theorem 4.14. *Let $\overline{SP}$ be a SIVFB (SIVFII, SIVF(1, 2)-ideal) of a semigroup $\mathfrak{G}$. Then the $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\overline{SP}$ are bi-ideals (interior ideals, (1,2)-ideals) of $\mathfrak{G}$, for every $\overline{\beta} \in Im(\overline{\eta}) \cap Im(\overline{\vartheta}) \cap Im(\overline{\varpi}) \subseteq \mathcal{C}$.*

Proof. Let $\overline{\beta} \in \mathrm{Im}(\overline{\eta}) \cap \mathrm{Im}(\overline{\vartheta}) \cap \mathrm{Im}(\overline{\varpi}) \subseteq \mathcal{C}$ and $u, v, w \in \overline{SP}$. Since $\overline{SP}$ is a SIVFB we have $\overline{SP}$ is a SIVFS. Thus, by Theorem 4.11, $\overline{\eta}_{\overline{\beta}}$, $\overline{\vartheta}_{\overline{\beta}}$, and $\overline{\varpi}_{\overline{\beta}}$ are subsemigroups of $\mathfrak{G}$. Let $v \in \mathfrak{G}$. If $u, w \in \overline{\eta}_{\overline{\beta}}$, then $\overline{\eta}(u) \succeq \overline{\beta}$ and $\overline{\eta}(w) \succeq \overline{\beta}$. Thus, $\overline{\eta}(uvw) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w) \succeq \overline{\beta}$. Hence, $uvw \in \overline{\eta}_{\overline{\beta}}$. If $u, w \in \overline{\vartheta}_{\overline{\beta}}$, then $\overline{\vartheta}(u) \succeq \overline{\beta}$ and $\overline{\vartheta}(w) \succeq \overline{\beta}$. Thus, $\overline{\vartheta}(uvw) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w) \succeq \overline{\beta}$. Hence, $uvw \in \overline{\vartheta}_{\overline{\beta}}$. If $u, w \in \overline{\varpi}_{\overline{\beta}}$, then $\overline{\varpi}(u) \preceq \overline{\beta}$ and $\overline{\varpi}(w) \preceq \overline{\beta}$. Thus, $\overline{\varpi}(uvw) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(w) \preceq \overline{\beta}$. Hence, $uvw \in \overline{\varpi}_{\overline{\beta}}$. Therefore, $\overline{\eta}_{\overline{\beta}}$, $\overline{\vartheta}_{\overline{\beta}}$, and $\overline{\varpi}_{\overline{\beta}}$ are bi-ideals of $\mathfrak{G}$. □

Theorem 4.15. *Let $\overline{SP}$ be a SIVFQ of a semigroup $\mathfrak{G}$. Then the $\overline{\eta}$-level $\overline{\beta}$-cut, $\overline{\vartheta}$-level $\overline{\beta}$-cut, and $\overline{\varpi}$-level $\overline{\beta}$-cut of $\overline{SP}$ are quasi-ideals of $\mathfrak{G}$, for every $\overline{\beta} \in Im(\overline{\eta}) \cap Im(\overline{\vartheta}) \cap Im(\overline{\varpi}) \subseteq \mathcal{C}$.*

