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General types of sup-hesitant fuzzy ideals of ternary semigroups

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Abstract

General types of the paper [P. Julatha, A. Iampan, Int. J. Fuzzy Log. Intell. Syst., **21** (2021), 169–175] are discussed. The concepts of $\sup_{\gamma}^{+}$-hesitant and $\sup_{\delta}^{-}$-hesitant fuzzy (resp., left, right, lateral) ideals of ternary semigroups are introduced, and their properties are investigated. Characterizations of the concepts are given in terms of fuzzy sets, Łukasiewicz (anti-) fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets, and interval-valued fuzzy sets. Moreover, we show relationships among interval-valued, hesitant, sup-hesitant, $\sup_{\gamma}^{+}$-hesitant and $\sup_{\delta}^{-}$-hesitant fuzzy ideals.

Keywords: General type of sup-hesitant fuzzy ideal, sup-hesitant fuzzy ideal, hesitant fuzzy ideal, interval-valued fuzzy ideal, Pythagorean fuzzy ideal, Łukasiewicz (anti-) fuzzy set.

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1. Introduction

After the idea of fuzzy sets (FSs) was introduced by Zadeh [38], he and others have found applications in many branches: computer science, automata, control engineering, robotics and theories of (Γ-) rings, (Γ-) groups, BCK/BCI-algebras, UP-algebras, (Γ-) semigroups, ternary semigroups and other algebras. In the case that different sources of imprecise and vague information appear simultaneously, there are limitations for using FSs to deal with them. In order to solve this problem, Zadeh and researchers have been studied and developed a lot of extended and general concepts of FSs such as hesitant fuzzy sets (HFSs) (see [33, 34]), Pythagorean fuzzy sets (PFSs) (see [35, 36]), interval-valued fuzzy sets (IvFSs) (see [30, 39]), bipolar fuzzy sets (see [2, 37]), cubic sets (see [11, 19]), (fuzzy) soft sets (see [7, 9, 23]), and extentions of fuzzy soft sets (see [8, 10, 12]).

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Ternary semigroup theory has been studied and developed by FSs and generalizations of FSs. Kar and Sarkar [21] introduced fuzzy (resp., lateral, left, right) ideals of ternary semigroups and characterized (intra-) regular ternary semigroups by their FSs. Shabir and Rehman [28] introduced anti-types of FSs: anti-fuzzy (resp., lateral, left, right) ideals of ternary semigroups and used the anti-types to characterize different classes of ternary semigroups. Chinram and Panityakul [5] studied PFSs on ternary semigroups and investigated charecterizations of Pythagorean fuzzy (resp., lateral, left, right) ideals of ternary semigroups. Suebsung and Chinram [30] used the concept of IvFSs to develope ternary semigroup theory. They studied interval-valued fuzzy (resp., lateral, left, right) ideals of ternary semigroups and their extensions.

The idea of HFSs introduced Tora [33]. This notion is an abstract of FSs, PFSs, and IvFSs. Studying HFSs on ternary semigroups, Talee and et al. [32] introduced hesitant fuzzy (resp., lateral, left, right) ideals of ternary semigroups, and characterize some classes of ternary semigroups in terms of HFSs. Julatha and Iampan [15] studied HFSs in the meaning of supremum, and introduced sup-types of HFSs: sup-hesitant fuzzy (resp., lateral, left, right) ideals of ternary semigroups. Further, the sup-types were characterized in terms of FSs, HFSs, and IvFSs. In the literature, HFSs on ternary semigroups and related algebras have been considered by many authores (see [1, 6, 13, 16, 24–26, 31]).

In 2022, Jittburus et al. [14] introduced $\sup^{+}_{\gamma}$-hesitant and $\sup^{-}_{\delta}$-hesitant fuzzy ideals, which are general types of sup-hesitant fuzzy ideals, of semigroups and investigated their characterizations via FSs, Łukasiewicz (anti-) fuzzy sets, PFSs, HFSs and IvFSs. Moreover, intra-regular, left (right) regular, completely regular, left (right) simple and simple semigroups were characterized in terms of the $\sup^{+}_{\gamma}$ and $\sup^{-}_{\delta}$-types of HFSs.

As previously stated, it motivated us to apply the $\sup^{+}_{\gamma}$- and $\sup^{-}_{\delta}$-types of HFSs to ternary semigroups. The concepts of $\sup^{+}_{\gamma}$-hesitant and $\sup^{-}_{\delta}$-hesitant fuzzy (resp., left, right, lateral) ideals of ternary semigroups are introduced, and their properties are investigated. Characterizations of the concepts are discussed in terms of FSs, PFSs, HFSs, IvFSs, and Łukasiewicz (anti-) fuzzy sets. Moreover, we consider the relationships among interval-valued, hesitant, sup-hesitant, $\sup^{+}_{\gamma}$-hesitant, and $\sup^{-}_{\delta}$-hesitant fuzzy ideals of ternary semigroups.

2. Preliminaries

In this section, we recall some important notions in our study. We divide this section into two subsections. The first subsection provides certain information on the mathematical tools dealing with uncertainties. The concepts of various kinds of fuzzy ideals in ternary semigroups are given in Subsection 2.2.

2.1. *Various kinds of fuzzy sets*

We are reminded of fuzzy sets and a few generalizations that will be considered in the following sections of the current study. The concept of fuzzy sets is the first mathematical object we will recall that can deal with unstable situations.

Let $\mathcal{S}$ be a nonempty set. A *fuzzy set* (FS) $\mathcal{A}$ in $\mathcal{S}$ is a pair $\mathcal{A} := (\mathcal{S}, \vartheta_{\mathcal{A}})$, where $\vartheta_{\mathcal{A}} \colon \mathcal{S} \to [0,1]$. The mapping $\vartheta_{\mathcal{A}}$ is called the *membership function* of $\mathcal{A}$ (see [38]). We can see that the membership function $\vartheta_{\mathcal{A}}$ determines the FS $\mathcal{A} := (\mathcal{S}, \vartheta_{\mathcal{A}})$ in $\mathcal{S}$. Therefore, we usually denote any FS $\mathcal{A} := (\mathcal{S}, \vartheta_{\mathcal{A}})$ by its membership function $\vartheta_{\mathcal{A}}$. Furthermore, if it is clear from the context, the FS $\vartheta_{\mathcal{A}}$ may be written by ϑ.

Any element a of $[0,1]$ can be regarded as an FS in $\mathcal{S}$ assigned by $a(s) := a$ for all $s \in \mathcal{S}$. For any subset $\mathcal{A}$ of $\mathcal{S}$, we define an FS $\chi_{\mathcal{A}}$ in $\mathcal{S}$ by $\chi_{\mathcal{A}}(s) = 1$ if $s \in \mathcal{A}$ and $\chi_{\mathcal{A}}(s) = 0$ if $s \notin \mathcal{A}$. This FS $\chi_{\mathcal{A}}$ is called the *characteristic function* of $\mathcal{A}$ in $\mathcal{S}$. Let ϑ and σ be FSs of $\mathcal{S}$. We define a binary relation $\subseteq$ on the set of all FSs of $\mathcal{S}$ by $\vartheta \subseteq \sigma$ if $\vartheta(s) \leqslant \sigma(s)$ for all $s \in \mathcal{S}$.

Remark 2.1. One can see that the set $\mathcal{S}$ can be regarded as the characteristic function $\chi_{\mathcal{S}}$. This reason illustrates that the notion of FSs is a generalization of sets.

By an *interval number* $\widetilde{t}$, we mean an interval $[t^L, t^U]$ and $0 \leqslant t^L \leqslant t^U \leqslant 1$. We denote the set of all interval numbers by $\mathcal{D}([0,1])$ and define a binary relation $\preceq$ on $\mathcal{D}([0,1])$ by

$$\widetilde{s} \preceq \widetilde{t} \quad \text{if and only if} \quad s^L \leqslant t^L \text{ and } s^U \leqslant t^U$$

for all $\widetilde{s} = [s^L, s^U], \widetilde{t} = [t^L, t^U] \in \mathcal{D}([0,1])$. The notation $\widetilde{t} \succeq \widetilde{s}$ stands for $\widetilde{s} \preceq \widetilde{t}$. If $\widetilde{s} \preceq \widetilde{t}$ and $\widetilde{t} \preceq \widetilde{s}$, then $\widetilde{s} = \widetilde{t}$. Moreover, we say that $\widetilde{s} \prec \widetilde{t}$ if $\widetilde{s} \preceq \widetilde{t}$ and $\widetilde{s} \neq \widetilde{t}$.

Ten years later, after introducing FSs, Zadeh defined an interval-valued fuzzy set (IvFS) in 1975 as follows. An IvFS $\mathcal{A}$ on $\mathcal{S}$ is a pair $\mathcal{A} := (\mathcal{S}, \widetilde{\varpi}_{\mathcal{A}})$, where $\widetilde{\varpi}_{\mathcal{A}} : \mathcal{A} \to \mathcal{D}([0,1])$. If it is evident from the context, the subscript of the mapping $\widetilde{\varpi}_{\mathcal{A}}$ will be omitted. Given an IvFS $\mathcal{A} := (\mathcal{S}, \widetilde{\varpi})$. We observe that the function $\widetilde{\varpi}$ is determined by FSs $\widetilde{\varpi}^L$ and $\widetilde{\varpi}^U$ in such a way that $\widetilde{\varpi}(s) := [\widetilde{\varpi}^L(s), \widetilde{\varpi}^U(s)] \in \mathcal{D}([0,1])$ for all $s \in \mathcal{S}$. Any interval number $[t^L, t^U]$ can be regarded as an IvFS defined by $[t^L(s), t^U(s)]$ for all $s \in \mathcal{S}$ (see [39]).

Remark 2.2. We observe that any FS ϑ in $\mathcal{S}$ can be considered as an IvFS defined by $[0(s), \vartheta(s)]$ for all $s \in \mathcal{S}$. By this explanation, we can think about IvFSs as a generalization of FSs.

