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Taylor-Maclaurin coefficients and the Fekete-Szegö inequalities for certain subclasses of bi-univalent functions involving the Gegenbauer polynomials

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Abstract

In this paper by using the idea of Gegenbauer polynomials, we introduced certain new subclasses of analytic and bi-univalent functions. Additionally, we determined the estimates for first two Taylor-Maclaurin coefficients and the Fekete-Szegö functional problems for each of the function classes we defined. In the concluding part, we recall the curious readers attention to the possibility of analyzing the result's q-generalizations presented in this article. Moreover, according to the proposed extension, the $(\mathfrak{p}, \mathfrak{q})$-extension will only be comparatively small and inconsequently change, as the additional parameter $\mathfrak{p}$ is redundant.

Keywords: Analytic function, bi-univalent function, Gegenbauer polynomials, coefficient estimates, subordination, Fekete-Szegö functional problems.

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1. Introduction and motivation

Let $\mathfrak{U} = \{z : z \in \mathfrak{C}, |z| < 1\}$, be a unit disk and $\mathfrak{A}$ be the class of analytical functions of the form

$$f(z) = z + \sum_{r=2}^{\infty} b_r z^r, \quad (z \in \mathfrak{U}), \tag{1.1}$$

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normalized by the condition

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Consider a class, $\mathcal{S} \subset \mathfrak{A}$ of holomorphic and univalent functions in $\mathfrak{U}$. Let $\mathcal{S}^*$ stand for the class of starlike functions in $\mathfrak{U}$, which consists of normalized functions $f \in \mathfrak{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad (\forall\, z \in \mathfrak{U}),$$

and let by $\mathcal{C}$, we identify the class of convex functions in $\mathfrak{U}$ that meet the inequality by having normalized functions $f \in \mathfrak{A}$,

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \quad (\forall\, z \in \mathfrak{U}).$$

Lewin [18] introduced this class of bi-univalent functions as a sub-class of $\mathfrak{A}$ and noted certain coefficient bounds for the class. He proved that: $|n_2| \leqslant 1.15$. Moreover, the Koebe 1/4 theorem (see [9]) specifies that the disk $d_\omega = \{\omega : |\omega| < 0.25\}$ is contained in every function's range $f \in \mathcal{S}$, hence, $\forall f \in \mathcal{S}$ with its inverse f^{-1}, such that

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{U})$$

and

$$f(f^{-1}(\omega)) = \omega, \quad (\omega : |\omega| < r_0(f); r_0(f) \geqslant 0.25)$$

where $f^{-1}(\omega)$ is expressed as

$$G(\omega) = \omega - b_2\omega^2 + (2b_2^2 - b_3)\omega^3 - (5b_2^3 - 5b_2b_3 + b_4)\omega^4 + \cdots. \tag{1.2}$$

So, the function $f \in \mathfrak{A}$ is said to be bi-univalent in $\mathfrak{U}$ if $f(z)$ and $G(z)$ are univalent in $\mathfrak{U}$. Let Σ stand for the class of holomorphic and bi-univalent functions in $\mathfrak{U}$. We are aware, some well-known functions $f \in \mathcal{S}$ like the Koebe function

$$\kappa(z) = z/(1-z)^2,$$

its rotation function

$$\kappa_\sigma(z) = z/(1-e^{i\sigma}z)^2, \quad f(z) = z - z^2/2,$$

and

$$f(z) = z/(1-z^2),$$

don't belong to Σ. For more details see [1, 2, 6–8, 12, 13, 29].

The groundbreaking research of Srivastava et al. [27] in fact, in recent years, revitalized the study of bi-univalent functions. Following the study of Srivastava et al. [27], numerous unique subclasses of the bi-univalent function class were presented and similarly explored by numerous authors. The function classes $H_\Sigma(\gamma, \varepsilon, \mu.\sigma; \alpha)$ and $H_\Sigma(\gamma, \varepsilon, \mu.\sigma; \beta)$ as an illustration, were defined and Srivastava et al. [25] produced estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Caglar et al. [23] were able to determine the upper bounds for the second Hankel determinant for specific subclasses of analytic and bi-univalent functions. By Tang et al. [24] and Srivastava et al. [26] several new subclasses of the class of m-fold symmetric bi-univalent functions were introduced, and the initial estimates of the Taylor-Maclaurin series as well as some Fekete-Szegö functional problems for each of their defined function classes were obtained. Several more prominent mathematicians provided their research on this topic see for example [5, 14–16].

