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Boundary control associated with a parabolic equation

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Abstract

In this paper, we consider a boundary control problem associated with a parabolic equation. On the part of the border of the considered domain, the value of the solution with control parameter is given. Restrictions on the control are given in such a way that the average value of the solution in some part of the considered domain gets a given value. The auxiliary problem is solved by the method of separation of variables, while the problem in consideration is reduced to the Volterra integral equation. By Laplace transform method, the existence theorem for admissible control is proved.

Keywords: Parabolic equation, integral equation, initial-boundary value problem, admissible control, Laplace transform.

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1. Introduction

Consider the following heat exchange process along the interval $0 < x < l$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right), \quad (x,t) \in \Omega_T = (0,l) \times (0,\infty), \tag{1.1}$$

with boundary value conditions

$$u(0,t) = \mu(t), \quad u(l,t) = 0, \quad t > 0, \tag{1.2}$$

and initial value condition

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant l. \tag{1.3}$$

Assume that the function $k(x) \in C^1\big([0,l]\big)$ satisfies condition

$$k(x) > 0, \quad 0 \leqslant x \leqslant l.$$

Definition 1.1. If function $\mu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\mu(0) = 0$ and $|\mu(t)| \leqslant 1$, we say that this function is an *admissible control*.

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We consider also the *weight function* $\rho(x)$, which is smooth on the interval $(0;l)$ and defined as follows

$$\rho(x) = \int_x^l \frac{dy}{k(y)} \Big(\int_0^l \frac{dy}{k(y)} \Big)^{-1}. \tag{1.4}$$

One of the urgent problems for the equations of mathematical physics is the problem of mathematical modeling of processes associated with various partial differential equations. In particular, mathematical modeling of the heat exchange process and the control of this process. Control in this situation is made by changing the heat flux entering to the region under consideration from a part of its boundary. It is natural to achieve an average temperature in the whole area. Therefore, it is important to control the boundary flow to reach the average temperature in any part of the area, and in the case of a delta-like distribution, at a fixed point in the area.

In this article, we consider the control problem associated with a parabolic equation. On part of the boundary of the segment $[0,l]$, there is a source with a given flow. It is required to find such a mode of operation of the source so that the average value of the solution in some part of the segment $[0,l]$ takes the specified value.

In the present work we consider the following problem.

Problem 1.2. For the given function $\theta(t)$, problem consists of looking for the admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1.1)-(1.3) exists and for all $t \geqslant 0$ satisfies the equation

$$\int_0^l \rho(x)\, u(x,t) dx = \theta(t), \tag{1.5}$$

where weight function $\rho(x)$ is defined by (1.4). More information about the function $\theta(t)$ will be given in Section 2.

From the increasing interest in physics and mathematics, a lot of effects have been devoted to the studies of boundary control problems for the parabolic equations in recent years. The optimal control problem for the second order parabolic type equations was studied by Friedman and Fattorin [15, 19]. Time-optimal problems with control on the boundary for the second order parabolic equation have been treated by Egorov [14]. He proved a bang-bang principle in the special case.

The boundary control problem for a second order parabolic type equation with a piecewise smooth boundary in a n-dimensional domain was studied by Albeverio and Alimov [1, 2] and an estimate for the minimum time required to reach a given average temperature was found. In [3], a boundary control problem with a positive weight function placed under the integral in the n-dimensional domain for the homogeneous heat conduction equation was studied.

The latest results on boundary control problems for the second order parabolic type equations are studied in works [5, 6, 8, 17, 21]. These articles are mainly devoted to the problems of finding the boundary control function for the heat transfer equation in one and two-dimensional domain. In [7], an estimate of the minimum time for a given average temperature in a two-dimensional domain was obtained.

Boundary control problems for pseudo-parabolic equations are studied in works [9, 10, 16]. In these works, the control problem is studied for the pseudo-parabolic type equation, but the proof of the control function is proved using the Laplace transform method.

A lot of information about optimal control problems is given in detail in the monographs of Friedman et al. [18, 20, 22]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [4]. The practical approaches to optimal control of the heat equation are described in publications like [13].

In previous works, boundary control problems for parabolic type equations were also considered. For example, the time-optimal control for a heat transfer equation was studied in [11, 12]. In our work, it is proved that the control function exists when the weight function is under integral. In Section 2, the given problem is reduced to the Volterra integral equation of the first kind. Finally, required estimate for the kernel is obtained and proved the existence of the control function using the Laplace transform method in the Section 3.