Another generalization of FSs is the idea of Pythagorean fuzzy sets (PFSs). Yager and Abbasov [35] presented the concept of PFSs in 2013. A PFS $\mathcal{A}$ in $\mathcal{S}$ is $\mathcal{A} := (\mathcal{S}, \vartheta_{\mathcal{A}}, \sigma_{\mathcal{A}})$, where $\vartheta_{\mathcal{A}}$ and $\sigma_{\mathcal{A}}$ are FSs in $\mathcal{S}$ such that

$$0 \leqslant (\vartheta_{\mathcal{A}}(s))^2 + (\sigma_{\mathcal{A}}(s))^2 \leqslant 1$$

for all $s \in \mathcal{S}$. From now on, we omit the subscript of any PFS $\mathcal{A} := (\mathcal{S}, \vartheta_{\mathcal{A}}, \sigma_{\mathcal{A}})$ if the context is clear. Since any PFS $\mathcal{A} := (\mathcal{S}, \vartheta, \sigma)$ is considered by the mappings ϑ and σ, we usually denote $\mathcal{A} := (\mathcal{S}, \vartheta, \sigma)$ by (ϑ, σ) (see [36]).

Remark 2.3. We observe that any FS ϑ in $\mathcal{S}$ can be regarded as a PFS $(\vartheta, 0)$.

We will now revisit the idea of hesitant fuzzy sets (HFSs). This notion is an abstract of FSs and IvFSs. Tora [33] introduced it. A HFS $\mathcal{A}$ on $\mathcal{S}$ is a pair $\mathcal{A} := (\mathcal{S}, \widehat{\xi}_{\mathcal{A}})$, where the notation $\mathcal{P}([0,1])$ is the set of all subsets of $[0,1]$ and $\widehat{\xi}_{\mathcal{A}} : \mathcal{S} \to \mathcal{P}([0,1])$ (see [34]). We see that the HFS $\mathcal{A} := (\mathcal{S}, \widehat{\xi}_{\mathcal{A}})$ is determined by $\widehat{\xi}_{\mathcal{A}}$. Thus, for simplicity, we denote the HFS $\mathcal{A} := (\mathcal{S}, \widehat{\xi}_{\mathcal{A}})$ by $\widehat{\xi}_{\mathcal{A}}$. The subscript of $\widehat{\xi}_{\mathcal{A}}$ can be omitted if the context is clear. Let $\mathcal{A}$ be any subset of $\mathcal{S}$. The *characteristic hesitant fuzzy set* (CHFS) $\widehat{\chi}_{\mathcal{A}}$ of $\mathcal{A}$ on $\mathcal{S}$ is given by:

$$\widehat{\chi}_{\mathcal{A}}(s) := \begin{cases} [0,1], & \text{if } s \in \mathcal{A}, \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $s \in \mathcal{S}$.

Remark 2.4. Extending FSs by HFSs is possible because the image of FSs can be viewed as a singleton set of the element in $[0,1]$. Moreover, every IvFS is an HFS, which is evident given that the image of IvFSs is a set.

In this work, HFSs are the main focus. Let us provide some of the notions and properties discussed later in the paper to the readers. We refer the readers to [13, 15, 26] for the literature in the developing of HFSs. We define $\sup^* : \mathcal{P}([0,1]) \to [0,1]$ by

$$\sup^* \Psi := \begin{cases} \sup \Psi, & \text{if } \Psi \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\Psi \subseteq [0,1]$. For each HFS $\widehat{\xi}$ on $\mathcal{S}$, the operation $\sup^*$ induces an FS $\mathcal{F}^{\widehat{\xi}}$ in $\mathcal{S}$ given by $\mathcal{F}^{\widehat{\xi}}(s) := \sup^* \widehat{\xi}(s)$ for all $s \in \mathcal{S}$. Let $\widehat{\xi}$ be an HFS on $\mathcal{S}$, Φ a subset of $[0,1]$ and $s \in \mathcal{S}$. We define the sets

$$\mathcal{S}[\widehat{\xi}; \Phi] := \{s \in \mathcal{S} : \sup^* \widehat{\xi}(s) \geqslant \sup^* \Phi\},$$

and

$$\mathcal{H}_{\sup^*}^{(\widehat{\xi}, \Phi)}(s) := \{\alpha \in \Phi : \sup^* \widehat{\xi}(s) \geqslant \alpha\}.$$

2.2. Various kinds of fuzzy ideals in ternary semigroups

Now, we recall the concept of ternary semigroups. In addition, we used the FSs developed in the preceding subsection to the ideals in ternary semigroups. By a *ternary semigroup* $(\mathfrak{T}; [\cdot])$ we mean a structure consisting of a nonempty set $\mathfrak{T}$ and a ternary operation $[\cdot]\colon \mathfrak{T}\times\mathfrak{T}\times\mathfrak{T}\to\mathfrak{T}$ satisfying the identity

$$[[t_1t_2t_3]t_4t_5] = [t_1[t_2t_3t_4]t_5] = [t_1t_2[t_3t_4t_5]]. \qquad \text{(associativity)}$$

The motivation in defining ternary semigroups, we refer the readers to [22]. It was known that any semigroup induces a ternary semigroup, but not conversely. This reason motivates us to study various kinds of fuzzy ideals in ternary semigroups. For any nonempty subsets $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ of a ternary semigroup $(\mathfrak{T}; [\cdot])$, we let

$$[\mathcal{X}\mathcal{Y}\mathcal{Z}] := \{[t_1t_2t_3] : t_1 \in \mathcal{X}, t_2 \in \mathcal{Y} \text{ and } t_3 \in \mathcal{Z}\}.$$

A ternary semigroup $(\mathfrak{T}; [\cdot])$ is usually denoted by its universe set $\mathfrak{T}$, and the notation $[t_1t_2t_3]$ is always written by $t_1t_2t_3$. This means that we always use the notation $\mathcal{X}\mathcal{Y}\mathcal{Z}$ instead of $[\mathcal{X}\mathcal{Y}\mathcal{Z}]$. Moreover, by the associativity of the ternary operation defined on $\mathfrak{T}$, we have that $t_1t_2\cdots t_{2n+1}$ is meaningful for all natural number n.

The conceptions of various types of ideals play crucial roles in understanding the structural characteristics of ternary semigroups. Following is a list of *ideals* (Ids), *left ideals* (LIds), *lateral ideals* (LtIds), and *right ideals* (RIds) of ternary semigroups. A nonempty subset $\mathcal{A}$ of $\mathfrak{T}$ is called:

(1) an LtId (resp., LId, RId) of $\mathfrak{T}$ if $\mathfrak{T}\mathcal{A}\mathfrak{T} \subseteq \mathcal{A}$ (resp., $\mathfrak{T}\mathfrak{T}\mathcal{A} \subseteq \mathcal{A}$, $\mathcal{A}\mathfrak{T}\mathfrak{T} \subseteq \mathcal{A}$);
(2) an Id of $\mathfrak{T}$ if $\mathcal{A}$ is an LtId, LId, and RId of $\mathfrak{T}$.

The notion of Ids of $\mathfrak{T}$ was studied in detail in [20, 29]. Moreover, in terms of these Ids, the characterizations of $\mathfrak{T}$ were provided in [27].

The idea of Ids is among the most widely utilized concepts in the study of ternary semigroup theory. We also present the notions of LtIds, LIds, RIds, and Ids of $\mathfrak{T}$ related to various FSs, as we have already mentioned several kinds of FSs: a *fuzzy ideal* (FId), a *fuzzy lateral ideal* (FLtId), a *fuzzy left ideal* (FLId), and a *fuzzy right ideal* (FRId). An FS ϑ in $\mathfrak{T}$ is said to be:

(1) an FLtId (resp., FLId, FRId) of $\mathfrak{T}$ if $\vartheta(svz) \geqslant \vartheta(v)$ (resp., $\vartheta(svz) \geqslant \vartheta(z)$, $\vartheta(svz) \geqslant \vartheta(s)$) for all $s, v, z \in \mathfrak{T}$;
(2) an FId of $\mathfrak{T}$ if ϑ is an FLtId, FLId, and FRId of $\mathfrak{T}$.

In 2012, Kar and Sarkar [21] conducted ground-breaking research on FIds in ternary semigroup theory. They defined the above FIds and applied them to characterize particular classes of ternary semigroups. Similarly, in ternary semigroups, Shabir and Rehman [28] defined related concepts, so-called *anti-fuzzy ideals* (AFIds), *anti-fuzzy lateral ideals* (AFLtIds), *anti-fuzzy left ideals* (AFLIds), and *anti-fuzzy right ideals* (AFRIds), as follows. An FS ϑ in $\mathfrak{T}$ is said to be:

(1) an AFLtId (resp., AFLId, AFRId) of $\mathfrak{T}$ if $\vartheta(svz) \leqslant \vartheta(v)$ (resp., $\vartheta(svz) \leqslant \vartheta(z)$, $\vartheta(svz) \leqslant \vartheta(s)$) for all $s, v, z \in \mathfrak{T}$;
(2) an AFId of $\mathfrak{T}$ if ϑ is an AFLtId, AFLId, and AFRId of $\mathfrak{T}$.

Some regularities of ternary semigroups were described using these AFIds in [28].

In ternary semigroups, Suebsung and Chinram [30] studied the ideas of *interval-valued fuzzy ideals* (IvFIds), *interval-valued fuzzy lateral ideals* (IvFLtIds), *interval-valued fuzzy left ideals* (IvFLIds) and *interval-valued fuzzy right ideals* (IvFRIds) defined as follows. An IvFS $\widetilde{\varpi}$ on $\mathfrak{T}$ is said to be:

(1) an IvFLtId (resp., IvFLId, IvFRId) of $\mathfrak{T}$ if $\widetilde{\varpi}(svz) \succeq \widetilde{\varpi}(v)$ (resp., $\widetilde{\varpi}(svz) \succeq \widetilde{\varpi}(z)$, $\widetilde{\varpi}(svz) \succeq \widetilde{\varpi}(s)$) for all $s, v, z \in \mathfrak{T}$;
(2) an IvFId of $\mathfrak{T}$ if $\widetilde{\varpi}$ is an IvFLtId, IvFLId, and IvFRId of $\mathfrak{T}$.