From [9], let $s(z)$ and $S(z)$ belongs to class $\mathfrak{A}$, then

$$s(z) \prec S(z) \quad (z \in \mathfrak{U}),$$

suppose ω holomorphic in $\mathfrak{U}$, such that

$$\omega(0) = 0, \quad |\omega(z)| < 1, \quad \text{and} \quad s(z) = S(\omega(z)).$$

Consequently, if the function $S(z)$ is univalent in $\mathfrak{U}$,

$$s(z) \prec S(z) \Rightarrow s(0) = S(0) \quad \text{and} \quad s(\mathfrak{U}) \subset S(\mathfrak{U}).$$

This conclusion is known as the subordination principle.

Amourah et al. [4] have lately studied the Gegenbauer polynomials $\mathcal{H}_\phi(t,z)$, which are determined by the recurrence relation. A generating function of Gegenbauer polynomials is defined by for nonzero real constant ϕ,

$$\mathcal{H}_\phi(t,z) = \frac{1}{(1-2tz+z^2)^\phi},$$

where $-1 \leqq t \leqq 1$ and $z \in \mathfrak{U}$. Applying Taylor series expansion, the holomorphic function $\mathcal{H}_\phi$ can be express in the following form

$$\mathcal{H}_\phi(t,z) = \sum_{r=0}^{\infty} \mathfrak{G}_r^\phi(t) z^r,$$

where t is fixed and $\mathfrak{G}_r^\phi(t)$ is Gegenbauer polynomials of degree r. When $\phi = 0$, $\mathcal{H}_\phi$ obviously produces nothing. As a result, the Gegenbauer polynomial's generating function is set to

$$\mathfrak{G}_r^\phi(t) = \frac{1}{r}\left\{2t(r+\phi-1)\mathfrak{G}_{r-1}^\phi(t) - (r+2\phi-2)\mathfrak{G}_{r-1}^\phi(t)\right\},$$

using the starting values

$$\mathfrak{G}_0^\phi(t) = 1, \;\; \mathfrak{G}_1^\phi(t) = 2\phi t, \;\; \text{and} \;\; \mathfrak{G}_2^\phi(t) = 2\phi(1+\phi)t^2 - \phi. \tag{1.3}$$

Remark 1.1. First of all, if in polynomial $\mathfrak{G}_r^\phi(t)$, we put $\phi = 1$, then we have the Chebyshev polynomial. Secondly, for $\phi = \frac{1}{2}$, polynomials $\mathfrak{G}_r^\phi(t)$, we have the Legendre polynomial.

In recent years, many researchers have been studying how orthogonal polynomials and bi-univalent functions interact including for example in [11, 19, 30] the second derivative sequences of Fibonacci and Lucas polynomials have been studied. Also in [17, 20] some properties of the (p,q)-Fibonacci and (p,q)-Lucas polynomials have been studied. On the other hand, in [3, 28], the classes of Lucas-Lehmer polynomials have been introduced. Since, there is little work in the literature's related to bi-univalent functions for the Gegenbauer polynomial. The primary goal of this study is to launch an investigation into the properties of bi-univalent functions linked with Gegenbauer polynomial.

2. Coefficient bounds and Fekete-Szegö inequalities for the class $\mathfrak{S}_\Sigma(\delta, t, \phi)$

Definition 2.1. Let $0 \leqq \delta \leqq$, $\frac{1}{2} < t \leqq 1$. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{S}_\Sigma(\delta, t, \phi)$ if the following subordinations are fulfilled:

$$\left(\frac{zf'(z)}{f(z)}\right)^\delta \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\delta} \prec \mathcal{H}_\phi(t,z) = \frac{1}{(1-2tz+z^2)^\phi} \tag{2.1}$$

and

$$\left(\frac{zG'(\omega)}{G(\omega)}\right)^\delta \left(1 + \frac{\omega G''(\omega)}{G'(\omega)}\right)^{1-\delta} \prec \mathcal{H}_\phi(t,z) = \frac{1}{(1-2t\omega+\omega^2)^\phi}, \tag{2.2}$$

where the function $G(\omega)$ is defined by (1.2) and $0 \neq \phi$ is a real constant.

The initial Taylor coefficients $|b_2|$ and $|b_3|$ and the Fekete-Szegö inequality for the function class $\mathfrak{S}_\Sigma(\delta, t, \phi)$ are determined by the following theorem.