2. Main integral equation

Consider the following eigenvalue problem

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) = -\lambda_k v_k(x), \quad 0 < x < l, \tag{2.1}$$

with boundary value conditions

$$v_k(0) = v_k(l) = 0, \quad 0 \leqslant x \leqslant l, \quad k = 1, 2, \dots. \tag{2.2}$$

It is well-know that this problem is self-adjoint in $L_2(0,l)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \to \infty, \;\; k \to \infty.$$

The corresponding eigenfuctions v_k form a complete orthonormal system $\{v_k\}_{k\in N}$ in $L_2([0,l])$ and these functions belong to $C([0,l])$ (see [23, 24]).

Definition 2.1 ([11])**.** By the solution of the problem (1.1)-(1.3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \frac{l-x}{l}\mu(t) - w(x,t), \tag{2.3}$$

where the function $w(x,t) \in C^{2,1}_{x,t}(\Omega_T) \cap C(\overline{\Omega}_T)$ is the solution to the problem:

$$w_t = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial w}{\partial x}\right) + \frac{k'(x)}{l}\mu(t) + \frac{l-x}{l}\mu'(t),$$

with boundary value conditions

$$w(0,t) = 0, \quad w(l,t) = 0,$$

and initial value condition

$$w(x,0) = 0.$$

Set

$$\beta_k = \left(\lambda_k b_k - a_k\right)\gamma_k, \tag{2.4}$$

where

$$a_k = \int_0^l \frac{k'(x)}{l}\, v_k(x)dx, \quad b_k = \int_0^l \frac{l-x}{l} v_k(x)dx, \tag{2.5}$$

and

$$\gamma_k = \int_0^l \rho(x)\, v_k(x)dx, \quad k = 1, 2, \dots. \tag{2.6}$$

Consequently, we have

$$w(x,t) = \sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\left(\mu(s)a_k + \mu'(s)b_k\right)ds\right)v_k(x), \tag{2.7}$$

where a_k and b_k defined by (2.5) and the function $\nu_k(x)$ is the solution of problem (2.1)-(2.2), i.e., the eigenfunction. From (2.3) and (2.7), we get the solution of the problem (1.1)–(1.3) (see, [24]):

$$u(x,t)=\frac{l-x}{l}\mu(t)-\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\big(\mu(s)a_k+\mu'(s)b_k\big)\,ds\right)\nu_k(x).$$

We know that the eigenvalues λ_k of the boundary value problem (2.1)-(2.2) satisfy the inequalities $\lambda_k \geqslant 0$ (see [23, 24]). According to Jentsch's theorem, $\nu_1(x)>0$ (see [25]). Then from $k(x)>0$ and the equation (2.1), we have $\lambda_1>0$. According to condition (1.5) and the solution of the problem (1.1)-(1.3), we may write

$$\begin{aligned}
\theta(t)&=\int_0^l \rho(x)\,u(x,t)dx\\
&=\mu(t)\int_0^l \rho(x)\,\frac{l-x}{l}dx-\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\big(\mu(s)a_k+\mu'(s)b_k\big)\,ds\right)\int_0^l \rho(x)\,\nu_k(x)dx\\
&=\mu(t)\int_0^l \rho(x)\,\frac{l-x}{l}dx-\sum_{k=1}^{\infty}a_k\,\gamma_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds-\sum_{k=1}^{\infty}b_k\,\gamma_k\int_0^t e^{-\lambda_k(t-s)}\mu'(s)ds \qquad (2.8)\\
&=\mu(t)\int_0^l \rho(x)\,\frac{l-x}{l}dx-\sum_{k=1}^{\infty}a_k\,\gamma_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds\\
&\quad-\mu(t)\sum_{k=1}^{\infty}b_k\,\gamma_k+\sum_{k=1}^{\infty}\lambda_k\,b_k\,\gamma_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds,
\end{aligned}$$

where γ_k defined by (2.6). Note that

$$\sum_{k=1}^{\infty}b_k\,\gamma_k=\int_0^l\rho(x)\left(\sum_{k=1}^{\infty}b_k\,\nu_k(x)\right)dx=\int_0^l\rho(x)\,\frac{l-x}{l}dx. \qquad (2.9)$$

