J. Math. Computer Sci., 33 (2024), 137–145

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

The differential transform of the Caputo-Fabrizio fractional derivative

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Abstract

In this paper, we introduce the differential transform of Caputo-Fabeizio fractional derivatives which transforms the fractional derivatives into power series. This is an extension of the differential transform from ordinary derivatives to fractional ones. Interestingly, this transformation will help to solve fractional differential equations using the series method and by applying properties of difference operators.

Keywords: Differential transform, fractional derivatives, Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Laplace transform.

2020 MSC: 44A10, 26A33.

1. Introduction

In this section, we give the definitions and results needed afterwards.

Definition 1.1. For $\mathrm{Re}(s) > 0$, the lower incomplete gamma function is defined as:

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$$

and the upper incomplete gamma function is defined as

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt.$$

Clearly,

$$\Gamma(s,x) = \gamma(s) - \gamma(s,x)$$

Moreover, $\gamma(s,x) \to \gamma(s)$ as $x \to \infty$ and $\Gamma(s,0) = \Gamma(s)$.

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doi: 10.22436/jmcs.033.02.02

Received: 2023-06-29 *Revised:* 2023-09-23 *Accepted:* 2023-11-01

1.1. Differential transforms

The differential transform method is a mathematical method for resolving differential equations. It offers an alternate strategy to established approaches like analytical or numerical ones. The origins of the differential transform method can be traced back to the early 20th century. It is believed that the method was inspired by the Taylor series expansion, which is a fundamental concept in calculus. Researchers likely started exploring the idea of transforming differential equations into power series as a means of finding approximate solutions. For further details of the differential transform and its applications, see [5, 19, 22]. The differential transform is defined as following.

Definition 1.2 ([6])**.** Suppose the function $f(x)$ is analytical at $x_0 = 0$, then the differential transform $F(k)$ is defined as

$$F(k) = \frac{f^{(k)}(0)}{k!}.$$

The inverse differential transform of $F(k)$ is

$$f(x) = \sum_{k=0}^{\infty} F(k)x^k.$$

Clearly,

$$F(0) = f(0). \tag{1.1}$$

Hence,

$$\sum_{k=1}^{\infty} F(k)t^k = f(t) - f(0). \tag{1.2}$$

Example 1.3. Let $F(k)$ be the delta function $\delta(k - r)$, where

$$\delta(k-r) = \begin{cases} 1, & \text{if } k = r, \\ 0, & \text{if } k \neq r, \end{cases} \tag{1.3}$$

then the inverse differential transform of the delta function is

$$f(x) = \sum_{k=0}^{\infty} F(k)x^k = \sum_{k=0}^{\infty} \delta(k-r)x^k = x^r.$$

Let $U(k), G(k)$, and $H(k)$ be the differential transforms of $u(x), g(x)$, and $h(x)$, respectively at $x_0 = 0$. The key operations of the differential transform are presented in Table 1.

Table 1: Key operations of the differential transform.

Original function	Transformed function
$u(x) = g(x) + h(x)$	$U(k) = G(k) + H(k)$
$u(x) = cg(x)$	$U(k) = cG(k)$
$u(x) = \frac{d^n g(x)}{dx^n}$	$U(k) = \frac{(k+n)!}{k!} G(k+n)$
$u(x) = \int_0^x g(\xi)d\xi$	$U(k) = \frac{G(k-1)}{k}; k \geqslant 1$
$u(x) = g(x)h(x)$	$U(k) = \sum_{i=0}^{k} G(i)H(k-i)$
$u(x) = x^n$	$U(k) = \delta(k-n)$
$u(x) = \exp(cx)$	$U(k) = \frac{c^k}{k!}$
$u(x) = \cos(\omega x)$	$U(k) = \frac{\omega^k}{k!} \cos(\frac{k\pi}{2})$
$u(x) = \sin(\omega x)$	$U(k) = \frac{\omega^k}{k!} \sin(\frac{k\pi}{2})$

A related topic to differential transforms is the difference operators.