Proof. Let $\overline{\beta} \in \mathrm{Im}(\overline{\eta}) \cap \mathrm{Im}(\overline{\vartheta}) \cap \mathrm{Im}(\overline{\varpi}) \subseteq \mathcal{C}$ and $u \in \overline{SP}$. If $u \in \overline{\eta}_{\overline{\beta}}$, then $(\overline{\mathfrak{G}} \circ \overline{\eta}) \curlywedge (\overline{\eta} \circ \overline{\mathfrak{G}})(u) \succeq \overline{\beta}$. Thus, $\overline{\eta}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\eta}) \curlywedge (\overline{\eta} \circ \overline{\mathfrak{G}})(u) \succeq \overline{\beta}$. Hence, $u \in \overline{\eta}_{\overline{\beta}}$. If $u \in \overline{\vartheta}_{\overline{\beta}}$, then $(\overline{\mathfrak{G}} \circ \overline{\vartheta}) \curlywedge (\overline{\vartheta} \circ \overline{\mathfrak{G}})(u) \succeq \overline{\beta}$. Thus, $\overline{\vartheta}(u) \succeq (\overline{\mathfrak{G}} \circ \overline{\vartheta}) \curlywedge (\overline{\vartheta} \circ \overline{\mathfrak{G}})(u) \succeq \overline{\beta}$. Hence, $u \in \overline{\vartheta}_{\overline{\beta}}$. If $u \in \overline{\varpi}_{\overline{\beta}}$, then $(\overline{\mathfrak{G}} \circ \overline{\varpi}) \curlyvee (\overline{\varpi} \circ \overline{\mathfrak{G}})(u) \preceq \overline{\beta}$. Thus, $\overline{\varpi}(u) \preceq (\overline{\mathfrak{G}} \circ \overline{\varpi}) \curlyvee (\overline{\varpi} \circ \overline{\mathfrak{G}})(u) \preceq \overline{\beta}$. Hence, $u \in \overline{\varpi}_{\overline{\beta}}$. Therefore, $\overline{\eta}_{\overline{\beta}}$, $\overline{\vartheta}_{\overline{\beta}}$, and $\overline{\varpi}_{\overline{\beta}}$ are quasi-ideals of $\mathfrak{G}$. □

Definition 4.16. Let $\overline{SP}_1 = (\overline{\eta}, \overline{\vartheta}, \overline{\varpi})$ and $\overline{SP}_2 = (\overline{\tau}, \overline{\nu}, \overline{\alpha})$ be two SIVF-set of a non-empty set $\mathfrak{T}$. Then

(1) $\overline{SP}_1 \sqsubseteq \overline{SP}_2$ if $\overline{\eta}(u) \preceq \overline{\tau}(u)$, $\overline{\vartheta}(u) \preceq \overline{\nu}(u)$, and $\overline{\varpi}(u) \succeq \overline{\alpha}(u)$;
(2) $\overline{SP}_1 \sqcap \overline{SP}_2$ if $\overline{\eta}(u) \curlywedge \overline{\tau}(u)$ $\overline{\vartheta}(u) \curlywedge \overline{\nu}(u)$, and $\overline{\varpi}(u) \curlyvee \overline{\alpha}(u)$,

for all $u \in \mathfrak{T}$.

Theorem 4.17. *Let $\overline{SP}$ be SIVF-set of a semigroup $\mathfrak{G}$. Then $\overline{SP}$ is a SIVFS of $\mathfrak{G}$ if and on ly if $\overline{SP} \,\overline{\circ}\, \overline{SP} \sqsubseteq \overline{SP}$.*

Proof. Assume that $\overline{SP}$ is a SIVFS of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. If $F_u = \emptyset$. Then $(\overline{\eta} \,\overline{\circ}\, \overline{\eta})(u) = 0 \preceq \overline{\eta}(u)$, $(\overline{\vartheta} \,\overline{\circ}\, \overline{\vartheta})(u) = 0 \preceq \overline{\vartheta}(u)$, and $(\overline{\varpi} \,\overline{\circ}\, \overline{\varpi})(u) = 1 \succeq \overline{\varpi}(u)$. If $F_u \neq \emptyset$. Then

$$(\overline{\eta} \,\overline{\circ}\, \overline{\eta})(u) = \underset{(i,j) \in F_u}{\curlyvee} \overline{\eta}(i) \curlywedge \overline{\eta}(j) \preceq \underset{(i,j) \in F_u}{\curlyvee} \overline{\eta}(ij) = \overline{\eta}(u),$$
$$(\overline{\vartheta} \,\overline{\circ}\, \overline{\vartheta})(u) = \underset{(i,j) \in F_u}{\curlyvee} \overline{\vartheta}(i) \curlywedge \overline{\vartheta}(j) \preceq \underset{(i,j) \in F_u}{\curlyvee} \overline{\vartheta}(ij) = \overline{\vartheta}(u),$$

$$(\overline{\varpi}\,\bar{\circ}\,\overline{\varpi})(u) = \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\varpi}(i) \curlyvee \overline{\varpi}(j) \succeq \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\varpi}(ij) = \overline{\varpi}(u).$$