As a result, from (2.8) and (2.9), we obtain

$$\theta(t)=\sum_{k=1}^{\infty}\big(\lambda_k\,b_k-a_k\big)\,\gamma_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds.$$

Set

$$B(t)=\sum_{k=1}^{\infty}\beta_k\,e^{-\lambda_k t},\quad t>0, \qquad (2.10)$$

where β_k defined by (2.4). Then we get the main integral equation

$$\int_0^t B(t-s)\,\mu(s)ds=\theta(t),\quad t>0. \qquad (2.11)$$

Denote by $W(M)$ the set of function $\theta\in W_2^2(-\infty,+\infty)$, $\theta(t)=0$ for $t\leqslant 0$, which satisfies the condition

$$\|\theta\|_{W_2^2(\mathbb{R}_+)}\leqslant M.$$

Theorem 2.2. *There exists $M>0$ such that for any function $\theta\in W(M)$ the solution $\mu(t)$ of the equation (2.11) exists and satisfies condition*

$$|\mu(t)|\leqslant 1.$$

3. Proof of the Theorem 2.2

Lemma 3.1. *For the coefficients* $\{\beta_k\}_{k=1}^{\infty}$ *the estimate*

$$0 \leqslant \beta_k \leqslant C_1, \quad k = 1, 2, \dots,$$

is valid, where $C_1 = \text{const} > 0$.

Proof.

Step 1. We use from (2.1) and (2.5), consider the following equality

$$\begin{aligned}
\lambda_k\, b_k = \int_0^l \frac{l-x}{l} \lambda_k\, \nu_k(x)\, dx &= -\int_0^l \frac{l-x}{l} \frac{d}{dx}\left(k(x)\, \frac{d\nu_k(x)}{dx}\right) dx \\
&= -\left(\frac{l-x}{l} k(x)\nu_k'(x)\Big|_{x=0}^{x=l} + \frac{1}{l}\int_0^l k(x)\, \nu_k'(x)\, dx\right) \\
&= k(0)\nu_k'(0) - \frac{1}{l}\int_0^l k(x)\, \nu_k'(x)\, dx \\
&= k(0)\nu_k'(0) - \frac{1}{l}\Big(k(l)\nu_k(l) - k(0)\nu_k(0)\Big) + \int_0^l \frac{k'(x)}{l}\, \nu_k(x)\, dx = k(0)\nu_k'(0) + a_k.
\end{aligned}$$

Then we have

$$\lambda_k\, b_k - a_k = k(0)\nu_k'(0), \quad k = 1, 2, \dots. \tag{3.1}$$

Step 2. It can be seen that from Eq. (2.1) we have the following relation

$$k(0)\nu_k'(0) = \lambda_k \int_0^l \rho(\tau)\nu_k(\tau)\, d\tau, \quad k = 1, 2, \dots, \tag{3.2}$$

where function $\rho(\tau)$ defined by (1.4). Consequently, from (3.1) and (3.2), we get the following estimate

$$\beta_k = (\lambda_k\, b_k - a_k)\, \gamma_k = k(0)\nu_k'(0) \cdot \int_0^l \rho(x)\, \nu_k(x)\, dx = \lambda_k \left(\int_0^l \rho(x)\, \nu_k(x)\, dx\right)^2 \geqslant 0, \quad k = 1, 2, \dots.$$

Step 3. It is clear that if $k(x) \in C^1([0, l])$, we may write the estimate (see [23, 26])

$$\max_{0 \leqslant x \leqslant l} |\nu_k'(x)| \leqslant C_2\, \lambda_k^{1/2}, \quad k = 1, 2, \dots.$$

Therefore,

$$|\nu_k'(0)| \leqslant C_2\, \lambda_k^{1/2}, \quad |\nu_k'(l)| \leqslant C_2\, \lambda_k^{1/2}. \tag{3.3}$$

According to (3.2) and (3.3), we have the estimate

$$|\gamma_k| = \left|\int_0^l \rho(x)\, \nu_k(x)\, dx\right| = \frac{1}{\lambda_k} k(0)\, |\nu_k'(0)| \leqslant C_3\, \lambda_k^{-1/2}.$$