Suebsung and Chinram research a variety of IvFIds' properties. The primitive of such notions were also investigated. Moreover, the extensions of IvFSs were introduced and investigated (see [30]).

Remark 2.5. By Remark 2.2, we have that any FLtId (resp., FLId, FRId, FId) can be considered as an IvFLtId (resp., IvFLId, IvFRId, IvFId).

In 2020, Chinram and Panityakul [5] presented ideas and properties of Pythagorean fuzzy ideals (PFIds), Pythagorean fuzzy lateral ideals (PFLtIds), Pythagorean fuzzy left ideals (PFLIds) and Pythagorean fuzzy right ideals (PFRIds) of ternary semigroups. In order to characterize these PFSs they established, they defined a product of PFSs in ternary semigroups (see [5]). The following are the definitions of a PFLtId, a PFLId, a PFRId, and a PFId of Ŧ. A PFS (ϑ, σ) in Ŧ is said to be:

(1) a PFLtId (resp., PFLId, PFRId) of Ŧ if ϑ is an FLtId (resp., FLId, FRId) of Ŧ and σ is an AFLtId (resp., AFLId, AFRId) of Ŧ;
(2) a PFId of Ŧ if (ϑ, σ) is a PFLtId, PFLId and PFRId of Ŧ.

Remark 2.6. We can produce PFLtIds (resp., PFLIds, PFRIds, PFIds) in ternary semigroups by using FLtIds (resp., FLIds, FRIds, FIds) as indicated in Remark 2.3.

Talee et al. [32] established the concepts of hesitant fuzzy ideals (HFIds), hesitant fuzzy lateral ideals (HFLtIds), hesitant fuzzy left ideals (HFLIds), and hesitant fuzzy right ideals (HFRIds), which were then used to apply the concept of HFSs to ternary semigroups. The following are formal definitions of these concepts. An HFS $\widehat{\xi}$ on Ŧ is said to be:

(1) an HFLtId (resp., HFLId, HFRId) of Ŧ if $\widehat{\xi}(s\nu z) \supseteq \widehat{\xi}(\nu)$ (resp., $\widehat{\xi}(s\nu z) \supseteq \widehat{\xi}(z)$, $\widehat{\xi}(s\nu z) \supseteq \widehat{\xi}(s)$) for all $s, \nu, z \in$ Ŧ;
(2) an HFId of Ŧ if $\widehat{\xi}$ is an HFLtId, HFLId, and HFRId of Ŧ.

Remark 2.7. We can see, by Remark 2.4, that the notion of HFLtIds (resp., HFLIds, HFRIds, HFIds) is an abstraction of FLtIds (resp., FLIds, FRIds, FIds).

Remark 2.8 ([15]). In general, an IvFId of Ŧ is not an HFId of Ŧ and a HFId of Ŧ is not an IvFId of Ŧ.

We will now discuss the general notion of HFIds and IvFIds in ternary semigroup theory. In 2021, Julatha and Iampan [15] introduced sup-hesitant fuzzy ideals (SHFIds), sup-hesitant fuzzy lateral ideals (SHFLtIds), sup-hesitant fuzzy left ideals (SHFLIds), and sup-hesitant fuzzy right ideals (SHFRIds) of ternary semigroups seen in Definition 2.9, and showed that the notion of SHFIds is a general type of HFIds and IvFIds.

Definition 2.9 ([15])**.** An HFS $\widehat{\xi}$ on Ŧ is said to be:

(1) a SHFLtId of Ŧ if $\mathcal{S}[\widehat{\xi}; \Phi]$ is either an empty set or an LtId of Ŧ for all $\Phi \subseteq [0, 1]$;
(2) a SHFLId of Ŧ if $\mathcal{S}[\widehat{\xi}; \Phi]$ is either an empty set or an LId of Ŧ for all $\Phi \subseteq [0, 1]$;
(3) a SHFRId of Ŧ if $\mathcal{S}[\widehat{\xi}; \Phi]$ is either an empty set or an RId of Ŧ for all $\Phi \subseteq [0, 1]$;
(4) a SHFId of Ŧ if $\mathcal{S}[\widehat{\xi}; \Phi]$ is a SHFLtId, SHFLId, and SHFRId of Ŧ.

Many concepts of FSs were used to describe SHFIds in 2021 by Julatha and Iampan (see [15]). They also identified a connection between Ids and SHFIds. Additionally, the following relationships between SHFIds and other FIds were demonstrated.

Lemma 2.10 ([15])**.** *The following are true for a ternary semigroup Ŧ.*

(1) *Every HFLtId (resp., HFLId, HFRId, HFId) of Ŧ is a SHFLtId (resp., SHFLId, SHFRId, SHFId).*
(2) *Every IvFLtId (resp., IvFLId, IvFRId, IvFId) of Ŧ is a SHFLtId (resp., SHFLId, SHFRId, SHFId).*

The following result demonstrates that in some contexts, the concepts of LtIds, LIds, RIds and Ids by FSs, IvFSs and HFSs share some closed connections.

Theorem 2.11 ([15]). *The following statements are equivalent for an HFS $\widehat{\xi}$ on $Ŧ$.*

(1) $\widehat{\xi}$ *is a SHFLtId (resp., SHFLId, SHFRId, SHFId) of* $Ŧ$.
(2) $F^{\widehat{\xi}}$ *is an FLtId (resp., FLId, FRId, FId) of* $Ŧ$.
(3) $\mathcal{H}_{\sup^*}^{(\widehat{\xi},\Phi)}$ *is an HFLtId (resp., HFLId, HFRId, HFId) of* $Ŧ$ *for all* $\Phi \subseteq [0,1]$.
(4) $\mathcal{H}_{\sup^*}^{(\widehat{\xi},[0,1])}$ *is an IvFLtId (resp., IvFLId, IvFRId, IvFId) of* $Ŧ$.
(5) $\mathcal{H}_{\sup^*}^{(\widehat{\xi},[0,1])}$ *is an HFLtId (resp., HFLId, HFRId, HFId) of* $Ŧ$.
(6) $\mathcal{H}_{\sup^*}^{(\widehat{\xi},[0,1])}$ *is a SHFLtId (resp., SHFLId, SHFRId, SHFId) of* $Ŧ$.

Using the function $\sup^*$ and the equivalence (1) $\Leftrightarrow$ (2) of Theorem 2.11 above, we can determine the criteria for obtaining SHFLtIds, SHFLIds, SHFRIds and SHFIds.

Theorem 2.12. *Let* $\widehat{\kappa}$ *be an HFS on* $Ŧ$. *Then we obtain the following statements.*

(1) $\widehat{\kappa}$ *is a SHFLtId (resp., SHFLId, SHFRId) of* $Ŧ$ *if and only if* $\sup^* \widehat{\kappa}(s\nu z) \geqslant \sup^* \widehat{\kappa}(\nu)$ *(resp.,* $\sup^* \widehat{\kappa}(s\nu z) \geqslant \sup^* \widehat{\kappa}(z)$, $\sup^* \widehat{\kappa}(s\nu z) \geqslant \sup^* \widehat{\kappa}(s)$*) for all* $s, \nu, z \in Ŧ$.
(2) $\widehat{\kappa}$ *is a SHFId of* $Ŧ$ *if and only if* $\max\{\sup^* \widehat{\kappa}(s), \sup^* \widehat{\kappa}(\nu), \sup^* \widehat{\kappa}(z)\} \leqslant \sup^* \widehat{\kappa}(s\nu z)$ *for all* $s, \nu, z \in Ŧ$.

We note here that the concept of SHFIds is not only present in ternary semigroups, but also in semigroups and Γ-semigroups (see [13]). The so-called $\sup^+_\gamma$- and $\sup^-_\delta$-hesitant fuzzy ideals, which are new varieties of SHFIds in ternary semigroups, are defined in the following section. We use an example to demonstrate how these new ideas about HFIds are an extension of the one introduced in [15].

3. Main results

Now, we come to the central part of the paper. In the following section, we present two types of HFSs in ternary semigroups, which are generalizations of SHFIds. This section is divided into four subsections. In what follows, let $\gamma, \delta \in [0,1]$, unless otherwise specified. In Subsection 3.1, the concepts of $\sup^+_\gamma$-hesitant fuzzy ideals (S^+_γHFIds), $\sup^+_\gamma$-hesitant fuzzy lateral ideals (S^+_γHFLtIds), $\sup^+_\gamma$-hesitant fuzzy left ideals (S^+_γHFLIds), $\sup^+_\gamma$-hesitant fuzzy right ideals (S^+_γHFRIds) are presented. In ternary semigroups, we demonstrate that each SHFId (resp., SHFLId, SHFLtId, SHFRId) is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId). We give an example to illustrate that SHFIds and S^+_γHFIds are different. It is examined how the notion of S^+_γHFIds and the other recalled from the preceding section compare.

The second subsection introduces the ideas of $\sup^-_\delta$-hesitant fuzzy ideals (S^-_δHFIds), $\sup^-_\delta$-hesitant fuzzy lateral ideals (S^-_δHFLtIds), $\sup^-_\delta$-hesitant fuzzy left ideals (S^-_δHFLIds) and $\sup^-_\delta$-hesitant fuzzy right ideals (S^-_δHFRIds) in ternary semigroups. We demonstrate that each SHFId (resp., SHFLId, SHFLtId, SHFRId) is a S^-_δHFId (resp., S^-_δHFLId, S^-_δHFLtId, S^-_δHFRId). The same approaches used for S^+_γHFIds (resp., S^+_γHFLIds, S^+_γHFLtIds, S^+_γHFRIds) are employed to analyze S^-_δHFIds (resp., S^-_δHFLIds, S^-_δHFLtIds, S^-_δHFRIds).

Subsections 3.3 and 3.4 provide characterizations of S^+_γHFIds and S^-_δHFIds by means of Łukasiewicz (anti-) fuzzy sets and PFSs, HFSs and IvFSs, respectively.