Hence, $\overline{SP}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$. Conversely, assume that $\overline{SP}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$ and $u, v \in \mathfrak{G}$. Then

$$\begin{aligned}
\overline{\eta}(uv) &\succeq (\overline{\eta}\,\bar{\circ}\,\overline{\eta})(uv) = \mathop{\curlyvee}\limits_{(i,j)\in F_{uv}} \overline{\eta}(i) \curlywedge \overline{\eta}(j) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(v),\\
\overline{\vartheta}(uv) &\succeq (\overline{\vartheta}\,\bar{\circ}\,\overline{\vartheta})(uv) = \mathop{\curlyvee}\limits_{(i,j)\in F_{uv}} \overline{\vartheta}(i) \curlywedge \overline{\vartheta}(j) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(v),\\
\overline{\varpi}(uv) &\preceq (\overline{\varpi}\,\bar{\circ}\,\overline{\varpi})(uv) = \mathop{\curlywedge}\limits_{(i,j)\in F_{uv}} \overline{\varpi}(i) \curlyvee \overline{\varpi}(j) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(v).
\end{aligned}$$

Thus, $\overline{SP}$ is a SIVFS of $\mathfrak{G}$. □

Theorem 4.18. *Let* $\overline{SP}$ *be SIVF-set of a semigroup* $\mathfrak{G}$. *Then* $\overline{SP}$ *is a SIVFL (SIVFR) of* $\mathfrak{G}$ *if and on ly if* $\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}(\overline{SP}\,\bar{\circ}\,\overline{\mathfrak{G}} \sqsubseteq \overline{SP})$.

Proof. Assume that $\overline{SP}$ is a SIVFL of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. If $F_u = \emptyset$, then $(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u) = 0 \preceq \overline{\eta}(u)$, $(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u) = 0 \preceq \overline{\vartheta}(u)$, and $(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u) = 1 \succeq \overline{\varpi}(u)$. If $F_u \neq \emptyset$. Then

$$\begin{aligned}
(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u) &= \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\mathfrak{G}}(i) \curlywedge \overline{\eta}(j) \preceq \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\eta}(j) = \overline{\eta}(u),\\
(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u) &= \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\mathfrak{G}}(i) \curlywedge \overline{\vartheta}(j) \preceq \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\vartheta}(j) = \overline{\vartheta}(u),\\
(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u) &= \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\mathfrak{G}}(i) \curlyvee \overline{\varpi}(j) \succeq \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\varpi}(j) = \overline{\varpi}(u).
\end{aligned}$$

Hence, $\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$. Conversely, assume that $\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$ and $u, v \in \mathfrak{G}$. Then

$$\begin{aligned}
\overline{\eta}(uv) &\succeq (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(uv) = \mathop{\curlyvee}\limits_{(i,j)\in F_{uv}} \overline{\mathfrak{G}}(i) \curlywedge \overline{\eta}(j) \succeq \overline{\eta}(v),\\
\overline{\vartheta}(uv) &\succeq (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(uv) = \mathop{\curlyvee}\limits_{(i,j)\in F_{uv}} \overline{\mathfrak{G}}(i) \curlywedge \overline{\vartheta}(j) \succeq \overline{\vartheta}(v),\\
\overline{\varpi}(uv) &\preceq (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(uv) = \mathop{\curlywedge}\limits_{(i,j)\in F_{uv}} \overline{\mathfrak{G}}(i) \curlyvee \overline{\varpi}(j) \preceq \overline{\varpi}(v).
\end{aligned}$$

Thus, $\overline{SP}$ is a SIVFL of $\mathfrak{G}$. □

Corollary 4.19. *Let* $\overline{SP}$ *be SIVF-set of a semigroup* $\mathfrak{G}$. *Then* $\overline{SP}$ *is a SIVFI of* $\mathfrak{G}$ *if and on ly if* $\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$ *and* $\overline{SP}\,\bar{\circ}\,\overline{\mathfrak{G}} \sqsubseteq \overline{SP}$.