Theorem 2.2. *Let* $f \in \mathfrak{S}_\Sigma(\delta, t, \phi)$. *Then*

$$|b_2| \leqslant 2|\phi|t\sqrt{\frac{2\phi t}{2\phi^2t^2(\delta^2-3\delta+4)-(2-\delta)^2\phi(2(1+\phi)t^2-1)}}, \qquad |b_3| \leqslant \frac{4\phi^2t^2}{(2-\delta)^2}+\frac{\phi t}{3-2\delta},$$

and for $\chi \in \mathcal{R}$,

$$\left|d_3-\chi d_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|3-2\delta|}, & |\chi-1| \leqq |D|, \\ \frac{8\phi^3t^3|1-\chi|}{2\phi^2x^2(\delta^2-3\delta+4)-(2-\delta)^2\phi(2(1+\phi)t^2-1)}, & |\chi-1| \geqq |D|, \end{cases}$$

where

$$D = \frac{2\phi x^2(\delta^2-3\delta+4)-(2-\delta)^2(2(1+\phi)t^2-1)}{8\phi t^2(3-2\delta)}.$$

Proof. Let $f \in \mathfrak{S}_\Sigma(\delta, t, \phi)$. From (2.1) and (2.2), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\delta} = 1+\mathfrak{G}_1^{\phi}(t)s_1z+[\mathfrak{G}_1^{\phi}(t)s_2+\mathfrak{G}_2^{\phi}(t)s_1^2]z^2+\cdots \tag{2.3}$$

and

$$\left(\frac{zG'(\omega)}{G(\omega)}\right)^{\delta}\left(1+\frac{\omega G''(\omega)}{G'(\omega)}\right)^{1-\delta} = 1+\mathfrak{G}_1^{\phi}(t)l_1\omega+[\mathfrak{G}_1^{\phi}(t)l_2+\mathfrak{G}_2^{\phi}(t)l_1^2]\omega^2+\cdots \tag{2.4}$$

for some holomorphic functions

$$u(z) = s_1z+s_2z^2+s_3z^3+\cdots, \qquad v(\omega) = l_1\omega+l_2\omega^2+l_3\omega^3+\cdots,$$

such that

$$u(0) = v(0) = 0, \quad |s(z)| < 1, \quad \text{and} \quad |v(\omega)| < 1 \quad (z, \omega \in \mathfrak{U}).$$

Therefore, we have

$$|s_k| \leqq 1 \quad \text{and} \quad |l_k| \leqq 1.$$

When the equivalent coefficients in (2.3) and (2.4) are compared, we get

$$(2-\delta)b_2 = \mathfrak{G}_1^{\phi}(t)s_1, \tag{2.5}$$

$$2(3-2\delta)b_3+(\delta^2+5\delta-8)\frac{b_2^2}{2} = \mathfrak{G}_1^{\phi}(t)s_2+\mathfrak{G}_2^{\phi}(t)s_1^2, \tag{2.6}$$

$$-(2-\delta)b_2 = \mathfrak{G}_1^{\phi}(t)l_1, \tag{2.7}$$

$$(\delta^2-11\delta+16)\frac{b_2^2}{2}-2(3-2\delta)b_3 = \mathfrak{G}_1^{\phi}(t)l_2+\mathfrak{G}_2^{\phi}(t)l_1^2. \tag{2.8}$$

From (2.5) and (2.7), we have

$$s_1 = -l_1,$$

$$b_2^2 = \frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(2-\delta)^2}, \qquad s_1^2+l_1^2 = \frac{2(2-\delta)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}. \tag{2.9}$$

Summation of (2.6) and (2.8) gives

$$(\delta^2-3\delta+4)b_2^2 = \mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\mathfrak{G}_2^{\phi}(t)(s_1^2+l_1^2) = \mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\mathfrak{G}_2^{\phi}(t)\left[\frac{2(2-\delta)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}\right]. \tag{2.10}$$

Applying (2.9) in (2.10), yields

$$[(\delta^2-3\delta+4)[\mathfrak{G}_1^{\phi}(t)]^2-2(2-\delta)^2\mathfrak{G}_2^{\phi}(t)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2) \tag{2.11}$$

and

$$[4\phi^2x^2(\delta^2-3\delta+4)-2(2-\delta)^2\phi(2(1+\phi)t^2-1)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2),$$

which gives

$$|b_2|\leqslant 2|\phi|t\sqrt{\frac{2\phi t}{2\phi^2t^2(\delta^2-3\delta+4)-(2-\delta)^2\phi(2(1+\phi)t^2-1)}}.$$

Hence, (2.8) minus (2.6) gives us

$$4(3-2\delta)b_3-4(3-2\delta)b_2^2=\mathfrak{G}_1^{\phi}(t)(s_2-l_2)+\mathfrak{G}_2^{\phi}(t)(s_1^2-l_1^2). \tag{2.12}$$

Then, using (1.3), (2.9), and (2.12), we get

$$b_3=b_2^2+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\delta)}=\frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(2-\delta)^2}+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\delta)}. \tag{2.13}$$

Applying (1.3), yields

$$|b_3|\leqslant\frac{4\phi^2t^2}{(2-\delta)^2}+\frac{\phi t}{3-2\delta}.$$

From (2.13), for $\chi\in\mathcal{R}$, we have

$$b_3-\chi b_2^2=(1-\chi)b_2^2+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\delta)}. \tag{2.14}$$