3.1. $\sup^+_\gamma$-hesitant fuzzy ideals

We define the function $\sup^+_\gamma$ on $\mathcal{P}([0,1])$ by

$$\sup^+_\gamma \Phi := \min\{\sup^* \Phi + \gamma, 1\}$$

for all $\Phi \subseteq [0,1]$. By the definition of $\sup^+_\gamma$, we define a relation $\trianglelefteq^+_\gamma$ on $\mathcal{P}([0,1])$ by

$$\Psi \trianglelefteq^+_\gamma \Phi \quad \text{if and only if} \quad \sup^+_\gamma \Psi \leqslant \sup^+_\gamma \Phi$$

for all $\Phi, \Psi \subseteq [0,1]$. We may denote $\Phi \trianglelefteq^{+}_{\gamma} \Psi$ by $\Psi \trianglerighteq^{+}_{\gamma} \Phi$. We say that $\Phi \approxeq^{+}_{\gamma} \Psi$ if $\Phi \trianglelefteq^{+}_{\gamma} \Psi$ and $\Psi \trianglelefteq^{+}_{\gamma} \Phi$. Moreover, $\Phi \triangleleft^{+}_{\gamma} \Psi$ if $\sup^{+}_{\gamma} \Phi < \sup^{+}_{\gamma} \Psi$. If $\gamma = 0$, then we denote $\trianglelefteq^{+}_{0}$ (resp., $\triangleleft^{+}_{0}$) by $\trianglelefteq$ (resp., $\triangleleft$). We observe here that:

(1) $\sup^{+}_{0} = \sup^{*}$;
(2) if $\widetilde{s}, \widetilde{t} \in \mathcal{D}([0,1])$ and $\widetilde{s} \preceq \widetilde{t}$, then $\widetilde{s} \trianglelefteq^{+}_{\alpha} \widetilde{t}$ for all $\alpha \in [0,1]$.

For more results of function $\sup^{+}_{\gamma}$ and relation $\trianglelefteq^{+}_{\gamma}$, the readers are referred to [14].

The relation $\trianglelefteq^{+}_{\gamma}$ is used to define new kinds of HFSs: $\sup^{+}_{\gamma}$-hesitant fuzzy ideals (S^{+}_{γ}HFIds), $\sup^{+}_{\gamma}$-hesitant fuzzy left ideals (S^{+}_{γ}HFLIds), $\sup^{+}_{\gamma}$-hesitant fuzzy lateral ideals (S^{+}_{γ}HFLtIds) and $\sup^{+}_{\gamma}$-hesitant fuzzy right ideals (S^{+}_{γ}HFRIds) of ternary semigroups as follows.

Definition 3.1. An HFS $\widehat{\xi}$ on Ŧ is said to be:

(1) a S^{+}_{γ}HFLtId (resp., S^{+}_{γ}HFLId, S^{+}_{γ}HFRId) of Ŧ if $\widehat{\xi}(v) \trianglelefteq^{+}_{\gamma} \widehat{\xi}(svz)$ (resp., $\widehat{\xi}(z) \trianglelefteq^{+}_{\gamma} \widehat{\xi}(svz)$, $\widehat{\xi}(s) \trianglelefteq^{+}_{\gamma} \widehat{\xi}(svz)$) for all $s, v, z \in$ Ŧ;
(2) a S^{+}_{γ}HFId of Ŧ if $\widehat{\xi}$ is a S^{+}_{γ}HFLtId, S^{+}_{γ}HFLId and S^{+}_{γ}HFRId of Ŧ.

We provide the following examples to help the readers better grasp the above definition.

Example 3.1. Let us consider a ternary semigroup Ŧ $= \{(t,s),(s,s),(t,t),(s,t)\}$ defined by

$$[(s_1,t_1)(s_2,t_2)(s_3,t_3)] := (s_1,t_3)$$

for all (s_1,t_1), (s_2,t_2), $(s_3,t_3) \in$ Ŧ. We define HFSs $\widehat{\kappa}$ and $\widehat{\xi}$ on Ŧ as follows:

$$\widehat{\kappa}(z) := \begin{cases} [0.2,0.8] & \text{if } z = (s,s), \\ [0,0.3) & \text{if } z = (t,t), \\ \{0.3\} & \text{if } z = (s,t), \\ \{0.1,0.3,0.5,0.7\} & \text{if } z = (t,s), \end{cases} \quad \text{and} \quad \widehat{\xi}(z) := \begin{cases} \emptyset & \text{if } z = (s,s), \\ [0.3,0.6] & \text{if } z = (t,t), \\ \{0\} & \text{if } z = (s,t), \\ \{0.3,0.5,0.6,0.8\} & \text{if } z = (t,s), \end{cases}$$

for all $z \in$ Ŧ. Then we can carefully calculate that $\widehat{\xi}$ is a S^{+}_{γ}HFRId of Ŧ for all $\gamma \in [0.4,1]$, and $\widehat{\kappa}$ is a S^{+}_{γ}HFLId of Ŧ for all $\gamma \in [0.3,1]$. On the contrary, $\widehat{\xi}$ is not a SHFRId of Ŧ since $\widehat{\xi}([(t,s)(t,s)(t,t)]) \triangleleft \widehat{\xi}((t,s))$. Moreover, since $\widehat{\kappa}([(t,t)(t,s)(s,s)]) \triangleleft \widehat{\kappa}((s,s))$, we have that $\widehat{\kappa}$ is not a SHFLId of Ŧ.

Example 3.2. We consider the ternary semigroup Ŧ $= \{-1,-2,-3,-4,\ldots\}$ under the usual ternary multiplication. Define an HFS $\widehat{\xi}$ on Ŧ by

$$\widehat{\xi}(v) := \begin{cases} [0,0.8], & \text{if } v \in 2Ŧ \setminus \{-2,-4,-6\}, \\ (0,1], & \text{if } v \in \{-2,-4,-6\}, \\ \emptyset, & \text{if } v \notin 2Ŧ, \end{cases}$$

for all $v \in$ Ŧ. Then $\widehat{\xi}$ is a S^{+}_{γ}HFId of Ŧ for all $\gamma \in [0.2,1]$ but not a SHFId of Ŧ. Indeed, we see that

(1) $\widehat{\xi}$ is not a SHFLtId of Ŧ because $\widehat{\xi}(p(-2)q) \triangleleft \widehat{\xi}(-2)$ for all $p, q \in$ Ŧ$\setminus\{-1\}$;
(2) $\widehat{\xi}$ is not a SHFLId of Ŧ because $\widehat{\xi}(pq(-4)) \triangleleft \widehat{\xi}(-4)$ for all $p, q \in$ Ŧ$\setminus\{-1\}$; and
(3) $\widehat{\xi}$ is not a SHFRId of Ŧ because $\widehat{\xi}((-6)pq) \triangleleft \widehat{\xi}(-6)$ for all $p, q \in$ Ŧ$\setminus\{-1\}$.

The above examples demonstrate how ternary semigroups' conceptions of S^{+}_{γ}HFIds differ from those of SHFIds. We note that the definition of SHFIds in ternary semigroups in Definition 2.9 is specified in terms of Ids. The following consequence allows us to formulate Definition 2.9 in terms of HFSs and the relation $\trianglelefteq$ after applying Theorem 2.12.

Proposition 3.2. *Let $\widehat{\xi}$ be an HFS on Ŧ. Then we obtain the following.*

(1) *$\widehat{\xi}$ is a SHFLtId of Ŧ if and only if $\widehat{\xi}(\nu) \trianglelefteq \widehat{\xi}(s\nu z)$ for all $s, \nu, z \in$ Ŧ.*
(2) *$\widehat{\xi}$ is a SHFLId of Ŧ if and only if $\widehat{\xi}(z) \trianglelefteq \widehat{\xi}(s\nu z)$ for all $s, \nu, z \in$ Ŧ.*
(3) *$\widehat{\xi}$ is a SHFRId of Ŧ if and only if $\widehat{\xi}(s) \trianglelefteq \widehat{\xi}(s\nu z)$ for all $s, \nu, z \in$ Ŧ.*

The following result demonstrates how the ideas behind our new HFIds are extended versions of SHFIds in ternary semigroups.

Proposition 3.3. *Every SHFId (resp., SHFLId, SHFLtId, SHFRId) of Ŧ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) for all $\gamma \in [0, 1]$.*

Proof. We demonstrate that every SHFLtId of Ŧ is a S^+_γHFLtId of Ŧ for all $\gamma \in [0, 1]$. Let $\widehat{\xi}$ be a SHFLtId of Ŧ, given $\gamma \in [0, 1]$, let $s, \nu, z \in$ Ŧ. Then, by our assumption, we have $\sup^* \widehat{\xi}(\nu) \leqslant \sup^* \widehat{\xi}(s\nu z)$. This implies that

$$\sup^+_\gamma \widehat{\xi}(s\nu z) = \min\{\sup^* \widehat{\xi}(s\nu z) + \gamma, 1\} \geqslant \min\{\sup^* \widehat{\xi}(\nu) + \gamma, 1\} = \sup^+_\gamma \widehat{\xi}(\nu).$$

Therefore, $\widehat{\xi}(s\nu z) \trianglerighteq^+_\gamma \widehat{\xi}(\nu)$. As a consequence, $\widehat{\xi}$ is a S^+_γHFLtId of Ŧ. Other kinds of idealities can be done similarly. □

From Examples 3.1 and 3.2, it is clear that the converse of Proposition 3.3 may not be valid. We can summarize the relationships between SHFIds and S^+_γHFIds by Figure 1.

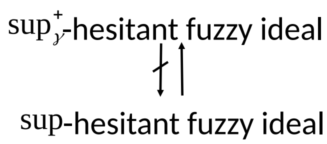

Figure 1: Relationships between sup- and $\sup^+_\gamma$-hesitant fuzzy ideals.

However, we might ask under which conditions a S^+_γHFId is a SHFId. The following result can answer our argument.