Theorem 4.20. *Let* $\overline{SP}$ *be SIVF-set of a semigroup* $\mathfrak{G}$. *Then* $\overline{SP}$ *is a SIVFB of* $\mathfrak{G}$ *if and on ly if* $\overline{SP}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$ *and* $\overline{SP}\,\bar{\circ}\,\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$.

Proof. Assume that $\overline{SP}$ is a SIVFB of $\mathfrak{G}$ let $u \in \mathfrak{G}$. Since $\overline{SP}$ is a SIVFB of $\mathfrak{G}$ we have $\overline{SP}$ is a SIVFS of $\mathfrak{G}$. Thus by Theorem 4.17, $\overline{SP}\,\bar{\circ}\,\overline{SP} \sqsubseteq \overline{SP}$. If $F_u = \emptyset$, then $(\overline{\eta}\,\bar{\circ}\,\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u) = 0 \preceq \overline{\eta}(u)$, $(\overline{\vartheta}\,\bar{\circ}\,\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u) = 0 \preceq \overline{\vartheta}(u)$, and $(\overline{\varpi}\,\bar{\circ}\,\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u) = 1 \succeq \overline{\varpi}(u)$. If $F_u \neq \emptyset$. Then

$$\begin{aligned}
\overline{\eta}\,\bar{\circ}\,(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(u) &= \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\eta}(i) \curlywedge (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\eta})(j) = \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\eta}(i) \curlywedge (\mathop{\curlyvee}\limits_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlywedge \overline{\eta}(r)) \preceq \mathop{\curlyvee}\limits_{(i,r)\in F_u} \overline{\eta}(i) \curlywedge \overline{\eta}(r) = \overline{\eta}(u),\\
\overline{\vartheta}\,\bar{\circ}\,(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(u) &= \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\vartheta}(i) \curlywedge (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\vartheta})(j) = \mathop{\curlyvee}\limits_{(i,j)\in F_u} \overline{\vartheta}(i) \curlywedge (\mathop{\curlyvee}\limits_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlywedge \overline{\vartheta}(r)) \preceq \mathop{\curlyvee}\limits_{(i,r)\in F_u} \overline{\vartheta}(i) \curlywedge \overline{\vartheta}(r) = \overline{\vartheta}(u),\\
\overline{\varpi}\,\bar{\circ}\,(\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(u) &= \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\varpi}(i) \curlyvee (\overline{\mathfrak{G}}\,\bar{\circ}\,\overline{\varpi})(j)\\
&= \mathop{\curlywedge}\limits_{(i,j)\in F_u} \overline{\varpi}(i) \curlywedge (\mathop{\curlyvee}\limits_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlyvee \overline{\varpi}(r)) \preceq \mathop{\curlywedge}\limits_{(i,r)\in F_u} \overline{\varpi}(i) \curlyvee \overline{\varpi}(r) = \overline{\varpi}(u).
\end{aligned}$$

