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Crank-Nicolson finite element methods for nonlocal problems with p-Laplace-type operator

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Abstract

A theoretical analysis of a Crank-Nicolson Galerkin finite element method for a class of nonlinear nonlocal diffusion problems associated with p-Laplace-type operator is presented here. It is shown, by a rigorous analysis that the unconditionally optimal error estimates for the fully discrete scheme are established. The presence of the nonlocal term in the models destroys the sparsity of the Jacobian matrices when solving the problem numerically using finite element method and Newton-Raphson method. As a consequence, computations consume more time and space in contrast to local problems. To overcome this difficulty, a new algorithm is proposed to avoid the full Jacobian matrix. Finally, some numerical simulations are presented to illustrate our theoretical analysis.

Keywords: Crank-Nicolson scheme, Newton-Raphson method, Galerkin finite element method, nonlocal diffusion term, p-Laplace operator, optimal error estimate.

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1. Introduction

Let Ω be a bounded open subset in $\mathbb{R}^d$, $d = 2, 3$ with smooth boundary $\partial\Omega$. The following parabolic problem with nonlocal nonlinearity is the focus of this research article:

$$\begin{cases} u_t - \nabla \cdot (a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega \times (0,T)\,, \\ u(x,t) = 0 \;\; \text{on } \partial\Omega \times (0,T)\,, \\ u(x,0) = u_0(x) \;\; \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\|\cdot\|_p$ denotes the L^p-norm, $1 < p < \infty$. $a(\cdot)$ and f are functions to be defined in the next section. This kind of model problem arises for instance in diffusion of bacteria: $u(x;t)$ is the density of population located at x at the time t, f is the density of bacteria supplied from outside, u_0 is the initial density of population, a is the diffusion rate (depending on $\|\nabla u\|_p^p$. The model is said to be nonlocal because of

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the presence of the term $a(\|\nabla u\|_p^p)$ which implies equations in (1.1) are no longer pointwise equalities. Furthermore, we have the presence of the p-Laplacian operator that appears in several areas of the science such as astronomy, glaciology, climatology, nonnewtonian fluids, petroleum extraction. Problems that involve these two terms, $a(\|\nabla u\|_p^p)$ and the p-Laplacian operator, present several difficulties such as uniqueness, regularity and degeneracy. The interest of the mathematicians on the study of problems associated with the p-Laplace operator has increased because they represent a variety of relevant physical and engineering situations which requires a nontrivial apparatus to solve them (see for instance [2, 10–12, 15]). In particular, a lot of attention has been devoted to nonlocal problems. One of the justifications of such models lies in the fact that in reality the measurements are not made pointwise but through some local average. Some interesting features of nonlocal problems and more motivation are described in [1, 5, 6, 9, 19] and in the references therein.

There are few numerical methods and analysis of the problems (1.1) except the case where $p = 2$ (the p-Laplace operator is a Laplacian operator and the main challenge is only the presence of the nonlocal term). One can mention for this case, the followings papers [14, 16] and references therein. The numerical approximation of such problem with $p \neq 2$ is much more challenging due to the presence of the nonlocal term and the p-Laplacian operator. The backward Euler Galerkin finite element approximation of the model (1.1) has been studied in [18] without numerical simulations. Compared with backward Euler scheme, the Crank-Nicolson scheme has higher order temporal accuracy. However, the strong nonlinearity of the problems produces extra difficulties in theoretical analyses for Crank-Nicolson scheme. Certain time stepsize restrictions were always required to derive the optimal error estimates of such scheme. In [20], Rachford established optimal L^2 error estimates of Crank-Nicolson Galerkin FEMs under a time stepsize restriction $\Delta t = Ch^{(r+1)/2}$. Several works developed and analyzed numerical schemes for nonlinear elliptic equations involving the p-Laplacian operator. We can cite the work of Feng et al. [13] based on the preconditioned steepest descent solvers for fourth and sixth order nonlinear elliptic equations that include p-Laplacian terms on periodic domains in 2D and 3D.

The main goal of this work is to establish the well-posedness and the error analysis of the numerical scheme in L2-norm associated to (1.1) for all $p \in (1, \infty)$ using the Crank-Nicolson scheme. The unconditionally optimal error analysis of such a scheme for the strongly nonlinear parabolic equations (1.1) is a challenging problem.

The fully-discrete formulation leads to a system of nonlinear equations which can be solved by Newton-Raphson method but the Jacobian matrix is full. To avoid this difficulty, the new reformulation which is an equivalent problem is proposed.

This paper is organized as follows. In the next section, we briefly describe the mathematical setting and recall the variational formulation. Section 3 is devoted to the spatial discretization and its convergence. The core of the paper is the Section 4 in which we present a thorough study of the proposed scheme with the error estimates results. Also, the equivalent formulation problem is given. Finally, in the last section, we report and discuss the numerical results.

2. Continuous formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with polygonal or polyhedral boundary $\partial\Omega = \Gamma$. We adopt the standard definitions from [4, 17] for the Sobolev spaces $W^{m,p}(\Omega)$ and their associated norm and semi-norm denoted respectively by $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$. The Lebesgue space is denoted as usual $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, with norms $\|\cdot\|_p$ (except the $L^2(\Omega)$-norm which is denoted by $\|\cdot\|$). The Bochner spaces, such as $L^q(0, T, X)$ with norm denoted by $\|\cdot\|_{L^q(X)}$, where X is an Hilbert space are also employed. Assuming that $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $f \in C([0, T], L^2(\Omega)) \cap L^q(0, T, L^q(\Omega))$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the weak formulation of (1.1) takes the form: find $u(t)$ for almost $t \in (0; T)$ such that

$$\langle u_t, v\rangle + a(\|\nabla u\|_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \langle f, v\rangle, \ \forall v \in W_0^{1,p}(\Omega), \tag{2.1}$$

$$u(x,0) = u_0(x) \text{ in } \Omega\,. \tag{2.2}$$

Chipot and Savitska [7] proved that if $a(\cdot)$ is a Lipshitz continuous and nondecreasing function and satisfies

$$a : \mathbb{R} \to \mathbb{R}^+ \quad \text{and there exists} \quad m > 0,\ M > 0, \quad 0 < m \leqslant a(s) \leqslant M \quad \text{for all} \quad s \in \mathbb{R}, \tag{2.3}$$

then the weak formulation (2.1)-(2.2) has the unique solution $u \in L^p(0,T,W_0^{1,p}(\Omega)) \cap \mathcal{C}([0,T];L^r(\Omega))$, where $r = \min\{2,p\}$.