Proposition 3.4. *Let $\widehat{\kappa}$ be a HFS on Ŧ and $0 < \alpha \leqslant 1$. We have that if $\widehat{\kappa}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ for all $\gamma \in (0, \alpha]$, then $\widehat{\kappa}$ is a SHFId (resp., SHFLId, SHFLtId, SHFRId) of Ŧ.*

Proof. Let $\widehat{\kappa}$ be a S^+_γHFLtId of Ŧ for all $\gamma \in (0, \alpha]$. We will show that $\widehat{\kappa}$ is a SHFLtId of Ŧ. We suppose that $\sup^* \widehat{\kappa}(s\nu z) < \sup^* \widehat{\kappa}(\nu)$ for some $s, \nu, z \in$ Ŧ. We put

$$\gamma := \min\left\{\frac{\sup^* \widehat{\kappa}(\nu) - \sup^* \widehat{\kappa}(s\nu z)}{2}, \alpha\right\}.$$

This implies that $\gamma \in (0, \alpha]$. Since $\sup^* \widehat{\kappa}(\nu) - \sup^* \widehat{\kappa}(s\nu z) > \gamma$, we get $1 \geqslant \sup^* \widehat{\kappa}(\nu) > \sup^* \widehat{\kappa}(s\nu z) + \gamma$ and

$$\sup^* \widehat{\kappa}(\nu) + \gamma > \sup^* \widehat{\kappa}(\nu) > \sup^* \widehat{\kappa}(s\nu z) + \gamma.$$

Hence,

$$\sup^+_\gamma \widehat{\kappa}(s\nu z) = \min\{\sup^* \widehat{\kappa}(s\nu z) + \gamma, 1\} \leqslant \sup^* \widehat{\kappa}(s\nu z) + \gamma < \min\{\sup^* \widehat{\kappa}(\nu) + \gamma, 1\} = \sup^+_\gamma \widehat{\kappa}(\nu).$$

This means that $\widehat{\kappa}(s\nu z) \vartriangleleft^+_\gamma \widehat{\kappa}(\nu)$. By our presumption, we see that $\widehat{\kappa}(s\nu z) \vartriangleleft^+_\gamma \widehat{\kappa}(\nu) \trianglelefteq^+_\gamma \widehat{\kappa}(s\nu z)$. Thus, $\sup^* \widehat{\kappa}(s\nu z) < \sup^* \widehat{\kappa}(s\nu z)$, which is a contradiction. Therefore, $\sup^* \widehat{\kappa}(s\nu z) \geqslant \sup^* \widehat{\kappa}(\nu)$ for all $s, \nu, z \in$ Ŧ. As a consequence, $\widehat{\kappa}$ is a SHFLtId of Ŧ. Similar procedures can be used for the other types ideals. □

The last result of this subsection is to provide an interconnection of S^+_γHFIds in ternary semigroups.

Proposition 3.5. *Let $\widehat{\xi}$ be a HFS on Ŧ, and $\gamma, \alpha \in [0,1]$ such that $\gamma \leqslant \alpha \leqslant 1$. We have that if $\widehat{\xi}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ, then $\widehat{\xi}$ is also a S^+_αHFId (resp., S^+_αHFLId, S^+_αHFLtId, S^+_αHFRId) of Ŧ.*

Proof. Let $\widehat{\xi}$ be a S^+_γHFLtId of Ŧ and $s, v, z \in$ Ŧ. By the assumption, we get that $\widehat{\xi}(svz) \trianglerighteq^+_\gamma \widehat{\xi}(v)$. That is,

$$\min\{\sup^* \widehat{\xi}(v) + \gamma, 1\} \leqslant \min\{\sup^* \widehat{\xi}(svz) + \gamma, 1\}. \tag{3.1}$$

If $\min\{\sup^* \widehat{\xi}(svz) + \gamma, 1\} = 1$, then

$$\sup^+_\alpha \widehat{\xi}(v) \leqslant 1 \leqslant \sup^* \widehat{\xi}(svz) + \gamma \leqslant \sup^* \widehat{\xi}(svz) + \alpha.$$

This means that $\sup^+_\alpha \widehat{\xi}(v) \leqslant \sup^+_\alpha \widehat{\xi}(svz)$. We suppose that $\min\{\sup^* \widehat{\xi}(svz) + \gamma, 1\} \neq 1$. Then we obtain that $\sup^* \widehat{\xi}(svz) + \gamma < 1$. By (3.1), we have that $\sup^* \widehat{\xi}(v) + \gamma \leqslant \sup^* \widehat{\xi}(svz) + \gamma$. This implies that $\sup^* \widehat{\xi}(v) \leqslant \sup^* \widehat{\xi}(svz)$. Thus, $\sup^+_\alpha \widehat{\xi}(v) \leqslant \sup^+_\alpha \widehat{\xi}(svz)$. Hence, $\widehat{\xi}(v) \trianglelefteq^+_\alpha \widehat{\xi}(svz)$. Altogether, $\widehat{\xi}$ is a S^+_αHFLtId of Ŧ. Similar procedures can be used for the other types ideals. □

3.2. $\sup^-_\delta$-hesitant fuzzy ideals

We introduce different varieties of SHFIds in this subsection, similar to those provided in Subsection 3.1. The structure of this subsection is the same as that of the previous one. Define the function $\sup^-_\delta$ and a binary relation $\trianglelefteq^-_\delta$ on $\mathcal{P}([0,1])$ by

$$\sup^-_\delta \Phi := \max\{\sup^* \Phi - \delta, 0\}$$

for all $\Phi \subseteq [0,1]$, and

$$\Phi \trianglelefteq^-_\delta \Psi \quad \text{if and only if} \quad \sup^-_\delta \Phi \leqslant \sup^-_\delta \Psi$$

for all $\Psi, \Phi \subseteq [0,1]$. By $\Psi \trianglerighteq^-_\delta \Phi$ we mean $\Phi \trianglelefteq^-_\delta \Psi$. We write $\Phi \approxeq^-_\delta \Psi$ if $\Phi \trianglelefteq^-_\delta \Psi$ and $\Psi \trianglelefteq^-_\delta \Phi$. Furthermore, we denote by $\Phi \triangleleft^-_\delta \Psi$ if $\sup^-_\delta \Phi < \sup^-_\delta \Psi$. If $\delta = 0$, then we denote $\trianglelefteq^-_\delta$ (resp., $\triangleleft^-_\delta$) by $\trianglelefteq$ (resp., $\triangleleft$), that is, $\trianglelefteq^+_\gamma = \trianglelefteq = \trianglelefteq^-_\delta$ whenever $\gamma = 0 = \delta$. The related results of the function $\sup^-_\delta$ and the relation $\trianglelefteq^-_\delta$ can be found in [14].

The relation $\trianglelefteq^-_\delta$ is used to define new types of HFSs: $\sup^-_\delta$-hesitant fuzzy ideals (S^-_δHFIds), $\sup^-_\delta$-hesitant fuzzy left ideals (S^-_δHFLIds), $\sup^-_\delta$-hesitant fuzzy lateral ideals (S^-_δHFLtIds) and $\sup^-_\delta$-hesitant fuzzy right ideals (S^-_δHFRIds) of ternary semigroups as follows.

Definition 3.6. An HFS $\widehat{\xi}$ on Ŧ is said to be:

(1) a S^-_δHFLtId (resp., S^-_δHFLId, S^-_δHFRId) of T if $\widehat{\xi}(v) \trianglelefteq^-_\delta \widehat{\xi}(svz)$ (resp., $\widehat{\xi}(z) \trianglelefteq^-_\delta \widehat{\xi}(svz)$, $\widehat{\xi}(s) \trianglelefteq^-_\delta \widehat{\xi}(svz)$) for all $s, v, z \in$ Ŧ;
(2) a S^-_δHFId of Ŧ if $\widehat{\xi}$ is a S^-_δHFLId, S^-_δHFLtId and S^-_δHFRId of Ŧ.

Let us give an example to guarantee the existence of the concepts that we established before proceeding.

Example 3.3. We consider the ternary semigroup Ŧ$:= \{-i, i, 0\}$ under the usual multiplication over complex numbers. Next, we define an HFS $\widehat{\xi}$ on Ŧ as follows:

$$\widehat{\xi}(z) := \begin{cases} [0.5, 0.6], & \text{if } z = -i, \\ \emptyset, & \text{if } z = i, \\ \{0, 0.5, 1\}, & \text{if } z = 0, \end{cases}$$

for all $z \in$ Ŧ. We can carefully calculate that $\widehat{\xi}$ is a S^-_δHFId of Ŧ for all $\delta \in [0.6, 1]$.

Proposition 3.7. *Every SHFId (resp., SHFLId, SHFLtId, SHFRId) of Ŧ is a S_δ^-HFId (resp., S_δ^-HFLId, S_δ^-HFLtId, S_δ^-HFRId) of Ŧ for all $\delta \in [0,1]$.*

Proof. Let $\widehat{\xi}$ be a SHFLtId of Ŧ, given $\delta \in [0,1]$, then, by our assumption, we have that $\sup^* \widehat{\xi}(svz) \geqslant \sup^* \widehat{\xi}(v)$ for each $s, v, z \in$ Ŧ. This implies that

$$\sup_\delta^- \widehat{\xi}(svz) = \max\{\sup^* \widehat{\xi}(svz) - \delta, 0\} \geqslant \max\{\sup^* \widehat{\xi}(v) - \delta, 0\} = \sup_\delta^- \widehat{\xi}(v)$$

for each $s, v, z \in$ Ŧ. Therefore, $\widehat{\xi}$ is a S_δ^-HFLtId of Ŧ. Other kinds of idealities can be done similarly. □

The above proposition illustrates the generality of S_δ^-HFIds in ternary semigroups. Example 3.4 (1) examines that the converse of Proposition 3.7 does not hold.

Example 3.4. By Example 3.3, we can carefully calculate that $\widehat{\xi}$ is neither a SHFId nor a S_γ^+HFId ($\gamma \in [0,1)$) of Ŧ. In fact, we see that:

(1) $\widehat{\xi}$ is not a SHFLtId (resp, SHFLId, SHFRId) of Ŧ since $\widehat{\xi}(i(-i)i) \lhd \widehat{\xi}(-i)$ (resp., $\widehat{\xi}(i^2(-i)) \lhd \widehat{\xi}(-i)$, $\widehat{\xi}((-i)i^2) \lhd \widehat{\xi}(-i)$);
(2) $\widehat{\xi}$ is not a S_γ^+HFLtId (resp, S_γ^+HFLId, S_γ^+HFRId) ($\gamma \in [0,1)$) of Ŧ since $\widehat{\xi}((-i)^3) \lhd_\gamma^+ \widehat{\xi}(-i)$.