By substituting (2.11) in (2.14), we have

$$\begin{aligned}b_3-\chi b_2^2&=\frac{(1-\chi)[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2)}{(\delta^2-3\delta+4)[\mathfrak{G}_1^{\phi}(t)]^2-2(2-\delta)^2\mathfrak{G}_2^{\phi}(t)}+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\delta)}\\&=\mathfrak{G}_1^{\phi}(t)\left\{\left(G(\chi)+\frac{1}{4(3-2\delta)}\right)s_2+\left(G(\chi)-\frac{1}{4(3-2\delta)}\right)l_2\right\},\end{aligned}$$

where

$$G(\chi)=\frac{(1-\chi)[\mathfrak{G}_1^{\phi}(t)]^2}{(\delta^2-3\delta+4)[\mathfrak{G}_1^{\phi}(t)]^2-2(2-\delta)^2\mathfrak{G}_2^{\phi}(t)}.$$

Thus, according to (1.3), we have

$$\left|b_3-\chi b_2^2\right|\leqslant\begin{cases}\frac{|\mathfrak{G}_1^{\phi}(t)|}{2(3-2\delta)}, & 0\leqq|G(\chi)|\leqq\frac{1}{4(3-2\delta)},\\ 2|G(\chi)||\mathfrak{G}_1^{\phi}(t)|, & |G(\chi)|\geqq\frac{1}{4(3-2\delta)},\end{cases}$$

hence, after some calculations, gives

$$\left|b_3-\chi b_2^2\right|\leqslant\begin{cases}\frac{|\phi|t}{|3-2\delta|}, & |\chi-1|\leqq|D|,\\ \frac{8\phi^3t^3|1-\chi|}{2\phi^2x^2(\delta^2-3\delta+4)-(2-\delta)^2\phi(2(1+\phi)t^2-1)}, & |\chi-1|\geqq|D|.\end{cases}$$

□

3. Coefficient bounds and Fekete-Szegö inequalities for the class $\mathfrak{M}_\Sigma(\varphi, t, \phi)$

Definition 3.1. Let $\varphi \in [0, 1]$, $1/2 < t \leqq 1$. A function $f \in \mathfrak{M}_\Sigma(\varphi, t, \phi)$, if the following subordinations are fulfilled:

$$\varphi\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\varphi)\frac{zf'(z)}{f(z)} \prec \mathcal{H}_\phi(t, z) = \frac{1}{(1 - 2tz + z^2)^\phi} \tag{3.1}$$

and

$$\varphi\left(1 + \frac{\omega G''(z)}{G'(z)}\right) + (1-\varphi)\frac{\omega G'(z)}{G(z)} \prec \mathcal{H}_\phi(t, z) = \frac{1}{(1 - 2t\omega + \omega^2)^\phi}, \tag{3.2}$$

where the function $G(\omega)$ is defined by (1.2) and $\phi \neq 0$ is a real constant.

The initial Taylor coefficients $|b_2|$ and $|b_3|$ and Fekete-Szegö inequality for the function class $\mathfrak{M}_\Sigma(\varphi, t, \phi)$ are determined by the following theorem.

Theorem 3.2. *Let $f \in \mathfrak{M}_\Sigma(\varphi, t, \phi)$. Then*

$$|b_2| \leqslant 2|\phi|t\sqrt{\frac{2\phi t}{|4\phi^2t^2(1+\varphi) - (1+\varphi)^2\phi(2(1+\phi)t^2 - 1)|}}, \qquad |b_3| \leqslant \frac{4\phi^2t^2}{(1+\varphi)^2} + \frac{\phi t}{1 + 2\varphi},$$

and for $\vartheta \in \mathcal{R}$

$$\left|b_3 - \vartheta b_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|1+2\varphi|}, & |\vartheta - 1| \leqq |N|, \\ \frac{8\phi^3t^3|1-\vartheta|}{4\phi^2t^2(1+\varphi)-(1+\varphi)^2\phi(2(1+\phi)t^2-1)}, & |\vartheta - 1| \geqq |N|, \end{cases}$$

where

$$N = \frac{4\phi t^2(1+\varphi) - (1+\varphi)^2(2(1+\phi)t^2 - 1)}{8\phi t^2(1+2\varphi)}.$$