We recall some useful lemmas which will be used in this work. Their proofs can be found in [3, 7].

Lemma 2.1. *Given $p > 1$ and $\alpha \geqslant 0$, there exists positive constants C_1 and C_2 such that for all u, $v \in \mathbb{R}^d$ with $d \geqslant 1$ we have*

$$||u|^{p-2}u - |v|^{p-2}v| \leqslant C_1|u-v|^{1-\alpha}(|u|+|v|)^{p-2+\alpha}, \tag{2.4}$$

$$(|u|^{p-2}u - |v|^{p-2}v)\cdot(u-v) \geqslant C_2|u-v|^{2+\alpha}(|u|+|v|)^{p-2-\alpha}. \tag{2.5}$$

Lemma 2.2. *Given $p > 1$ there exists and ϵ_0 such that for all $x, y, z \geqslant 0$ and for all $\epsilon \in (0, \epsilon_0)$*

$$(x+y)^{p-2}yz \leqslant \epsilon(x+y)^{p-2}y^2 + C(\epsilon^{-1})(x+z)^{p-2}z^2. \tag{2.6}$$

Lemma 2.3. *Given two positive real numers a and b, for all $p > 1$,*

$$|a^p - b^p| \leqslant p|a-b|(a+b)^{p-1}.$$

Lemma 2.4. *Let $p > 1$. For all $\eta, \zeta \in \mathbb{R}^d$,*

$$|\eta|^p \geqslant |\zeta|^p + p|\zeta|^{p-2}\zeta\cdot(\eta-\zeta), \tag{2.7}$$

$$\frac{1}{2}(|\eta|+|\zeta|) \leqslant (|\eta-\zeta|+|\zeta|) \leqslant 2(|\zeta|+|\eta|). \tag{2.8}$$

Remark 2.5. As consequence of (2.8), there exists $K_1 > 0$ and $K_2 > 0$ such that

$$K_1(|\eta|+|\zeta|)^{p-2} \leqslant (|\eta-\zeta|+|\zeta|)^{p-2} \leqslant K_2(|\eta|+|\zeta|)^{p-2}. \tag{2.9}$$

3. Spatial discretization

Let $\mathcal{T}_h$ be a family of regular triangular partition of Ω into triangles T following finite element method theory [8, 21]. We assume that the diameter of T is not greater than $0 < h < 1$. We define the following finite element spaces:

$$S_h = \{v_h \in \mathcal{C}(\bar{\Omega}),\ v_h|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_h\}; \quad V_h = \{v_h \in S_h,\ v_h = 0 \text{ on } \partial\Omega\}.$$

We have the following result from [21].

Lemma 3.1.

(a) *If $q, s \in [1,\infty]$ with $W^{2,s}(\Omega) \subset W^{m,q}(\Omega)$,*

$$\|w - \Pi_h w\|_{m,q} \leqslant Ch^{2-m+d(1/q-1/s)}\|w\|_{2,s}, \quad \forall w \in W_0^{2,s}(\Omega).$$

(b) *If $q > d$*

$$\|w - \Pi_h w\|_{m,q} \leqslant Ch^{1-m}\|w\|_{1,q}, \quad \forall w \in W_0^{1,q}(\Omega),$$

where C is a positive constant that does not depend on h and $m \in \{0,1\}$. $\Pi_h : C(\overline{\Omega}) \to S_h$ is an interpolation operator.

The space discrete approximation of the weak formulations(2.1)-(2.2) is given by: find $u_h(t) \in V_h$ for $t \geqslant 0$ such that

$$\langle (u_h)_t, v_h\rangle + a(\|\nabla u_h\|_p^p)\int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla v_h\, dx = \langle f, v_h\rangle,\ \forall v_h \in V_h, \qquad u_h(0) = \Pi_h u_0. \tag{3.1}$$

Mbehou in [18] proved the following convergence results.

Theorem 3.2. *The semi discrete problem* (3.1) *admits a unique solution* $u_h \in V_h$, *which satisfies*

$$\|(u_h)_t\|^2_{L^2(L^2(\Omega))} + \|u_h\|^p_{L^\infty(W_0^{1,p}(\Omega))} \leqslant C(\|f\|^2_{L^2(L^2(\Omega))} + \|u_0\|^p_{1,p}).$$

If $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ *and* $f \in C([0,T], L^2(\Omega)) \cap L^q(0,T,L^q(\Omega))$ *and* (2.3) *holds and if* u *is the unique solution of problem* (2.1)-(2.2), *then*

(a) *for* $p \in (1,2)$ *and* $u \in L^2(0,T,W^{2,p}(\Omega))$,

$$\|u - u_h\|^2_{L^\infty(L^2(\Omega))} + \|u - u_h\|^2_{L^2(W_0^{1,p}(\Omega))} \leqslant Ch^p;$$

(b) *for* $p \in [2,\infty)$ *and* $u \in L^2(0,T,W^{2,p}(\Omega)) \cap L^\infty(0,T,W^{1,\infty}(\Omega))$,

$$\|u - u_h\|^2_{L^\infty(L^2(\Omega))} + \|u - u_h\|^p_{L^p(W_0^{1,p}(\Omega))} \leqslant Ch^2.$$

4. Fully discrete approximation

Let $\{t_n | t_n = n\delta;\ 0 \leqslant n \leqslant N\}$ be a uniform partition of $[0,T]$ with time step $\delta = T/N$. We write $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$ and $w^n = w(x,t_n)$. u_h^n denotes the approximation of u_h at time $t = t_n$. For the sequence of functions $\{w^n\}_{n=0}^N$, we define

$$\bar\partial w^n = \frac{w^n - w^{n-1}}{\delta}, \qquad \bar w^n = \frac{1}{2}(w^n + w^{n-1}).$$

The fully discrete approximation of (3.1) is as follows. Find $u_h^n \in V_h$ with $n = 1, \dots, N$ such that

$$(\bar\partial u_h^n, v_h) + a(\|\nabla \bar u_h^n\|_p^p)\int_\Omega |\nabla \bar u_h^n|^{p-2}\nabla \bar u_h^n\cdot\nabla v_h\, dx = \langle f^{n-1/2}, v_h\rangle,\ \forall v_h \in V_h, \qquad u_h^0 = \Pi_h u_0. \tag{4.1}$$

Lemma 4.1. *The fully discrete problem* (4.1) *has a unique solution* $\{u_h^n\}_{n=0}^N$, *which satisfies*

$$\sup_{1\leqslant n\leqslant N} \|u_h^n\|^2 + \delta\sum_{n=1}^N \|\bar u_h^n\|^p_{1,p} \leqslant C\left\{\delta\sum_{n=1}^N \|f^{n-1/2}\|_q^q + \|u_0\|^2\right\}, \tag{4.2}$$

$$\delta^2\sum_{n=1}^N \|\bar\partial u_h^n\|^2 \leqslant C\left\{\delta\sum_{n=1}^N \|f^{n-1/2}\|_q^q + \delta\sum_{n=1}^N \|f^{n-1/2}\|^2 + \|u_0\|^2\right\}. \tag{4.3}$$

To show the existence of solution to the discrete problem (4.1), we introduce the following result which is the consequence of the Brouwer fixed point theorem.