Example 3.5. By Example 3.2, we can carefully calculate that $\widehat{\xi}$ is not a S_δ^-HFId of Ŧ when $\delta \in [0,1)$. In fact, we see that $\widehat{\xi}$ is not a S_δ^-HFLtId (resp, S_δ^-HFLId, S_δ^-HFRId) of Ŧ because $\widehat{\xi}((-4)^3) \lhd_\delta^- \widehat{\xi}(-4)$.

By Examples 3.2, 3.3, 3.4, 3.5 and Propositions 3.3 and 3.7, we have that the relationships of SHFIds, S_γ^+HFIds and S_δ^-HFIds in ternary semigroups are obtained by Figure 2.

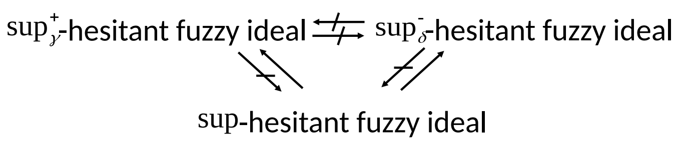

Figure 2: Relationships between sup-, $\sup_\gamma^+$- and $\sup_\delta^-$-hesitant fuzzy ideals.

We now ask the criterion that converts each S_δ^-HFId into a SHFId.

Proposition 3.8. *Let $\widehat{\kappa}$ be an HFS on Ŧ and $0 < \alpha \leqslant 1$. We have that if $\widehat{\kappa}$ is a S_δ^-HFId (resp., S_δ^-HFLId, S_δ^-HFLtId, S_δ^-HFRId) of Ŧ for all $\delta \in (0, \alpha]$, then $\widehat{\kappa}$ is a SHFId (resp., SHFLId, SHFLtId, SHFRId) of Ŧ.*

Proof. Let $\widehat{\kappa}$ be a S_δ^-HFLtId of Ŧ for all $\delta \in (0, \alpha]$. We prove that $\widehat{\kappa}$ is a SHFLtId of Ŧ. Suppose that $\sup^* \widehat{\kappa}(szv) < \sup^* \widehat{\kappa}(z)$ for some $s, v, z \in$ Ŧ. We put

$$\delta := \min\{\frac{\sup^* \widehat{\kappa}(z) - \sup^* \widehat{\kappa}(szv)}{2}, \alpha\}.$$

This implies that $\delta \in (0, \alpha]$. Since $\sup^* \widehat{\kappa}(z) - \sup^* \widehat{\kappa}(szv) > \delta$, we get $\sup^* \widehat{\kappa}(z) - \delta > \sup^* \widehat{\kappa}(szv) \geqslant 0$ and

$$\sup^* \widehat{\kappa}(z) - \delta > \sup^* \widehat{\kappa}(szv) > \sup^* \widehat{\kappa}(szv) - \delta.$$

Therefore,

$$\sup_\delta^- \widehat{\kappa}(z) = \max\{\sup^* \widehat{\kappa}(z) - \delta, 0\} \geqslant \sup^* \widehat{\kappa}(z) - \delta > \max\{\sup^* \widehat{\kappa}(szv) - \delta, 0\} = \sup_\delta^- \widehat{\kappa}(szv).$$

This means that $\widehat{\kappa}(szv) \lhd_\delta^- \widehat{\kappa}(z)$. By our assumption, we obtain $\widehat{\kappa}(z) \unlhd_\delta^- \widehat{\kappa}(szv) \lhd_\delta^- \widehat{\kappa}(z)$. That is, $\sup^* \widehat{\kappa}(z) > \sup^* \widehat{\kappa}(z)$ which is a contradiction. Hence, $\sup^* \widehat{\kappa}(szv) \geqslant \sup^* \widehat{\kappa}(z)$ for all $s, v, z \in$ Ŧ. As a consequence, $\widehat{\kappa}$ is a SHFLtId of Ŧ. The other types can be proven similarly. □

A common setting connects S_δ^-HFIds with S_α^-HFIds, as shown by the following result.

Proposition 3.9. *Let $\widehat{\kappa}$ be an HFS on Ŧ, and $\delta, \alpha \in [0,1]$ such that $\delta \leqslant \alpha \leqslant 1$. We have that if $\widehat{\kappa}$ is a S_δ^-HFId (resp., S_δ^-HFLId, S_δ^-HFLtId, S_δ^-HFRId) of Ŧ, then $\widehat{\kappa}$ is also a S_α^-HFId (resp., S_α^-HFLId, S_α^-HFLtId, S_α^-HFRId) of Ŧ.*

Proof. Let $\widehat{\kappa}$ be a S_δ^-HFLtId of Ŧ and $s, \nu, z \in$ Ŧ. We have that $\widehat{\kappa}(s\nu z) \trianglerighteq_\delta^- \widehat{\kappa}(\nu)$. That is,

$$\max\{\sup{}^* \widehat{\kappa}(s\nu z) - \delta, 0\} \geqslant \max\{\sup{}^* \widehat{\kappa}(\nu) - \delta, 0\}. \tag{3.2}$$

If $\max\{\sup{}^* \widehat{\kappa}(\nu) - \delta, 0\} = 0$, then

$$\sup{}_\alpha^- \widehat{\kappa}(s\nu z) \geqslant 0 \geqslant \sup{}^* \widehat{\kappa}(\nu) - \delta \geqslant \sup{}^* \widehat{\kappa}(\nu) - \alpha.$$

This means that $\sup_\alpha^- \widehat{\kappa}(s\nu z) \geqslant \sup_\alpha^- \widehat{\kappa}(\nu)$. We suppose that $\max\{\sup^* \widehat{\kappa}(\nu) - \delta, 0\} \neq 0$. Then, $\max\{\sup^* \widehat{\kappa}(\nu) - \delta, 0\} = \sup^* \widehat{\kappa}(\nu) - \delta > 0$. Using (3.2), we get $\sup^* \widehat{\kappa}(s\nu z) - \delta \geqslant \sup^* \widehat{\kappa}(\nu) - \delta$. Thus, $\sup^* \widehat{\kappa}(s\nu z) \geqslant \sup^* \widehat{\kappa}(\nu)$. Hence, $\sup_\alpha^- \widehat{\kappa}(s\nu z) \geqslant \sup_\alpha^- \widehat{\kappa}(\nu)$, which is that $\widehat{\kappa}(s\nu z) \trianglerighteq_\alpha^- \widehat{\kappa}(\nu)$. Therefore, $\widehat{\kappa}$ is a S_α^-HFLtId of Ŧ. Similarly procedures can be used for the other types. □

By Propositions 3.3, 3.4, 3.7, and 3.8, we obtain the equivalence of SHFIds, S_γ^+HFIds, and S_δ^-HFIds in ternary semigroups.

Corollary 3.10. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a SHFId (resp., SHFLId, SHFLtId, SHFRId) of Ŧ.*
(2) *$\widehat{\xi}$ is a S_γ^+HFId (resp., S_γ^+HFLId, S_γ^+HFLtId, S_γ^+HFRId) of Ŧ for all $\gamma \in [0,1]$.*
(3) *$\widehat{\xi}$ is a S_δ^-HFId (resp., S_δ^-HFLId, S_δ^-HFLtId, S_δ^-HFRId) of Ŧ for all $\delta \in [0,1]$.*

3.3. *Characterizations by Łukasiewicz (anti-) and Pythagorean fuzzy sets*

In this part, we characterize many kinds of S_γ^+HFIds and S_δ^-HFIds in ternary semigroups via Łukasiewicz (anti-) fuzzy sets and PFSs. Since the concept of PFSs is defined in preliminaries section. Let us recall the notion of Łukasiewicz (anti) fuzzy sets. Let ϑ be an FS in $\mathcal{S}$. Then FSs ϑ_δ^- and ϑ_γ^+ in $\mathcal{S}$, defined by

$$\vartheta_\delta^-(s) := \max\{\vartheta(s) - \delta, 0\} \quad \text{and} \quad \vartheta_\gamma^+(s) := \min\{\vartheta(s) + \gamma, 1\}$$

for all $s \in \mathcal{S}$, are called an *α-Łukasiewicz fuzzy set* [17] when $\alpha := 1 - \delta$ and a *γ-Łukasiewicz anti fuzzy set* [18] of ϑ in $\mathcal{S}$, respectively. We recall that for any HFS $\widehat{\xi}$ on $\mathcal{S}$, the FS $\mathcal{F}^{\widehat{\xi}}$ in $\mathcal{S}$ is defined by

$$\mathcal{F}^{\widehat{\xi}}(s) := \sup{}^* \widehat{\xi}(s)$$

for all $s \in \mathcal{S}$. The properties of $\mathcal{F}^{\widehat{\xi}}$ were given by Łukasiewicz (anti-) fuzzy sets as follows.

Lemma 3.11 ([14]). *Let $\widehat{\xi}$ be an HFS on $\mathcal{S}$. Then we obtain the following statements.*

(1) *$(\mathcal{F}^{\widehat{\xi}})_\gamma^+$ is a γ-Łukasiewicz anti-fuzzy set of $\mathcal{F}^{\widehat{\xi}}$ in $\mathcal{S}$.*
(2) *$(\mathcal{F}^{\widehat{\xi}})_\delta^-$ is a $1-\delta$-Łukasiewicz fuzzy set of $\mathcal{F}^{\widehat{\xi}}$ in $\mathcal{S}$.*
(3) *For any $s \in \mathcal{S}$, we have $(\mathcal{F}^{\widehat{\xi}})_\gamma^+(s) := \sup_\gamma^+ \widehat{\xi}(s)$ and $(\mathcal{F}^{\widehat{\xi}})_\delta^-(s) := \sup_\delta^- \widehat{\xi}(s)$.*

We obtain the characterizations of S_γ^+HFIds, S_γ^+HFLIds, S_γ^+HFLtIds and S_γ^+HFRIds via the Łukasiewicz anti-fuzzy set using Lemma 3.11 and the results in the preceding subsections.