1.2. *Forward difference operators*

The difference operator is defined as follows

Definition 1.4. Let N be the set of the natural numbers and let $S(N)$ be the set of all sequences over N. Define the difference operator $\Delta : S(N) \to S(N)$ as

$$(\Delta u)(n) = u(n+1) - u(n).$$

The forward difference operators are extensively used in numerical analysis and numerical techniques for solving differential equations. Particularly useful when dealing with discrete data or when there is no analytical expression. It is easy to prove that the difference operator Δ is a linear operator and it satisfies the following proposition

Proposition 1.5. *For* $u \in S(N)$, $\sum_{k=m}^{n-1} (\Delta u)(k) = u(n) - u(m)$.

Proposition 1.6. *For* $u \in S(N)$, $\Delta(\sum_{k=m}^{n-1} u(k)) = \sum_{k=m}^{n} u(k) - \sum_{k=m}^{n-1} u(k) = u(n)$.

The difference operators and their applications were well-studied by many authors. For example, see [1, 21].

1.3. *Fractional calculus and fractional derivatives*

Fractional calculus has resurfaced and gained momentum due to its potential in engineering systems, multidisciplinary fields, biology, medicine, and applied sciences. Its wide range of applications includes areas like linear anomalous diffusion equation and its characteristics [14], respiratory tissue, RC circuits [4], and drug diffusion [16]. At the same time, fractional calculus has found its way to sensors, analogue filters, and digital filters [18]. Furthermore, mathematical models of biological systems have been thoroughly examined using standard fractional derivatives. For example, Li et al. [17] investigated the PWD, pine wilt disease, using the Caputo fractional derivative. Also, fractional calculus is applied to interpret hydrological cycles or analyze underground water chemical reactions, see [20]. As discussed by Du et al. [13], fractional calculus introduces memory into modeling processes, which in turn is required for specific approaches in, e.g., biology or psychology. Thus the researchers suggest interpreting the fractional order of a process as a degree of memory of this process.

The aforementioned derivatives have a unique kernel, which is thought to cause certain issues. Caputo and Fabrizio [11] presented a definition of a fractional derivative with a non-singular kernel as follows.

Definition 1.7. For a real smooth function f and for $\alpha \in [0, 1)$, Caputo-Fabrizio fractional derivative [12] is defined as

$$({}^{CF}D^{\alpha}f)(t) = \begin{cases} \frac{1}{1-\alpha}\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-u)} f'(u)du, & 0 \leqslant \alpha < 1, \\ f'(t), & \alpha = 1. \end{cases} \tag{1.4}$$

Caputo-Fabrizio fractional derivative was used by many authors for modeling various problems in engineering sciences (see, e.g., [8, 10]). Using (1.4), if $g(t) = ({}^{CF}D^{\alpha}f)(t)$, then (1.4) implies $g(0) = 0$ and

$$(1-\alpha)e^{\frac{\alpha t}{1-\alpha}} g(t) = \int_0^t e^{\frac{\alpha u}{1-\alpha}} f'(u)du. \tag{1.5}$$

Differentiating both sides of (1.5) and simplifying we get:

$$(1-\alpha)g'(t) + \alpha g(t) = f'(t).$$

Therefore, we have the following result.

Proposition 1.8. *Let* ${}^{CF}D^{\alpha}$ *be the Caputo-Fabrizio fractional derivative. Let* $g(t) = ({}^{CF}D^{\alpha}f)(t)$, *then* $g(t)$ *is a solution for the IVP*

$$(1-\alpha)g'(t) + \alpha g(t) = f'(t);\ g(0) = 0.$$

Using this proposition, the following are the Caputo-Fabrizio fractional derivatives for some elementary functions.

Proposition 1.9. *For* $f(t) = c$, *then* $({}^{CF}D^{\alpha}f)(t) = 0$.