Proposition 4.2. *Let* H *be a finite dimensional Hilbert space with scalar product* $(\cdot,\cdot)$ *and norm* $|\cdot|$. *Let* $R : H \to H$ *be a continuous map with the following property: there exists* $\rho > 0$ *such that*

$$(R(v), v) > 0, \quad \forall v \in H \quad \textit{with} \quad |v| = \rho.$$

Then, there exists $w \in H$ *such that*

$$R(w) = 0, \quad |w| \leqslant \rho.$$

Proof of Lemma 4.1. Note that problem (4.1) can be written as

$$(\bar{u}_h^n, v_h) + \frac{\delta}{2} a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla v_h \, dx - \frac{1}{2}(f^{n-1/2}, v_h) - (u_h^{n-1}, v_h) = 0.$$

For fixed n, given $u_h^{n-1} \in V_h$, define $F : V_h \to V_h$ by

$$(F(w_h), v_h) = (w_h, v_h) + \frac{\delta}{2} a(\|\nabla w_h\|_p^p) \int_\Omega |\nabla w_h|^{p-2} \nabla w_h \cdot \nabla v_h \, dx - \frac{\delta}{2}(f^{n-1/2}, v_h) - (u_h^{n-1}, v_h). \tag{4.4}$$

Clearly, F is a continuous map on V_h. Setting $v_h = w_h$ in (4.4), we have

$$\begin{aligned} (F(w_h), w_h) &\geqslant \|w_h\|^2 + \frac{m\delta}{2} \|\nabla w_h\|_p^p - \frac{\delta}{2} \|f^{n-1/2}\| \|w_h\| - \|u_h^{n-1}\| \|w_h\| \\ &\geqslant \left(\|w_h\| - \frac{\delta}{2} \|f^{n-1/2}\| - \|u_h^{n-1}\| \right) \|w_h\|. \end{aligned}$$

Therefore $(F(w_h), w_h) > 0$ for all $w_h \in V_h$ with $\|w_h\| = \rho$, where ρ is any number such that $\rho > \left(\frac{\delta}{2}\|f^{n-1/2}\|_q + \|u_h^{n-1}\|\right)$. Then, by Proposition 4.2, we conclude the existence of the solution. The uniqueness follows from the Lipschitz continuity of $a(\cdot)$ and the relation (2.4).

To prove (4.2), taking $v_h = 2\bar{u}_h^n$ and applying the lower bound of $a(\cdot)$, (2.3), we have

$$\begin{aligned} \|u_h^n\|^2 - \|u_h^{n-1}\|^2 + 2m\delta \|\nabla \bar{u}_h^n\|_p^p &\leqslant 2\delta \|f^{n-1/2}\|_q \|\bar{u}_h^n\|_p \\ (\text{ applying Poincaré-Fredrichs inequality}) &\leqslant C\delta \|f^{n-1/2}\|_q \|\nabla \bar{u}_h^n\|_p, \end{aligned}$$

which with Young's inequality, lead to the following inequality:

$$\|u_h^n\|^2 - \|u_h^{n-1}\|^2 + m\delta \|\nabla \bar{u}_h^n\|_p^p \leqslant C\delta \|f^{n-1/2}\|_q^q.$$

Summing the above equation from $n = 1, \dots, J$,

$$\|u_h^J\|^2 - \|u_h^0\|^2 + m\delta \sum_{n=1}^{J} \|\nabla \bar{u}_h^n\|_p^p \leqslant \delta \sum_{n=1}^{J} \|f^{n-1/2}\|_q^q \quad \text{for all} \quad 1 \leqslant J \leqslant N.$$

For the relation (4.3), taking $v_h = \bar{u}_h^n - u_h^{n-1} = \frac{u_h^n - u_h^{n-1}}{2}$, we have

$$\|u_h^n - u_h^{n-1}\|^2 + 2\delta a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla(\bar{u}_h^n - u_h^{n-1}) dx = \delta(f^{n-1/2}, u_h^n - u_h^{n-1}). \tag{4.5}$$

Using the relation (2.7) with $\eta = \nabla \bar{u}_h^n$ and $\zeta = \nabla u_h^{n-1}$, we have

$$|\nabla u_h^{n-1}|^p - |\nabla \bar{u}_h^n|^p \geqslant p |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla(u_h^{n-1} - \bar{u}_h^n),$$

that is

$$\||\nabla \bar{u}_h^n|^p - \nabla u_h^{n-1}|^p \leqslant p |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla(\bar{u}_h^n - u_h^{n-1}).$$

By integrating the above relation over Ω, we obtain

$$\frac{1}{p}(\|\nabla \bar{u}_h^n\|_p^p - \|\nabla u_h^{n-1}\|_p^p) \leqslant \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla(\bar{u}_h^n - u_h^{n-1}) dx. \tag{4.6}$$

Taking (4.6) into (4.5) and using (2.3),

$$\begin{aligned} \|u_h^n - u_h^{n-1}\|^2 + \frac{2}{p} m\delta \|\nabla \bar{u}_h^n\|_p^p &\leqslant \delta(f^{n-1/2}, u_h^n - u_h^{n-1}) + \frac{2}{p} M\delta \|\nabla u_h^{n-1}\|_p^p \\ &\leqslant C\delta^2 \|f^{n-1/2}\|^2 + \frac{1}{2}\|u_h^n - u_h^{n-1}\|^2 + \frac{2}{p} M\delta \|\nabla u_h^{n-1}\|_p^p. \end{aligned}$$