Theorem 3.12. *The following statements are equivalent for an HFS $\widehat{\xi}$ on Ŧ.*

(1) *$\widehat{\xi}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ.*

(2) *$(\mathcal{F}^{\widehat{\xi}})^+_\alpha$ is an FId (resp., FLId, FLtId, FRId) of Ŧ for all $\alpha \in [\gamma, 1]$.*

(3) *$(\mathcal{F}^{\widehat{\xi}})^+_\gamma$ is an FId (resp., FLId, FLtId, FRId) of Ŧ.*

Proof. Only the LtIds procedure is taken into account in this proof. A similar proof can be made for the others.

(1) ⇒ (2). Let $\widehat{\xi}$ be a S^+_γHFLtId of Ŧ and $\alpha \in [\gamma, 1]$. By Proposition 3.5, we have that $\widehat{\xi}$ is also a S^+_αHFLtId of Ŧ. By Lemma 3.11, we have that

$$(\mathcal{F}^{\widehat{\xi}})^+_\alpha(v) = \sup^+_\alpha \widehat{\xi}(v) \leqslant \sup^+_\alpha \widehat{\xi}(svz) = (\mathcal{F}^{\widehat{\xi}})^+_\alpha(svz)$$

for all $s, v, z \in$ Ŧ. This means that $(\mathcal{F}^{\widehat{\xi}})^+_\alpha$ is an FLtId of Ŧ.

(2) ⇒ (3). This implication is clear.

(3) ⇒ (1). Let $(\mathcal{F}^{\widehat{\xi}})^+_\gamma$ be an FLtId of Ŧ. By using Lemma 3.11, we get

$$\sup^+_\gamma \widehat{\xi}(z) = (\mathcal{F}^{\widehat{\xi}})^+_\gamma(z) \leqslant (\mathcal{F}^{\widehat{\xi}})^+_\gamma(szv) = \sup^+_\gamma \widehat{\xi}(szv)$$

for all $s, v, z \in$ Ŧ. This means that $\widehat{\xi}(z) \trianglelefteq^+_\gamma \widehat{\xi}(szv)$ for all $s, v, z \in$ Ŧ. Thus, $\widehat{\xi}$ is a S^+_γHFLtId of Ŧ. □

Similarly, we obtain the following characterization.

Theorem 3.13. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a S^-_δHFId (resp., S^-_δHFLId, S^-_δHFLtId, S^-_δHFRId) of Ŧ.*

(2) *$(\mathcal{F}^{\widehat{\xi}})^-_\alpha$ is an FId (resp., FLId, FLtId, FRId) of Ŧ for all $\alpha \in [\delta, 1]$.*

(3) *$(\mathcal{F}^{\widehat{\xi}})^-_\delta$ is an FId (resp., FLId, FLtId, FRId) of Ŧ.*

The properties of Łukasiewicz fuzzy sets being AFIds provide another way to describe SHFIds. To do that the following information is necessary. Let $\widehat{\xi}$ be an HFS on a set $\mathcal{S}$. An HFS $\widehat{\kappa}$ on $\mathcal{S}$ is said to be a *supremum complement* of $\widehat{\xi}$ if $\sup^* \widehat{\kappa}(s) = 1 - \sup^* \widehat{\xi}(s)$ for all $s \in \mathcal{S}$. The notation $\mathsf{SC}(\widehat{\xi})$ stands for the set of all supremum complement of $\widehat{\xi}$. For any HFS $\widehat{\xi}$ on $\mathcal{S}$, we define an HFS $\widehat{\xi}^\star$ on $\mathcal{S}$ by $\widehat{\xi}^\star(s) := \{1 - \sup^* \widehat{\xi}(s)\}$ for all $s \in \mathcal{S}$.

Lemma 3.14 ([14]). *Let $\widehat{\xi}$ be an HFS on a set $\mathcal{S}$ and $\alpha \in [0, 1]$. Then:*

(1) $\sup^+_\alpha \widehat{\kappa}(s) = 1 - \sup^-_\alpha \widehat{\xi}(s)$*;*

(2) $\sup^-_\alpha \widehat{\kappa}(s) = 1 - \sup^+_\alpha \widehat{\xi}(s)$*,*

for all $s \in \mathcal{S}$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$.

Theorem 3.15. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ.*

(2) *$(\mathcal{F}^{\widehat{\kappa}})^-_\alpha$ is an AFId (resp., AFLId, AFLtId, AFRId) of Ŧ for all $\alpha \in [\gamma, 1]$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$.*

(3) *$(\mathcal{F}^{\widehat{\xi}^\star})^-_\gamma$ is an AFId (resp., AFLId, AFLtId, AFRId) of Ŧ.*

Proof. Only the LtIds procedure is taken into account in this proof. A similar proof can be made for the others.

(1) ⇒ (2). Let $\widehat{\xi}$ be a S^+_γHFLtId of Ŧ, $\alpha \in [\gamma, 1]$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$. By Proposition 3.5, we have that $\widehat{\xi}$ is also a S^+_αHFLtId of Ŧ. Using Lemmas 3.11 and 3.14, we get that

$$(\mathcal{F}^{\widehat{\kappa}})^-_\alpha(v) = \sup^-_\alpha \widehat{\kappa}(v) = 1 - \sup^+_\alpha \widehat{\xi}(v) \geqslant 1 - \sup^+_\alpha \widehat{\xi}(svz) = \sup^-_\alpha \widehat{\kappa}(svz) = (\mathcal{F}^{\widehat{\kappa}})^-_\alpha(svz)$$

for all $s, v, z \in$ Ŧ. This means that $(\mathcal{F}^{\widehat{\xi}})^-_\alpha$ is an AFLtId of Ŧ.

(2) ⇒ (3). This implication is clear.

(3) ⇒ (1). Assume that $(\mathcal{F}^{\widehat{\xi}^\star})^-_\gamma$ is an AFLtId of Ŧ. By using Lemmas 3.11 and 3.14, we have that

$$\sup^+_\gamma \widehat{\xi}(v) = 1 - \sup^-_\gamma \widehat{\xi}^\star(v) = 1 - (\mathcal{F}^{\widehat{\xi}^\star})^-_\gamma(v) \leqslant 1 - (\mathcal{F}^{\widehat{\xi}^\star})^-_\gamma(svz) = 1 - \sup^-_\gamma \widehat{\xi}^\star(svz) = \sup^+_\gamma \widehat{\xi}(svz)$$

for all $s, v, z \in$ Ŧ. This means that $\widehat{\xi}(v) \trianglelefteq^+_\gamma \widehat{\xi}(svz)$ for all $s, v, z \in$ Ŧ. Therefore, $\widehat{\xi}$ is a S^+_γHFLtId of Ŧ. □

Similarly, we obtain the following theorem.

Theorem 3.16. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a S^-_δHFId (resp., S^-_δHFLId, S^-_δHFLtId, S^-_δHFRId) of Ŧ.*
(2) *$(\mathcal{F}^{\widehat{\kappa}})^+_\alpha$ is an AFId (resp., AFLId, AFLtId, AFRId) of Ŧ for all $\alpha \in [\delta, 1]$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$.*
(3) *$(\mathcal{F}^{\widehat{\xi}^\star})^+_\delta$ is an AFId (resp., AFLId, AFLtId, AFRId) of Ŧ.*

As two results, we obtain the characterizations of S^+_γHFIds and S^-_δHFIds by PFIds in ternary semigroups.

Corollary 3.17. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ.*
(2) *$((\mathcal{F}^{\widehat{\xi}})^+_\gamma, (\mathcal{F}^{\widehat{\xi}^\star})^-_\gamma)$ is a PFId (resp., PFLId, PFLtId, PFRId) of Ŧ.*
(3) *$((\mathcal{F}^{\widehat{\xi}})^+_\alpha, (\mathcal{F}^{\widehat{\kappa}})^-_\alpha)$ is a PFId (resp., PFLId, PFLtId, PFRId) of Ŧ for all $\alpha \in [\gamma, 1]$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$.*

Proof. It follows from Theorems 3.12 and 3.15. □

Corollary 3.18. *Let $\widehat{\xi}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\xi}$ is a S^-_δHFId (resp., S^-_δHFLId, S^-_δHFLtId, S^-_δHFRId) of Ŧ.*
(2) *$((\mathcal{F}^{\widehat{\xi}})^-_\delta, (\mathcal{F}^{\widehat{\xi}^\star})^+_\delta)$ is a PFId (resp., PFLId, PFLtId, PFRId) of Ŧ.*
(3) *$((\mathcal{F}^{\widehat{\xi}})^-_\alpha, (\mathcal{F}^{\widehat{\kappa}})^+_\alpha)$ is a PFId (resp., PFLId, PFLtId, PFRId) of Ŧ for all $\alpha \in [\delta, 1]$ and $\widehat{\kappa} \in \mathsf{SC}(\widehat{\xi})$.*

Proof. It follows from Theorems 3.13 and 3.16. □

3.4. Characterizations by hesitant and interval-valued fuzzy sets

In our final main subsection, using the set $\mathcal{H}^{(\widehat{\xi},\Phi)}_{\sup^*}$, we characterize the concepts of S^+_γHFIds and S^-_δHFIds in ternary semigroups by HFSs and IvFSs. Recall that for any HFS $\widehat{\xi}$ on $\mathcal{S}$ and $\Phi \subseteq [0, 1]$, we define the HFS $\mathcal{H}^{(\widehat{\xi},\Phi)}_{\sup^*}$ on $\mathcal{S}$ by

$$\mathcal{H}^{(\widehat{\xi},\Phi)}_{\sup^*}(s) := \{\alpha \in \Phi : \sup^* \widehat{\xi}(s) \geqslant \alpha\}$$

for all $s \in \mathcal{S}$.

Below is a characterization of S^+_γHFIds in ternary semigroups.