Proof. Let $f \in \mathfrak{M}_\Sigma(\varphi, t, \phi)$. From (3.1) and (3.2), we have

$$\varphi\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\varphi)\frac{zf'(z)}{f(z)} = 1 + \mathfrak{G}_1^\phi(t)s_1z + [\mathfrak{G}_1^\phi(t)s_2 + \mathfrak{G}_2^\phi(t)s_1^2]z^2 + \cdots \tag{3.3}$$

and

$$\varphi\left(1 + \frac{\omega G''(z)}{G'(z)}\right) + (1-\varphi)\frac{\omega G'(z)}{G(z)} = 1 + \mathfrak{G}_1^\phi(t)l_1\omega + [\mathfrak{G}_1^\phi(t)l_2 + \mathfrak{G}_2^\phi(t)l_1^2]\omega^2 + \cdots \tag{3.4}$$

for some holomorphic functions

$$u(z) = s_1z + s_2z^2 + s_3z^3 + \cdots, \qquad v(\omega) = l_1\omega + l_2\omega^2 + l_3\omega^3 + \cdots,$$

such that

$$u(0) = v(0) = 0$$

and

$$|s(z)| < 1 \text{ and } |v(\omega)| < 1 \quad (z, \omega \in \mathfrak{U}).$$

Therefore, we have

$$|s_k| \leqq 1 \text{ and } |l_k| \leqq 1 \quad (\forall k \in \mathfrak{N}).$$

When the equivalent coefficients in (3.3) and (3.4) are compared, we get

$$(1+\varphi)b_2 = \mathfrak{G}_1^\phi(t)s_1, \tag{3.5}$$

$$2(1+2\varphi)b_3 - (1+3\varphi)b_2^2 = \mathfrak{G}_1^\phi(t)s_2 + \mathfrak{G}_2^\phi(t)s_1^2, \tag{3.6}$$

$$-(1+\varphi)b_2 = \mathfrak{G}_1^\phi(t)l_1, \tag{3.7}$$

$$(3+5\varphi)b_2^2-2(1+2\varphi)b_3=\mathfrak{G}_1^{\phi}(t)l_2+\mathfrak{G}_2^{\phi}(t)l_1^2. \tag{3.8}$$

From (3.5) and (3.7),

$$\begin{aligned} &s_1=-l_1,\\ &b_2^2=\frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(1+\varphi)^2}, \qquad s_1^2+l_1^2=\frac{2(1+\varphi)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}. \end{aligned} \tag{3.9}$$

Summation of (3.6) and (3.8) gives

$$2(1+\varphi)b_2^2=\mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\ \mathfrak{G}_2^{\phi}(t)(s_1^2+l_1^2)=\mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\mathfrak{G}_2^{\phi}(t)\left[\frac{2(1+\varphi)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}\right]. \tag{3.10}$$

Applying (3.9) in (3.10), yields

$$[2(1+\varphi)[\mathfrak{G}_1^{\phi}(t)]^2-2(1+\varphi)^2\mathfrak{G}_2^{\phi}(t)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2) \tag{3.11}$$

and

$$[8\phi^2x^2(1+\varphi)-2(1+\varphi)^2\phi(2(1+\phi)t^2-1)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2),$$

which gives

$$|b_2|\leqslant 2|\phi|t\sqrt{\frac{2\phi t}{|4\phi^2t^2(1+\varphi)-(1+\varphi)^2\phi(2(1+\phi)t^2-1)|}}.$$

Hence, (3.8) minus (3.6) gives us

$$4(1+2\varphi)b_3-4(1+2\varphi)b_2^2=\mathfrak{G}_1^{\phi}(t)(s_2-l_2)+\mathfrak{G}_2^{\phi}(t)(s_1^2-l_1^2).$$

Then, using (1.3) and (3.9), we get

$$b_3=b_2^2+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+2\varphi)}, \qquad b_3=\frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(1+\varphi)^2}+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+2\varphi)}. \tag{3.12}$$

Applying (1.3), yields

$$|b_3|\leqslant\frac{4\phi^2t^2}{(1+\varphi)^2}+\frac{\phi t}{1+2\varphi}.$$

From (3.12), for $\vartheta\in\mathcal{R}$, we have

$$b_3-\vartheta b_2^2=(1-\vartheta)b_2^2+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+2\varphi)}. \tag{3.13}$$

By substituting (3.11) in (3.13), we have

$$\begin{aligned} b_3-\vartheta b_2^2&=\frac{(1-\chi)[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2)}{2(1+\varphi)[\mathfrak{G}_1^{\phi}(t)]^2-2(1+\varphi)^2\mathfrak{G}_2^{\phi}(t)}+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+2\varphi)}\\ &=\mathfrak{G}_1^{\phi}(t)\left\{\left(G(\vartheta)+\frac{1}{4(1+2\varphi)}\right)s_2+\left(G(\vartheta)-\frac{1}{4(1+2\varphi)}\right)l_2\right\}, \end{aligned}$$

where

$$G(\vartheta)=\frac{(1-\chi)[\mathfrak{G}_1^{\phi}(t)]^2}{2(1+\varphi)[\mathfrak{G}_1^{\phi}(t)]^2-2(1+\varphi)^2\mathfrak{G}_2^{\phi}(t)}.$$