Therefore

$$\|u_h^n - u_h^{n-1}\|^2 \leqslant C\delta^2 \|f^{n-1/2}\|^2 + C\delta \|\nabla \bar{u}_h^n\|_p^p.$$

We obtain the desired result (4.3) by summing the above equation from $n = 1, \dots, N$ and using (4.2). □

4.1. A priori error estimates and convergence

In order to derive *a priori* error estimates, let us introduce the following function.

$$E_h^n = \Pi_h u^n - u_h^n.$$

Theorem 4.3. *Assume that $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $f \in C([0,T], L^2(\Omega)) \cap L^q(0,T,L^q(\Omega))$ and (2.3) holds. If u_h is the unique solution of problem (3.1) and u_h^n with $n = 0, \dots, N$ the unique solution of the fully discrete problem (4.1), then there exits a positive constant C which does not depend on h and δ such that*

$$\begin{aligned}\bar{\partial}\|E_h^n\|^2 &+ \int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx \\ &\leqslant C\Big\{\|u_t^{n-1/2} - \bar{\partial}(\Pi_h u^n)\|\|\bar{E}_h^n\| + \|\nabla(u^{n-1/2} - \overline{\Pi_h} u^n)\|_p^2 \\ &\quad + \int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h} u^n)|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h} u^n)|^2 dx\Big\}.\end{aligned} \tag{4.7}$$

Proof. Note that $E_h^n = (u^n - u_h^n) - (u^n - \Pi_h u^n)$. From equations (2.1) and (4.1), we have

$$\begin{aligned}&(\bar{\partial}E_h^n, \bar{E}_h^n) + a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n) \cdot \nabla(u^{n-1/2} - \bar{u}_h^n) dx \\ &\quad + (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} \cdot \nabla(u^{n-1/2} - \bar{u}_h^n) dx \\ &= (u_t^{n-1/2} - \bar{\partial}u_h^n, \bar{E}_h^n) + a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n) \cdot \nabla \bar{E}_h^n dx \\ &\quad + (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} \cdot \nabla \bar{E}_h^n dx \\ &\quad - (u_t^{n-1/2} - \bar{\partial}\Pi_h u^n, \bar{E}_h^n) \\ &\quad + a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n) \cdot \nabla(u^{n-1/2} - \overline{\Pi_h} u^n) dx \\ &\quad + (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} \cdot \nabla(u^{n-1/2} - \overline{\Pi_h} u^n) dx \\ &= -(u_t^{n-1/2} - \bar{\partial}\Pi_h u^n, \bar{E}_h^n) \\ &\quad + a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n) \cdot \nabla(u^{n-1/2} - \overline{\Pi_h} u^n) dx \\ &\quad + (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} \cdot \nabla(u^{n-1/2} - \overline{\Pi_h} u^n) dx.\end{aligned} \tag{4.8}$$

The left hand side of (4.8) is denoted by LHS, that is

$$\text{LHS} \equiv (\bar{\partial}E_h^n, \bar{E}_h^n) + L1 + L2,$$

where

$$\begin{aligned}L1 &= a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n) \cdot \nabla(u^{n-1/2} - \bar{u}_h^n) dx, \\ L2 &= (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} \cdot \nabla(u^{n-1/2} - \bar{u}_h^n) dx.\end{aligned}$$

If we apply the lower bound of $a(\cdot)$, the relations (2.5) with $\tau = 0$ and (2.9), we get

$$L1 \geqslant mC_2K_1 \int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx.$$

By the relation (2.7),

$$\frac{1}{p}\left(\|\nabla u^{n-1/2}\|_p^p - \|\nabla \bar{u}_h^n\|_p^p\right) \leqslant \int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2}\cdot\nabla(u^{n-1/2}-\bar{u}_h^n)dx.$$

Since $a(\cdot)$ is nondecreasing, without loss of generality, we assume that $(a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \geqslant 0$ (If $(a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p)) \leqslant 0$, we interchange the role of $u^{n-1/2}$ and $\bar{u}_n^n$), then

$$0 \leqslant \frac{1}{p}(a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p))\left(\|\nabla u^{n-1/2}\|_p^p - \|\nabla \bar{u}_h^n\|_p^p\right) \leqslant L2.$$

Therefore,

$$\text{LHS} \geqslant \bar{\partial}\|E_h^n\|^2 + mC_2K_1\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2}-\bar{u}_h^n)|)^{p-2}|\nabla(u^{n-1/2}-\bar{u}_h^n)|^2dx. \tag{4.9}$$

Denoting the right hand side of (4.8) by RHS, we have

$$\text{RHS} = S1 + S2 + S3,$$

where

$$\begin{aligned}
S1 &= -(u_t^{n-1/2} - \bar{\partial}\Pi_h u^n, \bar{E}_h^n) \leqslant \|u_t^{n-1/2} - \bar{\partial}\Pi_h u^n\|\|\bar{E}_h^n\|,\\
S2 &= a(\|\nabla \bar{u}_h^n\|_p^p)\int_\Omega (|\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2} - |\nabla \bar{u}_h^n|^{p-2}\nabla \bar{u}_h^n)\cdot\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)dx,\\
S3 &= (a(\|\nabla u^{n-1/2}\|_p^p) - a(\|\nabla \bar{u}_h^n\|_p^p))\int_\Omega |\nabla u^{n-1/2}|^{p-2}\nabla u^{n-1/2}\cdot\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)dx.
\end{aligned}$$

Applying the right bound of (2.3), the relations (2.4) with $\tau = 0$ and (2.9),

$$S2 \leqslant MC_1K_2\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2}-\bar{u}_h^n)|)^{p-2}|\nabla(u^{n-1/2}-\bar{u}_h^n)||\nabla(u^{n-1/2} - \overline{\Pi}_h u^n)|dx.$$

Now using (2.6) with $x = |\nabla u^{n-1/2}|$, $y = |\nabla(u^{n-1/2} - \bar{u}_h^n)|$ and $z = |\nabla(u^{n-1/2} - \overline{\Pi}_h u^n)|$, we end up with

$$\begin{aligned}
S2 \leqslant\; & MC_1K_2\varepsilon\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2}-\bar{u}_h^n)|)^{p-2}|\nabla(u^{n-1/2}-\bar{u}_h^n)|^2dx\\
&+ MC_1K_2C(\varepsilon^{-1})\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi}_h u^n)|)^{p-2}\|\nabla(u^{n-1/2} - \overline{\Pi}_h u^n)\|^2dx.
\end{aligned}$$