Proof. Using Proposition 1.8, if $f(t) = c$, then $g(t) = ({}^{CF}D^{\alpha}f)(t)$ satisfies

$$(1-\alpha)g'(t) + \alpha g(t) = 0;\ g(0) = 0. \tag{1.6}$$

Dividing (1.6) by $1-\alpha$,

$$g'(t) + \frac{\alpha}{1-\alpha}g(t) = 0. \tag{1.7}$$

Multiplying the equation (1.7) by the integrating factor $e^{\frac{\alpha t}{1-\alpha}}$,

$$e^{\frac{\alpha t}{1-\alpha}}g'(t) + \frac{\alpha}{1-\alpha}e^{\frac{\alpha t}{1-\alpha}}g(t) = 0. \tag{1.8}$$

Equation (1.8) is written as

$$(e^{\frac{\alpha t}{1-\alpha}}g(t))' = 0.$$

Integrating both sides from 0 to t,

$$e^{\frac{\alpha t}{1-\alpha}}g(t) - g(0) = 0.$$

Using the initial condition $g(0) = 0$, then $g(t) = 0$. Therefore, $({}^{CF}D^{\alpha}f)(t) = 0$. □

Proposition 1.10. *For* $f(t) = t$, *then* $({}^{CF}D^{\alpha}f)(t) = \frac{1-e^{-\frac{\alpha t}{1-\alpha}}}{\alpha}$.

Proof. Using Proposition 1.8, if $f(t) = t$, then $g(t) = ({}^{CF}D^{\alpha}f)(t)$ satisfies

$$(1-\alpha)g'(t) + \alpha g(t) = 1;\ g(0) = 0. \tag{1.9}$$

Dividing (1.9) by $1-\alpha$,

$$g'(t) + \frac{\alpha}{1-\alpha}g(t) = \frac{1}{1-\alpha}. \tag{1.10}$$

Multiplying the equation (1.10) by the integrating factor $e^{\frac{\alpha t}{1-\alpha}}$,

$$e^{\frac{\alpha t}{1-\alpha}}g'(t) + \frac{\alpha}{1-\alpha}e^{\frac{\alpha t}{1-\alpha}}g(t) = \frac{1}{1-\alpha}e^{\frac{\alpha t}{1-\alpha}}. \tag{1.11}$$

Equation (1.11) is written as

$$(e^{\frac{\alpha t}{1-\alpha}}g(t))' = \frac{1}{1-\alpha}e^{\frac{\alpha t}{1-\alpha}}.$$

Integrating both sides from 0 to t,

$$e^{\frac{\alpha t}{1-\alpha}}g(t) - g(0) = \frac{1}{\alpha}(e^{\frac{\alpha t}{1-\alpha}} - 1).$$

Using the initial condition $g(0) = 0$, then $g(t) = \frac{1-e^{-\frac{\alpha t}{1-\alpha}}}{\alpha}$. Therefore, $({}^{CF}D^{\alpha}f)(t) = \frac{1-e^{-\frac{\alpha t}{1-\alpha}}}{\alpha}$. □

Proposition 1.11. *For* $f(t) = t^r; \Re(r) > -1$, *then* $({}^{CF}D^{\alpha}f)(t) = \frac{r}{1-\alpha}e^{-\frac{\alpha t}{1-\alpha}}(\frac{\alpha}{\alpha-1})^{-r}\gamma(r, \frac{\alpha t}{\alpha-1})$, *where* $\gamma(r,x)$*is the incomplete gamma function (see* [2, 16]*).*

Proof. Set $g(t) = ({}^{CF}D^{\alpha}f)(t)$. Then $g(t)$ satisfies

$$(1-\alpha)g'(t) + \alpha g(t) = rt^{r-1}; \quad g(0) = 0. \tag{1.12}$$

Dividing both sides of (1.12) by $1-\alpha$ gives

$$g'(t) + \frac{\alpha}{1-\alpha}g(t) = \frac{1}{1-\alpha}rt^{r-1}. \tag{1.13}$$

Multiplying both sides of (1.13) by $e^{\frac{\alpha t}{1-\alpha}}$ to get

$$\frac{d}{dt}\left(e^{\frac{\alpha t}{1-\alpha}}g(t)\right) = \frac{1}{1-\alpha}rt^{r-1}e^{\frac{\alpha t}{1-\alpha}}. \tag{1.14}$$