Thus, according to (1.3), we have

$$\left|b_3 - \vartheta b_2^2\right| \leqslant \begin{cases} \frac{|\mathfrak{G}_1^{\phi}(t)|}{2(1+2\varphi)}, & 0 \leqq |G(\vartheta)| \leqq \frac{1}{4(1+2\varphi)}, \\ 2|G(\vartheta)||\mathfrak{G}_1^{\phi}(t)|, & |G(\chi)| \geqq \frac{1}{4(1-2\delta)}, \end{cases}$$

hence, after some calculations, gives

$$\left|b_3 - \vartheta b_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|1+2\varphi|}, & |\vartheta - 1| \leqq |N|, \\ \frac{8\phi^3 t^3 |1-\vartheta|}{4\phi^2 t^2(1+\varphi)-(1+\varphi)^2\phi(2(1+\phi)t^2-1)}, & |\vartheta - 1| \geqq |N|. \end{cases}$$ □

4. Coefficient bounds and Fekete-Szegö inequalities for the class $\mathfrak{H}_\Sigma(\psi, t, \phi)$

Definition 4.1. Let $\psi \geqq 0$, $1/2 < t \leqq 1$. A function $f \in \mathfrak{H}_\Sigma(\psi, t, \phi)$, if the following subordinations are fulfilled:

$$\psi \frac{z^2 f''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \prec \mathcal{H}_\phi(t, z) = \frac{1}{(1 - 2tz + z^2)^\phi}, \tag{4.1}$$

and

$$\psi \frac{\omega^2 G''(z)}{G'(z)} + \frac{\omega G'(z)}{G(z)} \prec \mathcal{H}_\phi(t, z) = \frac{1}{(1 - 2t\omega + \omega^2)^\phi}, \tag{4.2}$$

where the function $G(\omega)$ is defined by (1.2) and $\phi \neq 0$ is a real constant.

The initial Taylor coefficients $|b_2|$ and $|b_3|$ and Fekete-Szegö inequality for the function class $\mathfrak{H}_\Sigma(\psi, t, \phi)$ are determined by the following theorem.

Theorem 4.2. *Let $f \in \mathfrak{M}_\Sigma(\varphi, t, \phi)$. Then*

$$|b_2| \leqslant 2|\phi|t\sqrt{\frac{2\phi t}{|4\phi^2 t^2(1+4\psi) - (1+2\psi)^2\phi(2(1+\phi)t^2 - 1)|}}, \quad |b_3| \leqslant \frac{4\phi^2 t^2}{(1+2\psi)^2} + \frac{|\phi t|}{1+3\psi},$$

and for $\psi \in \mathcal{R}$,

$$\left|b_3 - \zeta b_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|1+3\psi|}, & |\zeta - 1| \leqq |R|, \\ \frac{8\phi^3 t^3 |1-\zeta|}{4\phi^2 t^2(1+4\psi)-(1+2\psi)^2\phi(2(1+\phi)t^2-1)}, & |\zeta - 1| \geqq |R|, \end{cases}$$

where

$$R = \frac{4\phi t^2(1+4\psi) - (1+2\psi)^2(2(1+\phi)t^2 - 1)}{8\phi t^2(1+3\psi)}.$$

Proof. Let $f \in \mathfrak{H}_\Sigma(\psi, t, \phi)$. From (4.1) and (4.2), we have

$$\psi \frac{z^2 f''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} = 1 + \mathfrak{G}_1^{\phi}(t)s_1 z + [\mathfrak{G}_1^{\phi}(t)s_2 + \mathfrak{G}_2^{\phi}(t)s_1^2]z^2 + \cdots$$

and

$$\psi \frac{\omega^2 G''(z)}{G'(z)} + \frac{\omega G'(z)}{G(z)} = 1 + \mathfrak{G}_1^{\phi}(t)l_1\omega + [\mathfrak{G}_1^{\phi}(t)l_2 + \mathfrak{G}_2^{\phi}(t)l_1^2]\omega^2 + \cdots$$

for some holomorphic functions

$$u(z) = s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \qquad v(\omega) = l_1\omega + l_2\omega^2 + l_3\omega^3 + \cdots,$$

such that

$$u(0) = v(0) = 0, \quad |s(z)| < 1, \quad \text{and} \quad |v(\omega)| < 1 \quad (z, \omega \in \mathfrak{U}).$$

Therefore, we have

$$|s_k| \leqq 1 \text{ and } |l_k| \leqq 1, \text{ for all } k \in \mathfrak{N}.$$