Concerning S3, we have

$$\begin{aligned}
S3 &\leqslant L(|\|\nabla u^{n-1/2}\|_p^p - \|\nabla \bar{u}_h^n\|_p^p|)\|\nabla u^{n-1/2}\|_p^{p-1}\|\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)\|_p \quad \text{(using Holder's inequality)}\\
&\leqslant L\left(\int_\Omega ||\nabla u^{n-1/2}|^p - |\nabla \bar{u}_h^n|^p|dx\right)\|\nabla u^{n-1/2}\|_p^{p-1}\|\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)\|_p\\
&\leqslant pL\int_\Omega (|\nabla u^{n-1/2}| + |\nabla \bar{u}_h^n|)^{p-1}|\nabla(u^{n-1/2} - \bar{u}_h^n)|dx\\
&\quad\times\|\nabla u^{n-1/2}\|_p^{p-1}\|\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)\|_p \quad \text{(applying Lemma 2.3)}\\
&\leqslant pL\left(\int_\Omega (|\nabla u^{n-1/2}| + |\nabla \bar{u}_h^n|)^p dx\right)^{1/2}\left(\int_\Omega (|\nabla u^{n-1/2}| + |\nabla \bar{u}_h^n|)^{p-2}|\nabla(u^{n-1/2} - \bar{u}_h^n)|^2dx\right)^{1/2}\\
&\quad\times\|\nabla u^{n-1/2}\|_p^{p-1}\|\nabla(u^{n-1/2} - \overline{\Pi_h}u^n)\|_p \quad \text{(using Holder's inequality)}\\
&\leqslant K_2^{1/2}pL\left(\int_\Omega (|\nabla u^{n-1/2}| + |\nabla \bar{u}_h^n|)^p dx\right)^{1/2}
\end{aligned}$$

$$\times \left(\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx\right)^{1/2}$$
$$\times \|\nabla u^{n-1/2}\|_p^{p-1} \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p \quad \text{(using the right hand side of (2.9))}$$
$$\leqslant \frac{mC_2K_1}{2}\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx \quad \text{(applying Young's inequality)}$$
$$+ \frac{K_2p^2L^2}{2mC_2K_1}\left(\int_\Omega (|\nabla u^{n-1/2}| + |\nabla \bar{u}_h^n|)^p\, dx\right) \|\nabla u^{n-1/2}\|_p^{2p-2} \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2.$$

Therefore,

$$\begin{aligned} \text{RHS} \leqslant & C\|u_t^{n-1/2} - \bar\partial \Pi_h u^n\| \|\bar{E}_h^n\| \\ & + (MC_1K_2\epsilon + \frac{mC_2K_1}{2})\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx \\ & + MC_1K_2C(\epsilon^{-1})\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^2 dx \\ & + C(\|\nabla u^{n-1/2}\|_p^2 + \|\nabla \bar{u}_h^n\|_p^2) \|\nabla u^{n-1/2}\|_p^{2p-2} \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2. \end{aligned} \tag{4.10}$$

From (4.9) and (4.10), with appropriate choice of ϵ, we end up with

$$\begin{aligned} & \bar\partial\|E_h^n\|^2 + \int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx \\ & \leqslant C\|u_t^{n-1/2} - \bar\partial \Pi_h u^n\| \|\bar{E}_h^n\| + C(\|u\|_{L^\infty(W_0^{1,p}(\Omega))}, \|u_h\|_{L^\infty(W_0^{1,p}(\Omega))}) \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2 \\ & \quad + C\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^2 dx. \end{aligned}$$

□

The main result of this work is the following.

Theorem 4.4. *We assume that*

$$\|u_t\|_{L^\infty(W_0^{2,p}(\Omega))} + \|u_{tt}\|_{L^\infty(W_0^{1,p}(\Omega))} + \|u_{ttt}\|_{L^\infty(L^2(\Omega))} \leqslant C.$$

If u is the unique solution of problem (2.1)-(2.2) and u_h^n with $n = 0, \dots, N$ the unique solution of the fully discrete problem (4.1), then there exits a positive constant C which does not depend on h and δ such that

(a) *for $p \in (1, 2)$ and $u \in L^2(0, T, W^{2,p}(\Omega))$,*

$$\|u^n - u_h^n\|^2 \leqslant C\{h^p + \delta^{2p}\};$$

(b) *for $p \in [2, \infty)$ and $u \in L^2(0, T, W^{2,p}(\Omega)) \cap L^\infty(0, T, W^{1,\infty}(\Omega))$,*

$$\|u^n - u_h^n\|^2 \leqslant C\{h^2 + \delta^4\}.$$

Proof. Let us decompose $u^n - u_h^n$ as

$$u^n - u_h^n = (u^n - \Pi_h u^n) + (\Pi_h u^n - u_h^n) \equiv (u^n - \Pi_h u^n) + E_h^n.$$

The estimation of $\|u^n - \Pi_h u^n\|$ is given by Lemma 3.1. To estimate E_h^n, we have from (4.7) that

$$\begin{aligned} & \frac{1}{2\delta}(\|E_h^n\|^2 - \|E_h^{n-1}\|^2) + \int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \bar{u}_h^n)|)^{p-2} |\nabla(u^{n-1/2} - \bar{u}_h^n)|^2 dx \\ & \leqslant C\delta\|u_t^{n-1/2} - \bar\partial(\Pi_h u^n)\|^2 + \frac{1}{4\delta}(\|E_h^n\|^2 + \|E_h^{n-1}\|^2) + C\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2 \\ & \quad + C\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^2 dx. \end{aligned} \tag{4.11}$$

Case $1 < p < 2$. Note that

$$\begin{aligned}\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^2 dx \\ \leqslant \int_\Omega |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^p dx = \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^p.\end{aligned} \tag{4.12}$$

Taking (4.12) into (4.11) and dropping the second term of the left hand side of (4.11), we have

$$\begin{aligned}\|E_h^n\|^2 \leqslant C\|E_h^{n-1}\|^2 + C\delta^2 \|u_t^{n-1/2} - \bar\partial(\Pi_h u^n)\|^2 \\ + C\delta\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2 + C\delta\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^p.\end{aligned} \tag{4.13}$$