Now, for $\Re(r) > 0$ (see [2]), we have

$$\int_0^t u^{r-1}e^{-au}du = a^{-r}\gamma(r, at). \tag{1.15}$$

Integrating both sides of (1.14) from 0 to t and applying (1.15) lead to

$$e^{\frac{\alpha t}{1-\alpha}}g(t) = \frac{r}{1-\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{-r}\gamma\left(r, \frac{\alpha t}{\alpha-1}\right).$$

Therefore, $({}^{CF}D^{\alpha}f)(t) = g(t) = \frac{r}{1-\alpha}e^{\frac{-\alpha t}{1-\alpha}}\left(\frac{\alpha}{\alpha-1}\right)^{-r}\gamma\left(r, \frac{\alpha t}{\alpha-1}\right).$ □

Proposition 1.12. *For* $f(t) = e^{bt}$, *then* $({}^{CF}D^{\alpha}f)(t) = \frac{b(e^{-\frac{at}{a-1}} - e^{bt})}{a(b-1)-b}$.

Proof. Set $g(t) = ({}^{CF}D^{\alpha}f)(t)$. Then $g(t)$ satisfies

$$(1-\alpha)g'(t) + \alpha g(t) = be^{bt}; \quad g(0) = 0.$$

Repeating the steps in the proof of Proposition 1.11 leads to

$$\left(e^{\frac{\alpha}{1-\alpha}}g(t)\right)' = \frac{b}{1-\alpha}e^{\left(b+\frac{\alpha}{1-\alpha}\right)t}.$$

Integrating both sides from 0 to t and using the initial condition $g(0) = 0$ give the desired result. □

More properties of the Caputo-Fabrizio fractional derivative are given in [9].

2. The differential transform of the Caputo-Fabrizio fractional derivative

This section gives a formula for the differential transform of the Caputo-Fabrizio fractional derivative.

Proposition 2.1. *For* $0 < \alpha < 1$, *let* $g(t) = ({}^{CF}D^{\alpha}f)(t)$ *be the Caputo-Fabrizio fractional derivative of* $f(t)$ *and let* $G(n)$ *be the differential transform of* $g(t)$. *Then* $G(n)$ *satisfies*

$$G(n) = \frac{-1}{\alpha}\sum_{m=1}^{n}\left(1-\frac{1}{\alpha}\right)^{m-n-1}\frac{m!}{n!}F(m) = \frac{1}{1-\alpha}\sum_{m=1}^{n}\left(1-\frac{1}{\alpha}\right)^{m-n}\frac{m!}{n!}F(m).$$

Proof. Since $g(t) = ({}^{CF}D^{\alpha}f)(t)$, then Proposition 1.8 implies $g(t)$ satisfies

$$(1-\alpha)g'(t) + \alpha g(t) = f'(t); \quad g(0) = 0.$$

Taking differential transform for both sides gives

$$(1-\alpha)(n+1)G(n+1) + \alpha G(n) = (n+1)F(n+1).$$

This equation is equivalent to

$$(n+1)G(n+1) + \frac{\alpha}{1-\alpha}G(n) = \frac{1}{1-\alpha}(n+1)F(n+1).$$

Multiply both sides by $n!(\frac{1}{\alpha}-1)^{n+1}$ to get

$$(n+1)!\left(1-\frac{1}{\alpha}\right)^{n+1} G(n+1) - \left(1-\frac{1}{\alpha}\right)^{n} n!G(n)(-1)^n = -\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{n}(n+1)!F(n+1).$$

This equation, using the forward difference operator from Definition 1.4, can be written as

$$\Delta\left(\left(1-\frac{1}{\alpha}\right)^{n} n!G(n)\right) = \frac{-1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{n}(n+1)!F(n+1).$$