Then

$$0 \leqslant \beta_k \leqslant k(0)\, \big|\nu_k'(0)\big|\, \big|\gamma_k\big| \leqslant C_1, \quad k = 1, 2, \dots. \qquad \square$$

It is known that if the function $k(x)$ is smooth and positive in the interval (0, l), then the following estimate is valid for the eigenvalues λ_k of the problem (2.1)-(2.2) (see, e.g., [26, Theorem 1]):

$$\lambda_k = \frac{k^2\pi^2}{p^2} + O(k^{-2}), \quad p = \int_0^l \frac{dx}{\sqrt{k(x)}}, \quad k = 1, 2, \ldots .$$

Proposition 3.2. *Let* $1/2 < \alpha < 1$. *Then for the function* $B(t)$ *defined by* (2.10) *the following estimate*

$$0 < B(t) \leqslant \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leqslant 1,$$

is valid.

Proof. It is clear that, from Lemma 3.1 function $B(t)$ is positive. Let $1/2 < \alpha < 1$ and $\lambda > 0$. Then the maximum value of the function $h(t, \lambda) = t^\alpha e^{-\lambda t}$ is reached at the point $t = \frac{\alpha}{\lambda}$ and this value is equal to $\frac{\alpha^\alpha}{\lambda^\alpha} e^{-\alpha}$. As a result, for any $1/2 < \alpha < 1$, we get the estimate

$$B(t) = \frac{1}{t^\alpha} \sum_{k=1}^\infty \beta_k t^\alpha e^{-\lambda_k t} \leqslant \frac{C_1}{t^\alpha} \sum_{k=1}^\infty \frac{\alpha^\alpha}{\lambda_k^\alpha} e^{-\alpha} \leqslant \frac{C_\alpha}{t^\alpha},$$

where

$$\sum_{k=1}^\infty \frac{1}{\lambda_k^\alpha} < +\infty, \quad 1/2 < \alpha < 1.$$

□

We rewrite integral equation (2.11)

$$\int_0^t B(t-s)\mu(s)ds = \theta(t), \quad t > 0.$$

We use the Laplace transform method to solve equation (2.11). We introduce the notation

$$\widetilde{\mu}(p) = \int_0^\infty e^{-pt} \mu(t)\, dt.$$

Then using Laplace transform we get

$$\widetilde{\mu}(p) = \frac{\widetilde{\theta}(p)}{\widetilde{B}(p)}, \quad \text{where } \ p = a + i\xi, \quad a > 0,$$

and

$$\mu(t) = \frac{1}{2\pi i} \int_{a-i\xi}^{a+i\xi} \frac{\widetilde{\theta}(p)}{\widetilde{B}(p)}\, e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\theta}(a+i\xi)}{\widetilde{B}(a+i\xi)}\, e^{(a+i\xi)t} d\xi. \tag{3.4}$$

Then we can write

$$\widetilde{B}(p) = \int_0^\infty B(t)e^{-pt}\, dt = \sum_{k=1}^\infty \beta_k \int_0^\infty e^{-(p+\lambda_k)t}\, dt = \sum_{k=1}^\infty \frac{\beta_k}{p+\lambda_k},$$

where $B(t)$ is defined by (2.10) and

$$\widetilde{B}(a+i\xi)=\sum_{k=1}^{\infty}\frac{\beta_k}{a+\lambda_k+i\xi}=\sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2}-i\xi\sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}=\mathrm{Re}\widetilde{B}(a+i\xi)+i\,\mathrm{Im}\widetilde{B}(a+i\xi),$$

where

$$\mathrm{Re}\widetilde{B}(a+i\xi)=\sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2},\quad \mathrm{Im}\widetilde{B}(a+i\xi)=-\xi\sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}.$$

We know that

$$(a+\lambda_k)^2+\xi^2\leqslant[(a+\lambda_k)^2+1](1+\xi^2),$$

and we have the following inequality

$$\frac{1}{(a+\lambda_k)^2+\xi^2}\geqslant\frac{1}{1+\xi^2}\,\frac{1}{(a+\lambda_k)^2+1}. \tag{3.5}$$