Hence, $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$. Conversely, assume that $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $u, v, w \in \mathfrak{G}$. Since $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ we have $\overline{\mathrm{SP}}$ is a SIVFS of $\mathfrak{G}$ by Theorem 4.17,

$$\begin{aligned}
\overline{\eta}(uvw) &\succeq \overline{\eta} \,\bar{\circ}\, (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\eta})(uvw) \\
&= \curlyvee_{(i,j)\in F_{uvw}} \overline{\eta}(i) \curlywedge (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\eta})(j) = \curlyvee_{(i,j)\in F_{uvw}} \overline{\eta}(i) \curlywedge \Big(\curlyvee_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlywedge \overline{\eta}(r) \Big) \succeq \overline{\eta}(u) \curlywedge \overline{\eta}(w), \\
\overline{\vartheta}(uvw) &\succeq \overline{\vartheta} \,\bar{\circ}\, (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\vartheta})(uvw) \\
&= \curlyvee_{(i,j)\in F_{uvw}} \overline{\vartheta}(i) \curlywedge (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\vartheta})(j) = \curlyvee_{(i,j)\in F_{uvw}} \overline{\vartheta}(i) \curlywedge \Big(\curlyvee_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlywedge \overline{\vartheta}(r) \Big) \succeq \overline{\vartheta}(u) \curlywedge \overline{\vartheta}(w), \\
\overline{\varpi}(uvw) &\preceq \overline{\varpi} \,\bar{\circ}\, (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\varpi})(uvw) \\
&= \curlywedge_{(i,j)\in F_{uvw}} \overline{\varpi}(i) \curlyvee (\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\varpi})(j) = \curlywedge_{(i,j)\in F_{uvw}} \overline{\varpi}(i) \curlyvee \Big(\curlywedge_{(k,r)\in F_j} \overline{\mathfrak{G}}(k) \curlyvee \overline{\varpi}(r) \Big) \preceq \overline{\varpi}(u) \curlyvee \overline{\varpi}(w).
\end{aligned}$$

Thus, $\overline{\mathrm{SP}}$ is a SIVFB of $\mathfrak{G}$. □

Theorem 4.21. *Let $\overline{\mathrm{SP}}$ be SIVF-set of a semigroup $\mathfrak{G}$. Then $\overline{\mathrm{SP}}$ is a SIVFII of $\mathfrak{G}$ if and on ly if $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$.*

Proof. It follows Theorem 4.20. □

Theorem 4.22. *Let $\overline{\mathrm{SP}}$ be SIVF-set of a semigroup $\mathfrak{G}$. Then $\overline{\mathrm{SP}}$ is a SIVF (1,2)-ideal of $\mathfrak{G}$ if and on ly if $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$.*

Proof. It follows Theorem 4.20. □

Theorem 4.23. *Let $\overline{\mathrm{SP}}$ be SIVF-set of a semigroup $\mathfrak{G}$. Then $\overline{\mathrm{SP}}$ is a SIVFQ of $\mathfrak{G}$ if and on ly if $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqcap \overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$.*

Proof. Assume that $\overline{\mathrm{SP}}$ is a SIVFQ of $\mathfrak{G}$ and let $u \in \mathfrak{G}$. It follows from Theorem 4.18, $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$. Thus, $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqcap \overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$.

Conversely, assume that $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqcap \overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$. Then $\overline{\mathfrak{G}} \,\bar{\circ}\, \overline{\mathrm{SP}} \sqsubseteq \overline{\mathrm{SP}}$ and $\overline{\mathrm{SP}} \,\bar{\circ}\, \overline{\mathfrak{G}} \sqsubseteq \overline{\mathrm{SP}}$. It follows from Theorem 4.18, $\overline{\mathrm{SP}}$ is a SIVFL and $\overline{\mathrm{SP}}$ is a SIVFR of $\mathfrak{G}$. Thus, by Theorem 3.22, $\overline{\mathrm{SP}}$ is a SIVFQ of $\mathfrak{G}$. □

5. Conclusion

Spherical interval valued fuzzy sets is one of the successful extensions of spherical fuzzy set for handling the uncertainties in the data. In this paper, we introduce the notion of spherical interval valued fuzzy ideals in semigroups. We desirable properties spherical interval valued fuzzy ideals and types spherical interval valued fuzzy ideals. We characterized necessary and sufficient conditions of coincidences spherical interval valued fuzzy ideals in semigroups. In continuity of this paper we will investigate about the spherical interval valued fuzzy of a ternary semigroup and their algebraic properties.

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