Taking the sum for both sides and applying Proposition 1.5,

$$\left(1-\frac{1}{\alpha}\right)^{n} n!G(n) - G(0) = -\frac{1}{\alpha}\sum_{m=0}^{n-1}\left(1-\frac{1}{\alpha}\right)^{m}(m+1)!F(m+1).$$

Now, using (1.1), then $G(0) = g(0) = 0$. Therefore,

$$G(n) = \frac{-1}{\alpha}\sum_{m=0}^{n-1}\left(1-\frac{1}{\alpha}\right)^{m-n}\frac{(m+1)!}{n!}F(m+1) = \frac{1}{1-\alpha}\sum_{m=1}^{n}\left(1-\frac{1}{\alpha}\right)^{m-n}\frac{m!}{n!}F(m).$$

□

As an application of this result, we have following example.

Example 2.2. For $p \in \mathbb{N}$, let $f(t) = t^p$. In this case, $F(n) = \delta(n-p)$, where $\delta(n-p)$ is the delta function given in (1.3). This implies

$$G(n) = \frac{1}{1-\alpha}\sum_{m=1}^{n}\left(1-\frac{1}{\alpha}\right)^{m-n}\frac{m!}{n!}\delta(m-p).$$

Hence,

$$G(n) = \frac{1}{1-\alpha}\left(1-\frac{1}{\alpha}\right)^{p-n}\frac{p!}{n!}; \ 1 \leqslant p \leqslant n. \tag{2.1}$$

Now, Proposition 2.2 in [7] (or Proposition 2.1 in [6]) gives

$$\sum_{n=0}^{N}\frac{a^n}{n!} = e^a\frac{\Gamma(N+1,a)}{N!}. \tag{2.2}$$

Equations (2.1) and (2.2) give

$$\begin{aligned}({}^{CF}D^{\alpha}f)(t) &= \frac{1}{1-\alpha}\left(1-\frac{1}{\alpha}\right)^{p} p!\sum_{n=p}^{\infty}\frac{(-t(\frac{\alpha}{1-\alpha}))^n}{n!} \\ &= \frac{1}{1-\alpha}\left(1-\frac{1}{\alpha}\right)^{p} p!e^{\frac{-\alpha t}{1-\alpha}}\frac{\gamma(p,\frac{-\alpha t}{1-\alpha})}{(p-1)!} = \frac{p}{1-\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{-p} e^{\frac{-\alpha t}{1-\alpha}}\gamma(p,\frac{-\alpha t}{1-\alpha}).\end{aligned}$$

3. Fractional calculus

Now, according to [15], we can change the ordinary time derivative operator by the fractional one as follows. We introduce a fractional time derivative operator as follows

$$\frac{d^\alpha}{dt^\alpha}, \quad 0 < \alpha \leqslant 1, \tag{3.1}$$

where α is an arbitrary parameter close to 1, this parameter represents the order of the derivative and the case $\alpha = 1$ is the ordinary derivative. However, the ordinary time operator has dimensions of inverse seconds s^{-1}. Hence, the expression (3.1) becomes

$$\left[\frac{d^\alpha}{dt^\alpha}\right] = \frac{1}{s^\alpha}, \quad 0 < \alpha \leqslant 1.$$

This is not an ordinary time derivative, because of the dimension $s^{-\alpha}$. To be consistent with dimensionality, we introduce a new parameter σ as

$$\left[\frac{1}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}\right] = \frac{1}{s}, \quad 0 < \alpha \leqslant 1, \tag{3.2}$$

such that when $\alpha = 1$ the expression (3.2) becomes an ordinary derivative. This is true if the parameter σ has dimensions of seconds, $[\sigma] = s$. Therefore, we can change the ordinary time derivative operator by the fractional as follows

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}, \quad 0 < \alpha \leqslant 1. \tag{3.3}$$