From numerical differentiation and interpolation theory, and on invoking Lemma 3.1, we can prove that

$$\begin{aligned}\|(u_t)^{n-1/2} - \bar\partial(\Pi_h u^n)\| &\leqslant \|(u_t)^{n-1/2} - \bar\partial u^n\| + \|\bar\partial(u^n - \Pi_h u^n)\| \\ &\leqslant C\delta^2 \|u_{ttt}\|_{L^\infty(L^2(\Omega))} + Ch^{2+d(\frac{1}{2}-\frac{1}{p})} \|u_t\|_{L^\infty(W^{2,p}(\Omega))},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p &\leqslant \|\nabla(u^{n-1/2} - \bar u^n)\|_p + \|\nabla(\bar u^n - \overline{\Pi_h u^n})\|_p \\ &\leqslant C\delta^2 \|u_{tt}\|_{L^\infty(W_0^{1,p}(\Omega))} + Ch\|u\|_{L^\infty(W_0^{2,p}(\Omega))}.\end{aligned} \tag{4.15}$$

Taking (4.14) and (4.15) into (4.13), we arrive at

$$\|E_h^n\|^2 \leqslant C\|E_h^{n-1}\|^2 + C\delta^2(\delta^4 + h^{2+d(\frac{1}{2}-\frac{1}{p})}) + C\delta(\delta^4 + h^2) + C\delta(\delta^{2p} + h^p)$$

and by induction, we conclude that

$$\|E_h^n\|^2 \leqslant C(\delta^{2p} + h^p).$$

Case $p \in [2, \infty)$.

$$\begin{aligned}\int_\Omega (|\nabla u^{n-1/2}| + |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|)^{p-2} |\nabla(u^{n-1/2} - \overline{\Pi_h u^n})|^2 dx \\ \leqslant C(\|u\|_{L^\infty(W_0^{1,p}(\Omega))})(\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|^2 + \|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^p).\end{aligned} \tag{4.16}$$

Taking (4.16) into (4.11) and dropping the second term of the left hand side of it,

$$\begin{aligned}\|E_h^n\|^2 \leqslant C\|E_h^{n-1}\|^2 + C\delta^2\|u_t^{n-1/2} - \bar\partial(\Pi_h u^n)\|^2 + C\delta\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^2 \\ + C\delta\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|^2 + C\delta\|\nabla(u^{n-1/2} - \overline{\Pi_h u^n})\|_p^p.\end{aligned}$$

Using (4.14), (4.15), and by induction, we conclude that

$$\|E_h^n\|^2 \leqslant C(\delta^4 + h^2).$$

□

4.2. Numerical method

At any time step, the fully discrete scheme (4.1) leads to a system of nonlinear equations. The Newton-Raphson iterative method is one of the attractive method to solve the nonlinear algebraic system, it is fast convergent, achieves the desired tolerance in a small number of iterations, and thus preserves the finite element order of convergence but in the present case, if we use Newton's method to solve (4.1), we observe that the sparsity of Jacobian matrices is lost due to the presence of nonlocal term in the equation. Indeed, (4.1) is equivalent to, for all $v_h \in V_h$,

$$(u_h^n, v_h) + \delta a(\|\nabla \bar u_h^n\|_p^p) \int_\Omega |\nabla \bar u_h^n|^{p-2} \nabla \bar u_h^n \cdot \nabla v_h \, dx - \delta(f^{n-1/2}, v_h) - (u_h^{n-1}, v_h) = 0. \tag{4.17}$$

Let N_p be the dimension and $\{\varphi_j\}_{j=1}^{N_p}$ be the canonical basis of V_h associated with the nodes of $\mathcal{T}_h$. u_h^n can be written as

$$u_h^n = \sum_{j=1}^{N_p} \beta_j^n \varphi_j, \quad \beta_j^n \in \mathbb{R}.$$

Set $\beta^n = [\beta_1^n, \dots, \beta_{N_p}^n]$, (4.17) leads to a system of nonlinear algebraic equations

$$R_i(\beta^n) = R_i(u_h^n) = 0, \quad 1 \leqslant i \leqslant N_p, \tag{4.18}$$

where

$$R_i(u_h^n) = (u_h^n, \varphi_i) + \delta a(\|\nabla \bar{u}_h^n\|_p^p) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_i \, dx - \delta(f^{n-1/2}, \varphi_i) - (u_h^{n-1}, \varphi_i), \quad 1 \leqslant i \leqslant N_p.$$

Any element of the Jacobian matrix J_1 takes the form

$$\begin{aligned}\frac{\partial R_i}{\partial \beta_j^n}(u_h^n) = (\varphi_j, \varphi_i) &+ p/2\delta a'(\|\nabla \bar{u}_h^n\|_p^p) \left(\int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_j \, dx\right) \left(\int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_i \, dx\right) \\ &+ \delta a(\|\nabla \bar{u}_h^n\|_p^p) \\ &\times \left((p-2)/2 \int_\Omega |\nabla \bar{u}_h^n|^{p-4} (\nabla \bar{u}_h^n \cdot \nabla \varphi_j)(\nabla \bar{u}_h^n \cdot \nabla \varphi_i) \, dx + 1/2 \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \varphi_j \cdot \nabla \varphi_i \, dx\right).\end{aligned}$$

Because of the first term of the equation above, we observe that the sparsity of the Jacobian matrix is lost (see also Figure 2). In order to avoid this difficulty, we adopt and extend the technique proposed in [14] for elliptic problems. The modified method is defined as follows: find $x \in \mathbb{R}$ and $u_h^n \in V_h$ such that

$$\begin{aligned}&\|\nabla \bar{u}_h^n\|_p^p - x = 0, \quad \forall v_h \in V_h, \\ &(u_h^n, v_h) + \delta a(x) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla v_h \, dx - \delta(f^{n-1/2}, v_h) - (u_h^{n-1}, v_h) = 0.\end{aligned} \tag{4.19}$$