When the equivalent coefficients in (3.3) and (3.4) are compared, we get

$$(1+2\psi)b_2 = \mathfrak{G}_1^{\phi}(t)s_1, \tag{4.3}$$

$$2(1+3\psi)b_3 - (1+2\psi)b_2^2 = \mathfrak{G}_1^{\phi}(t)s_2 + \mathfrak{G}_2^{\phi}(t)s_1^2, \tag{4.4}$$

$$-(1+2\psi)b_2 = \mathfrak{G}_1^{\phi}(t)l_1, \tag{4.5}$$

$$(3+10\psi)b_2^2 - 2(1+3\psi)b_3 = \mathfrak{G}_1^{\phi}(t)l_2 + \mathfrak{G}_2^{\phi}(t)l_1^2. \tag{4.6}$$

From (4.3) and (4.5),

$$s_1 = -l_1,$$

$$b_2^2 = \frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(1+2\psi)^2}, \qquad s_1^2+l_1^2 = \frac{2(1+2\psi)^2 b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}. \tag{4.7}$$

Summation of (4.4) and (4.6) gives

$$2(1+4\psi)b_2^2 = \mathfrak{G}_1^{\phi}(t)(s_2+l_2) + \mathfrak{G}_2^{\phi}(t)(s_1^2+l_1^2) = \mathfrak{G}_1^{\phi}(t)(s_2+l_2) + \mathfrak{G}_2^{\phi}(t)\left[\frac{2(1+2\psi)^2 b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}\right]. \tag{4.8}$$

Applying (4.7) in (4.8), yields

$$[2(1+4\psi)[\mathfrak{G}_1^{\phi}(t)]^2 - 2(1+2\psi)^2\mathfrak{G}_2^{\phi}(t)]b_2^2 = [\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2) \tag{4.9}$$

and

$$[8\phi^2x^2(1+4\psi) - 2(1+2\psi)^2\phi(2(1+\phi)t^2-1)]b_2^2 = [\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2),$$

which gives

$$|b_2| \leqslant 2|\phi|t\sqrt{\frac{2\phi t}{|4\phi^2t^2(1+4\psi) - (1+2\psi)^2\phi(2(1+\phi)t^2-1)|}}.$$

Hence, (4.6) minus (4.4) gives us

$$4(1+3\psi)b_3 - 4(1+3\psi)b_2^2 = \mathfrak{G}_1^{\phi}(t)(s_2-l_2) + \mathfrak{G}_2^{\phi}(t)(s_1^2-l_1^2). \tag{4.10}$$

Then, using (1.3) and (4.7), we get

$$b_3 = b_2^2 + \frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+3\psi)} = \frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(1+2\psi)^2} + \frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+3\psi)}.$$

Applying (1.3), yields

$$|b_3| \leqslant \frac{4\phi^2t^2}{(1+2\psi)^2} + \frac{|\phi t|}{1+3\psi}.$$

From (4.10), for $\zeta \in \mathfrak{R}$, we have

$$b_3 - \zeta b_2^2 = (1-\zeta)b_2^2 + \frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+3\psi)}. \tag{4.11}$$

By substituting (4.9) in (4.11), we have

$$b_3 - \zeta b_2^2 = \frac{(1-\zeta)[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2)}{2(1+4\psi)[\mathfrak{G}_1^{\phi}(t)]^2 - 2(1+2\psi)^2\mathfrak{G}_2^{\phi}(t)} + \frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(1+3\psi)}$$

$$= \mathfrak{G}_1^{\phi}(t)\left\{\left(G(\zeta)+\frac{1}{4(1+3\psi)}\right)s_2+\left(G(\zeta)-\frac{1}{4(1+3\psi)}\right)l_2\right\},$$

where

$$G(\zeta)=\frac{(1-\chi)[\mathfrak{G}_1^{\phi}(t)]^2}{2(1+4\psi)[\mathfrak{G}_1^{\phi}(t)]^2-2(1+2\psi)^2\mathfrak{G}_2^{\phi}(t)}.$$

Thus, according to (1.3), we have

$$\left|b_3-\zeta b_2^2\right| \leqslant \begin{cases} \frac{|\mathfrak{G}_1^{\phi}(t)|}{2(1+3\psi)}, & 0 \leqq |G(\zeta)| \leqq \frac{1}{4(1+3\psi)}, \\ 2|G(\zeta)||\mathfrak{G}_1^{\phi}(t)|, & |G(\chi)| \geqq \frac{1}{4(1-3\psi)}, \end{cases}$$

hence, after some calculations, gives

$$\left|b_3-\zeta b_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|1+3\psi|}, & |\zeta-1| \leqq |R|, \\ \frac{8\phi^3t^3|1-\zeta|}{4\phi^2t^2(1+4\psi)-(1+2\psi)^2\phi(2(1+\phi)t^2-1)}, & |\zeta-1| \geqq |R|. \end{cases}$$