4. Application example

Proposition 2.1 mapped the the Caputo-Fabrizio fractional differential equation into a difference equation, i.e., this Proposition allows us to find the series solution of a fractional differential equation. As an application of this approach, we investigate the falling body using the fractional derivative approach. The velocity $v(t)$ of a moving body and the position of this body $s(t)$ are related by the equation

$$v(t) = \frac{ds}{dt}. \tag{4.1}$$

By (3.3), equation (4.1) becomes

$$v(t) = \frac{1}{\sigma^{1-\alpha}}\frac{d^\alpha s}{dt^\alpha}, \quad 0 < \alpha \leqslant 1.$$

Therefore,

$$\frac{d^\alpha s}{dt^\alpha} = \sigma^{1-\alpha} v(t). \tag{4.2}$$

We will solve (4.2) in which we will find the position interm of the velocity. Taking the differential transform for both sides and applying Proposition 2.1,

$$\sum_{m=1}^{n}\left(1-\frac{1}{\alpha}\right)^m m!S(m) = n!(1-\alpha)\left(1-\frac{1}{\alpha}\right)^n \sigma^{1-\alpha}V(n),$$

where $S(n)$ and $V(n)$ are the differential transforms of $s(t)$ and $v(t)$, respectively. By Proposition 1.6 and applying the forward difference operator for both sides we get

$$\left(1-\frac{1}{\alpha}\right)^{n+1}(n+1)!S(n+1) = (n+1)!(1-\alpha)\left(1-\frac{1}{\alpha}\right)^{n+1}\sigma^{1-\alpha}V(n+1) - n!(1-\alpha)\left(1-\frac{1}{\alpha}\right)^n \sigma^{1-\alpha}V(n).$$

Therefore,

$$S(n) = (1-\alpha)\sigma^{1-\alpha}V(n) - (1-\alpha)\left(1-\frac{1}{\alpha}\right)^{-1}\sigma^{1-\alpha}\frac{V(n-1)}{n}; \quad n \geqslant 1.$$

Taking the inverse differential transform for both sides and applying (1.2) will lead to

$$s(t) = s(0) + \sum_{n=1}^{\infty} S(n)t^n = s(0) + \sigma^{1-\alpha}\left((1-\alpha)(\nu(t)-\nu(0)) + \alpha\int_0^t \nu(\xi)d\xi\right).$$

Therefore,

$$s(t) = s(0) + \sigma^{1-\alpha}\left((1-\alpha)(\nu(t)-\nu(0)) + \alpha\int_0^t \nu(\xi)d\xi\right). \tag{4.3}$$

Example 4.1. For the free-falling when $\nu(t) = \nu_0 - gt$ and initial position $s(0) = s_0$, then (4.3) becomes

$$s(t) = s_0 + \sigma^{1-\alpha}(\alpha\nu_0 - g(1-\alpha))t - \frac{1}{2}g\alpha\sigma^{1-\alpha}t^2.$$

As $\alpha \to 1$, then $s(t) \to s_0 + \nu_0 t - \frac{1}{2}gt^2$, which is the equation for the free falling using the ordinary derivative. For $\nu(0) = 5, s(0) = 100, \sigma = 2$, and $g = 9.81$, the height $s(t)$ is represented in Figure 1.

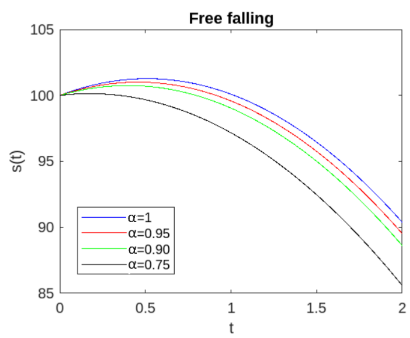

Figure 1: Free fall with fractional derivative as $\alpha \to 1$.

5. Conclusion

The differential transform of the Caputo-Fabrizio fractional derivative is obtained, this result transformed the fractional differential equations into algebraic equations. In addition, for initial conditions, we presented sets of solutions for different values of α between 0 and 1. An application for this result is provided.

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