To see the sparsity of the Jacobian matrix J_2 for (4.19), we define

$$\begin{aligned}R_i(u_h^n, x) &= (u_h^n, \varphi_i) + \delta a(x) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_i \, dx - \delta(f^{n-1/2}, \varphi_i) - (u_h^{n-1}, \varphi_i), \; 1 \leqslant i \leqslant N_p, \\ R_{N_p+1}(u_h^n, x) &= \|\nabla \bar{u}_h^n\|_p^p - x, \\ A_{ij} := \frac{\partial R_i}{\partial \beta_j^n}(u_h^n, x) &= (\varphi_j, \varphi_i) + \delta a(x) \left((p-2)/2 \int_\Omega |\nabla \bar{u}_h^n|^{p-4} (\nabla \bar{u}_h^n \cdot \nabla \varphi_j)(\nabla \bar{u}_h^n \cdot \nabla \varphi_i) dx \right. \\ &\left. + 1/2 \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \varphi_j \cdot \nabla \varphi_i \, dx\right), \quad 1 \leqslant i, j \leqslant N_p, \\ B_{i1} := \frac{\partial R_i}{\partial x}(u_h^n, x) &= \delta a'(x) \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_i \, dx, \quad 1 \leqslant i \leqslant N_p, \\ C_{1j} := \frac{\partial R_{N_p+1}}{\partial \beta_j^n}(u_h^n, x) &= p/2 \int_\Omega |\nabla \bar{u}_h^n|^{p-2} \nabla \bar{u}_h^n \cdot \nabla \varphi_j \, dx, \quad 1 \leqslant j \leqslant N_p, \\ \delta_{11} := \frac{\partial R_{N_p+1}}{\partial x}(u_h^n, x) &= -1.\end{aligned}$$

Note that $\|\nabla \bar{u}_h^n\|_p^p$ does not depend on x, then its derivative with respect to x is zero. Therefore, J_2 takes the form

$$J_2 := \begin{pmatrix} A & B \\ C & -1 \end{pmatrix}.$$

Given a vector function $\mathbf{w}$, the matrix A has the same form as the following

$$S_{ij} = \int_\Omega \varphi_j \varphi_i dx + \int_\Omega |\mathbf{w}|(\mathbf{w} \cdot \nabla\varphi_j)(\mathbf{w} \cdot \nabla\varphi_i) dx + \int_\Omega |\mathbf{w}|\nabla\varphi_j \cdot \nabla\varphi_i dx, \quad 1 \leqslant i, j \leqslant N_p,$$

which is the sparse matrix by the definition of basis functions φ_m. $B = (B_{i1})_{1\leqslant i\leqslant N_p}$, $C = (C_{1j})_{1\leqslant j\leqslant N_p}$ are respectively one column and one line matrices which can be full . Following [14], if we assume that A is invertible (which is the case here under assumption on $a(\cdot)$), then given a vector F and a scalar d, the following problem: find the vector X and the scalar y such that

$$\begin{bmatrix} A & B \\ C & -1 \end{bmatrix} \begin{bmatrix} X \\ y \end{bmatrix} = \begin{bmatrix} F \\ d \end{bmatrix},$$

leads to a unique solution

$$y = -\frac{d - CA^{-1}F}{1 + CA^{-1}B}, \quad X = A^{-1}F - A^{-1}By.$$

Note that $1 + CA^{-1}B \neq 0$ and we can compute $A^{-1}B$ and $A^{-1}F$ by using any iterative method.
It can easily be proven that

Theorem 4.5. *Given* u_h^{n-1}, *if* (u_h^n, x) *is a solution of* (4.19), *then* u_h^n *is a solution to* (4.17). *Conversely, if* u_h^n *is a solution of* (4.17), *then* $(u_h^n, \|\nabla\bar{u}_h^n\|_p^p)$ *is a solution of* (4.19).

5. Numerical simulations

In this section we present some numerical experiments to illustrate our theoretical analysis. All computations were performed using Matlab. Let $\Omega = (0, 1)^2$ and the final time $T = 1$. We consider problem (1.1) with

$$a(s) = 3 + \sin(s), \quad u_0(x, y) = xy(1 - x)(1 - y).$$

The function f is chosen corresponding to the exact solution

$$u_{ex}(x, y) = xy(1 - x)(1 - y) \exp(-t).$$

A uniform triangular partition with $N_p + 1$ nodes in each direction is used to generate the finite element method mesh with $h = 1/N_p$. We solve our nonlinear system (4.19) using the Newton-Raphson method. The initial guesses u_{guess} and x_{guess} are taking as follows:

$$\begin{cases} u_{\text{guess}} = u_0, \\ x_{\text{guess}} = \|u_0\|_p^p. \end{cases}$$

To analyze the convergence rates, this problem has been simulated for the values $p = 1.5$ and $p = 3$ respectively and with different combinations h and δ and the error results are represented in Figure 1. The error has been calculated using the L^2–norm. For the convergence with respect to the mesh size h (see Figure 1 (a)), δ is fixed at $\delta = 0.02$ and we solve problem (4.19) with different values of h ($h = 1/5; 1/10; 1/15; 1/20; 1/25$), from our theoretical analysis, the L^2-norm errors are in order $O(h^{p/2} + \delta^p) = O(h^{3/4} + \delta^{3/2}) \sim O(h^{3/4})$ for $p = 3/2$ and $O(h + \delta^2) \sim O(h)$ for $p = 3$. For the convergence with respect to the time step δ (see Figure 1 (b)), h is chosen as $h = \delta^2$ and we solve problem (4.19) with different time steps ($\delta = 1/4; 1/8; 1/16; 1/32$), the L^2-norm errors are in order $O(h^{p/2} + \delta^p) = O(\delta^{3/2} + \delta^{3/2}) \sim O(\delta^{3/2})$ for $p = 3/2$ and $O(h + \delta^2) = O(\delta^2 + \delta^2) \sim O(\delta^2)$ for $p = 3$. As expected the pictures show the convergence rates of $O(h^{3/4} + \delta^{3/2})$ for $1 < p = 3/2 < 2$ and $O(h + \delta^2)$ for $2 < p = 3$, which are in accordance with the theoretical analysis.

In summary, Figure 1 (a) shows the rate of convergence with respect to the mesh size h: for $p = 1.5$, the rate of convergence is 0.75 while for $p = 3$, the rate is 1. Figure 1 (b) shows the rate of convergence with respect to the time δ: for $p = 1.5$, the rate of convergence is 1.5 while for $p = 3$, the rate is 2.

To show numerically the non sparsity and the sparsity of the Jacobian matrices J_1 and J_2, respectively, we depict graphically J_1 and J_2 for different values of h, see Figures 2 and 3. In these figures, the numbers on the vertical and horizontal lines represent the row and column numbers of the Jacobian matrix, respectively. The number nz denotes the number of nonzero elements. Figures 2 and 3 give the structures of Jacobian matrices. Figure 2 shows that the Jacobian matrix J_1 is full when applying Newton-Raphson method directly to the problem (4.18) while Figure 3 shows that the Jacobian matrix is sparse when applying algorithm (4.19).