Theorem 3.19. *The following statements are equivalent for an HFS $\widehat{\kappa}$ on Ŧ.*

(1) *$\widehat{\kappa}$ is a S^+_γHFId (resp., S^+_γHFLId, S^+_γHFLtId, S^+_γHFRId) of Ŧ.*
(2) *$\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}$ is an HFId (resp., HFLId, HFLtId, HFRId) of Ŧ for all $\Phi \subseteq [0, 1-\gamma]$.*

Proof. Only the LtIds procedure is taken into account in this proof. A similar proof can be made for the others.

(1) $\Rightarrow$ (2). Let $\widehat{\kappa}$ be a S^{+}_{γ}HFLtId of Ŧ, $\Phi \subseteq [0, 1-\gamma]$ and $s, v, z \in \text{Ŧ}$. Suppose that $\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(z) \neq \emptyset$. Given $\alpha \in \mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(z)$. Then $\alpha \leqslant \min\{\sup^* \widehat{\kappa}(z), 1-\gamma\} = \sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma$. By our assumption, we obtain that

$$\sup^* \widehat{\kappa}(szv) = (\sup^* \widehat{\kappa}(szv) + \gamma) - \gamma \geqslant \sup^{+}_{\gamma} \widehat{\kappa}(szv) - \gamma \geqslant \sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma \geqslant \alpha.$$

That is, $\alpha \in \mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(szv)$. This means that $\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(z) \subseteq \mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(szv)$. Therefore, $\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}$ is an HFLtId of Ŧ for all $\Phi \subseteq [0, 1-\gamma]$.

(2) $\Rightarrow$ (1). Let $\mathcal{H}^{(\widehat{\kappa},\Psi)}_{\sup^*}$ be an HFLtId of Ŧ for all $\Psi \subseteq [0, 1-\gamma]$ and $s, v, z \in \text{Ŧ}$. Choose $\Phi := [0, 1-\gamma]$, we have that $\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}$ is a HFLtId of Ŧ. Then, we get that

$$\sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma = \min\{\sup^* \widehat{\kappa}(z), 1-\gamma\} \in \mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(z) \subseteq \mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}(szv)$$

and

$$\sup^{+}_{\gamma} \widehat{\kappa}(szv) - \gamma = \min\{\sup^* \widehat{\kappa}(szv), 1-\gamma\} \geqslant \sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma.$$

Hence, $\sup^{+}_{\gamma} \widehat{\kappa}(szv) \geqslant \sup^{+}_{\gamma} \widehat{\kappa}(z)$ which implies that $\widehat{\kappa}(szv) \trianglerighteq^{+}_{\gamma} \widehat{\kappa}(z)$. Therefore, we our claim as desire. □

Similarly, we obtain a characterization of S^{-}_{δ}HFIds in ternary semigroups by the following theorem.

Theorem 3.20. *The following statements are equivalent for an HFS $\widehat{\kappa}$ on Ŧ.*

(1) *$\widehat{\kappa}$ is a S^{-}_{δ}HFId (resp., S^{-}_{δ}HFLId, S^{-}_{δ}HFLtId, S^{-}_{δ}HFRId) of Ŧ.*
(2) *$\mathcal{H}^{(\widehat{\kappa},\Phi)}_{\sup^*}$ is an HFId (resp., HFLId, HFLtId, HFRId) of Ŧ for all $\Phi \subseteq (\delta, 1]$.*

Remark 3.21 ([14]). Let ϑ and σ be FSs in $\mathcal{S}$ such that $\vartheta \subseteq \sigma$. Then we obtain the following statements.

(1) If $\gamma \geqslant \delta$, then $[(\vartheta)^{-}_{\gamma}, (\sigma)^{-}_{\delta}]$ is an IvFS in $\mathcal{S}$.
(2) If $\gamma \leqslant \delta$, then $[(\vartheta)^{+}_{\gamma}, (\sigma)^{+}_{\delta}]$ is an IvFS in $\mathcal{S}$.

Being S^{+}_{γ}HFIds (resp., S^{+}_{γ}HFLIds, S^{+}_{γ}HFLtIds, S^{+}_{γ}HFRIds) is inconvenient for determining an HFS, according to Theorem 3.19. The following theorem provides a more practical approach to this viewpoint.

Theorem 3.22. *The following statements are equivalent for an HFS $\widehat{\kappa}$ on Ŧ.*

(1) *$\widehat{\kappa}$ is a S^{+}_{γ}HFId (resp., S^{+}_{γ}HFLId, S^{+}_{γ}HFLtId, S^{+}_{γ}HFRId) of Ŧ.*
(2) *$[(\mathcal{F}^{\widehat{\kappa}})^{+}_{\gamma}, (\mathcal{F}^{\widehat{\kappa}})^{+}_{\alpha}]$ is an IvFId (resp., IvFLId, IvFLtId, IvFRId) of Ŧ for all $\alpha \in [\gamma, 1]$.*
(3) *$[(\mathcal{F}^{\widehat{\kappa}})^{+}_{\gamma_1}, (\mathcal{F}^{\widehat{\kappa}})^{+}_{\gamma_2}]$ is an IvFId (resp., IvFLId, IvFLtId, IvFRId) of Ŧ for all $\gamma_1, \gamma_2 \in [\gamma, 1]$ with $\gamma_1 \leqslant \gamma_2$.*
(4) *$\mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}$ is an IvFId (resp., IvFLId, IvFLtId, IvFRId) of Ŧ.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is followed by Theorem 3.12 and Remark 3.21. Let us examine the equivalence (1) $\Leftrightarrow$ (4). Only the LtIds procedure is taken into account in this proof. A similar proof can be made for the others.

(1) $\Rightarrow$ (4). Let $\widehat{\kappa}$ be a S^{+}_{γ}HFLtId of Ŧ and $s, v, z \in \text{Ŧ}$. Consider

$$\mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}(z) = [0, \sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma] \preceq [0, \sup^{+}_{\gamma} \widehat{\kappa}(szv) - \gamma] = \mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}(szv).$$

Hence, $\mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}$ is an IvFLtId of Ŧ.

(4) $\Rightarrow$ (1). Assume that $\mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}$ is an IvFLtId of Ŧ. Let $s, v, z \in \text{Ŧ}$. Then

$$\mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}(z) = [0, \sup^{+}_{\gamma} \widehat{\kappa}(z) - \gamma] \preceq [0, \sup^{+}_{\gamma} \widehat{\kappa}(szv) - \gamma] = \mathcal{H}^{(\widehat{\kappa},[0,1-\gamma])}_{\sup^*}(szv).$$

This means that $\sup^{+}_{\gamma} \widehat{\kappa}(szv) \geqslant \sup^{+}_{\gamma} \widehat{\kappa}(z)$. Therefore, $\widehat{\kappa}$ is a S^{+}_{γ}HFLtId of Ŧ. □

Unfortunately, until now, we have been unable to characterize S_δ^-HFIds (resp., S_δ^-HFLIds, S_δ^-HFLtIds, S_δ^-HFRIds) using IvFIds (resp., IvFLIds, IvFLtIds, IvFRIds) in the HFS $\mathcal{H}_{\sup^*}^{(\widehat{\xi},\Phi)}$ form. However, it is worth stating the following result regarding IvFIds (resp., IvFLIds, IvFLtIds, IvFRIds) in ternary semigroups.

Corollary 3.23. *Let $\widehat{\kappa}$ be an HFS on Ŧ. The following statements are equivalent.*

(1) *$\widehat{\kappa}$ is a S_δ^-HFId (resp., S_δ^-HFLId, S_δ^-HFLtId, S_δ^-HFRId) of Ŧ.*
(2) *$[(\mathcal{F}^{\widehat{\kappa}})_\alpha^-, (\mathcal{F}^{\widehat{\kappa}})_\delta^-]$ is an IvFId (resp., IvFLId, IvFLtId, IvFRId) of Ŧ for all $\alpha \in [\delta, 1]$.*
(3) *$[(F^{\widehat{\kappa}})_{\delta_1}^-, (F^{\widehat{\kappa}})_{\delta_2}^-]$ is an IvFId (resp., IvFLId, IvFLtId, IvFRId) of Ŧ for all $\delta_1, \delta_2 \in [\delta, 1]$ with $\delta_2 \leqslant \delta_1$.*

Proof. We complete the proof by applying Theorem 3.13 and Remark 3.21. □

4. Conclusions and future work

In present paper, we have introduced S_γ^+HFIds (resp., S_γ^+HFLtIds, S_γ^+HFLIds, S_γ^+HFRIds) and S_δ^-HFIds (resp., S_δ^-HFLtIds, S_δ^-HFLIds, S_δ^-HFRIds) which are general types of SHFIds (resp., SHFLtIds, SHFLIds, SHFRIds) of ternary semigroups, and discussed their interesting properties. We can summarize the relationships between IvFIds, HFIds, SHFIds, S_δ^-HFIds and S_γ^+HFIds by Figure 4. Further, the general types of SHFIds have been characterized by HFSs, FSs, PFSs, IvFSs and Łukasiewicz (anti-) fuzzy sets.

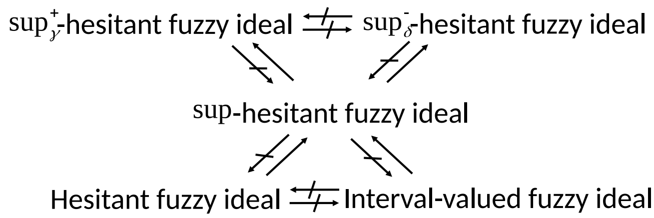

Figure 3: Relationships of IvFIds, HFIds, SHFIds, S_γ^+HFIds and S_δ^-HFIds.

The following are objectives for study and research in ternary semigroups and other algebras:

1. to study $\sup_\gamma^+$- and $\sup_\delta^-$-types of HFSs based on bi-ideals of ternary semigroups;
2. to study $\sup_\gamma^+$- and $\sup_\delta^-$-types of HFSs in Γ-semigroups and LA-semigroups;
3. to extend $\sup_\gamma^+$- and $\sup_\delta^-$-types of HFSs to BRK-algebras and G-algebras introduced by Bandaru [3, 4].

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