Consequently, according to Lemma 3.1 and estimate (3.5) we can obtain the following estimates

$$|\mathrm{Re}\widetilde{B}(a+i\xi)|=\sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2}\geqslant\frac{1}{1+\xi^2}\sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+1}=\frac{C_{1a}}{1+\xi^2}, \tag{3.6}$$

and

$$|\mathrm{Im}\widetilde{B}(a+i\xi)|=|\xi|\sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}\geqslant\frac{|\xi|}{1+\xi^2}\sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+1}=\frac{C_{2a}\,|\xi|}{1+\xi^2}, \tag{3.7}$$

where C_{1a}, C_{2a} are as follows

$$C_{1a}=\sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+1},\quad C_{2a}=\sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+1}.$$

From (3.6) and (3.7), we have the following estimate

$$|\widetilde{B}(a+i\xi)|^2=|\mathrm{Re}\widetilde{B}(a+i\xi)|^2+|\mathrm{Im}\widetilde{B}(a+i\xi)|^2\geqslant\frac{\min(C_{1a}^2,C_{2a}^2)}{1+\xi^2},$$

and

$$|\widetilde{B}(a+i\xi)|\geqslant\frac{C_a}{\sqrt{1+\xi^2}},\quad \text{where}\quad C_a=\min(C_{1a},C_{2a}). \tag{3.8}$$

Then, when $a\to 0$, from (3.4), we obtain

$$\mu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\theta}(i\xi)}{\widetilde{B}(i\xi)}\,e^{i\xi t}d\xi. \tag{3.9}$$

Lemma 3.3. *Let* $\theta(t)\in W(M)$. *Then for the image of the function* $\theta(t)$ *the following inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi\leqslant C_4\,\|\theta\|_{W_2^2(R_+)},$$

is valid, where $C_4=\mathrm{const}>0$.

Proof. It is known that the following relation is valid for the Laplace transform of the function $\theta(t)$:

$$(a+i\xi)\,\widetilde{\theta}(a+i\xi) = \int_0^\infty e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

and for $a \to 0$ we have

$$i\xi\,\widetilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\,\theta'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\,\theta''(t)\,dt.$$

Then we have

$$\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi \leqslant C_5\,\|\theta\|^2_{W_2^2(R_+)}. \tag{3.10}$$

Consequently, according to (3.10) we get the following estimate

$$\begin{aligned}\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi &= \int_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}} \\ &\leqslant \left(\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi\right)^{1/2}\left(\int_{-\infty}^{+\infty} \frac{1}{1+\xi^2}d\xi\right)^{1/2} \leqslant C_4\,\|\theta\|_{W_2^2(R_+)}.\end{aligned}$$

□

Proof of the Theorem 2.2. We prove that $\mu \in W_2^1(\mathbb{R}_+)$. Indeed, according to (3.8) and (3.9), we obtain

$$\int_{-\infty}^{+\infty} |\widetilde{\mu}(\xi)|^2(1+|\xi|^2)\,d\xi = \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\theta}(i\xi)}{\widetilde{B}(i\xi)}\right|^2 (1+|\xi|^2)\,d\xi \;\leqslant C\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+|\xi|^2)^2\,d\xi = C\|\theta\|^2_{W_2^2(\mathbb{R})}.$$

Further,

$$|\mu(t)-\mu(s)| = \left|\int_s^t \mu'(\tau)\,d\tau\right| \leqslant \|\mu'\|_{L_2}\sqrt{t-s}.$$

Hence, $\mu \in \mathrm{Lip}\,\zeta$, where $\zeta = 1/2$. Then from (3.8), (3.9), and Lemma 3.3, we can write

$$|\mu(t)| \leqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|}{|\widetilde{B}(i\xi)|}d\xi \leqslant \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi \leqslant \frac{C_4}{2\pi C_0}\|\theta\|_{W_2^2(R_+)} \leqslant \frac{C_4\,M}{2\pi C_0} = 1,$$

as M we took as

$$M = \frac{2\pi C_0}{C_4}.$$

It remains to verify the fulfillment of condition $\mu(0) = 0$. For that we rewrite Eq. (2.11) as follows:

$$\int_0^t B(s)\,\mu(t-s)ds = \theta(t).$$

By differentiating this equation, we have

$$B(t)\,\mu(0) + \int_0^t B(s)\,\mu'(t-s)ds = \theta'(t).$$

Let us tend $t \to 0$ in this correlation. Then, taking into account the conditions imposed on the function θ, and the unboundedness of the function $B(t)$ in the neighborhood of zero, we obtain the desired equality $\mu(0) = 0$. □

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