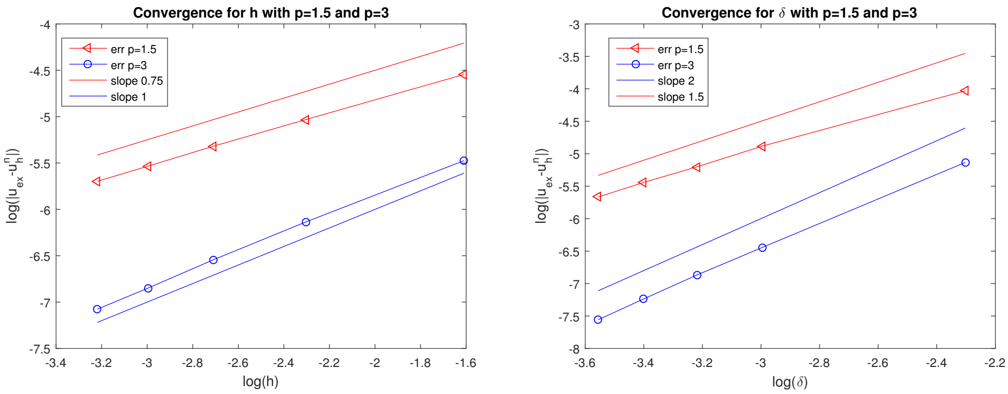

Figure 1: (a) h- rate for $p = 1.5$ and $p = 3$; (b) δ- rate for $p = 1.5$ and $p = 3$ (right).

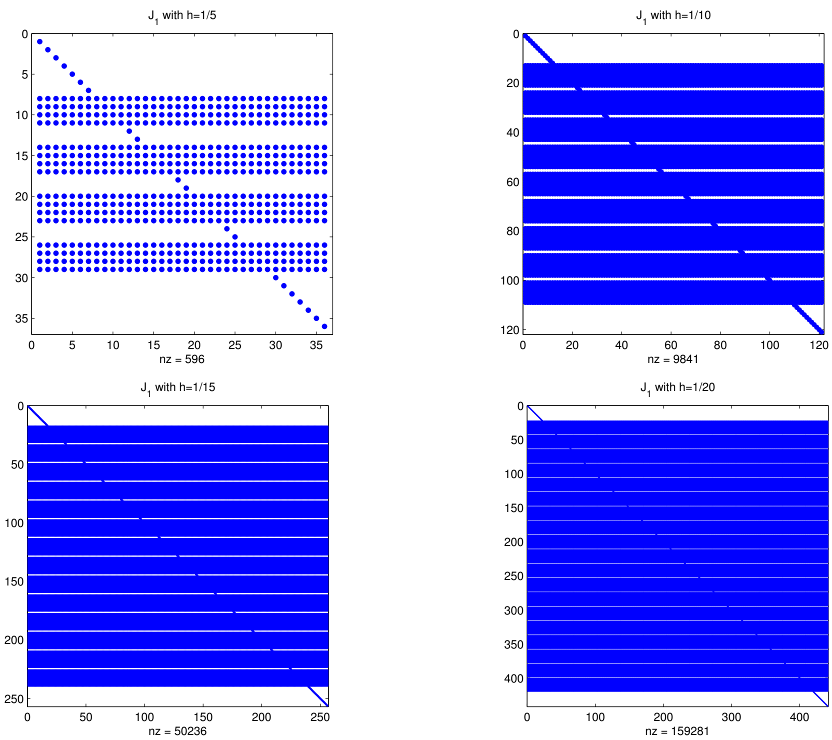

Figure 2: Jacobian matrices J_1.

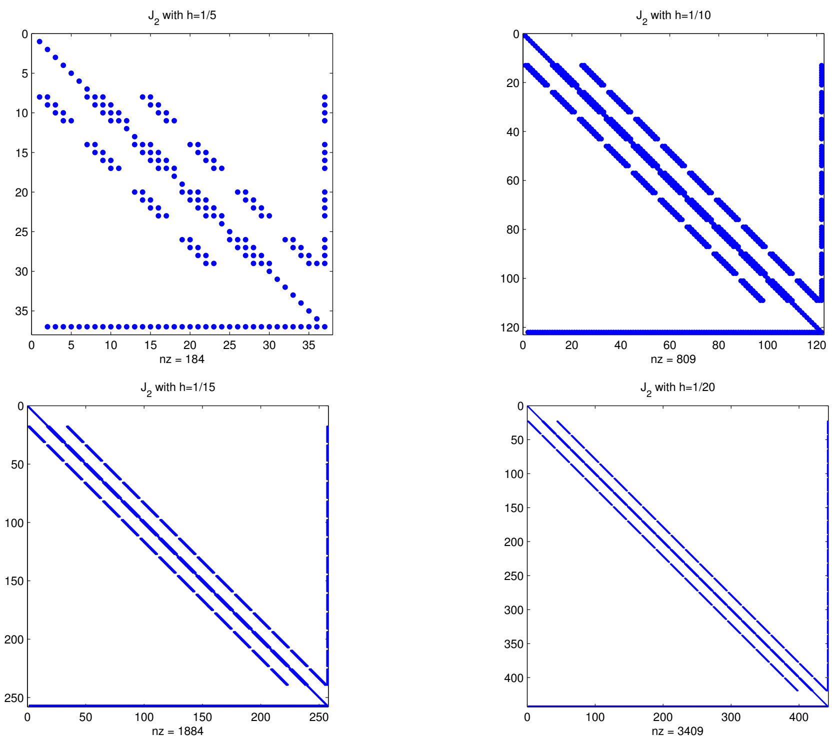

Figure 3: Jacobian matrices J_2.

6. Conclusion

In this article, a Crank-Nicolson Galerkin finite element method for a class of nonlinear nonlocal diffusion problems associated with p-Laplace-type operator has been presented here. Firstly, the semi discrete approximation and the fully-discrete formulation have been discussed. More important, the optimal order of convergence has been derived for the fully discrete approximation without any restriction of time step. Secondly, since the fully discrete approximation leads to the nonlinear system, the Jacobian matrix for the corresponding nonlinear system is full, the new equivalent problem whose Jacobian matrix is sparse has been proposed. Finally, we have exhibited some numerical experiments that validate the theoretical findings. Our next challenge is to study a posteriori error control for nonlocal problems in fluids under nonlinear slip boundary condition.

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