□

5. Coefficient bounds and Fekete-Szegö inequalities for the class $\mathfrak{BO}_\Sigma(\beta,t,\phi)$

Definition 5.1. Let $\beta \in [0,1]$, $1/2 < t \leqq 1$. A function $f \in \mathfrak{BO}_\Sigma(\beta,t,\phi)$, if the following subordinations are fulfilled:

$$\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}-\frac{\beta z^2f''(z)+zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}+1 \prec \mathcal{H}_\phi(t,z)=\frac{1}{(1-2tz+z^2)^\phi} \tag{5.1}$$

and

$$\frac{G''(\omega)}{G'(\omega)}+\frac{\omega G'(\omega)}{G(z)}-\frac{\beta\omega^2G''(\omega)+\omega G'(\omega)}{\beta\omega G'(\omega)+(1-\beta)G(\omega)}+1 \prec \mathcal{H}_\phi(t,z)=\frac{1}{(1-2t\omega+\omega^2)^\phi}, \tag{5.2}$$

where the function $G(\omega)$ is defined by (1.2) and $0 \neq \phi$ is a real constant.

The initial Taylor coefficients $|b_2|$ and $|b_3|$ and Fekete-Szegö inequality for the function class $\mathfrak{BO}_\Sigma(\beta,t,\phi)$ are determined by the following theorem.

Theorem 5.2. *Let* $f \in \mathfrak{BO}_\Sigma(\beta,t,\phi)$. *Then*

$$|b_2| \leqslant 2|\phi|t\sqrt{\frac{2\phi t}{4\phi^2t^2(1+(\beta-1)^2)-(2-\beta)^2\phi(2(1+\phi)t^2-1)}}, \quad |b_3| \leqslant \frac{4\phi^2t^2}{(2-\beta)^2}+\frac{\phi t}{3-2\beta},$$

and for $\eta \in \mathcal{R}$,

$$\left|b_3-\eta b_2^2\right| \leqslant \begin{cases} \frac{|\phi|t}{|3-2\beta|}, & |\eta-1| \leqq |W|, \\ \frac{8\phi^3t^3|1-\eta|}{4\phi^2x^2(1+(\beta-1)^2)-(2-\beta)^2\phi(2(1+\phi)t^2-1)}, & |\eta-1| \geqq |W|, \end{cases}$$

where

$$W=\frac{4\phi x^2(1+(\beta-1)^2)-(2-\beta)^2(2(1+\phi)t^2-1)}{8\phi t^2(3-2\beta)}.$$

Proof. Let $f \in \mathfrak{S}_\Sigma(\delta,t,\phi)$. From (5.1) and (5.2), we have

$$\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}-\frac{\beta z^2f''(z)+zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}+1=1+\mathfrak{G}_1^{\phi}(t)s_1z+[\mathfrak{G}_1^{\phi}(t)s_2+\mathfrak{G}_2^{\phi}(t)s_1^2]z^2+\cdots \tag{5.3}$$

and

$$\frac{G''(\omega)}{G'(\omega)}+\frac{\omega G'(\omega)}{G(z)}-\frac{\beta\omega^2G''(\omega)+\omega G'(\omega)}{\beta\omega G'(\omega)+(1-\beta)G(\omega)}+1=1+\mathfrak{G}_1^{\phi}(t)l_1\omega+[\mathfrak{G}_1^{\phi}(t)l_2+\mathfrak{G}_2^{\phi}(t)l_1^2]\omega^2+\cdots \tag{5.4}$$

for some holomorphic functions

$$u(z)=s_1z+s_2z^2+s_3z^3+\cdots,\quad v(\omega)=l_1\omega+l_2\omega^2+l_3\omega^3+\cdots,$$

such that

$$u(0)=v(0)=0,\ \ |s(z)|<1,\ \text{ and }\ |v(\omega)|<1\ \ (z,\omega\in\mathfrak{U}).$$

Now therefore

$$|s_k|\leqq 1\ \text{ and }\ |l_k|\leqq 1\ \ (k\in\mathfrak{N}).$$

When the equivalent coefficients in (5.3) and (5.4) are compared, we get

$$(2-\beta)b_2=\mathfrak{G}_1^{\phi}(t)s_1, \tag{5.5}$$

$$2(3-2\beta)b_3+(5-(\beta+1)^2)b_2^2=\mathfrak{G}_1^{\phi}(t)s_2+\mathfrak{G}_2^{\phi}(t)s_1^2, \tag{5.6}$$

$$(\beta-2)b_2=\mathfrak{G}_1^{\phi}(t)l_1, \tag{5.7}$$

$$(7-8\beta+(1+\beta)^2)b_2^2-2(3-2\beta)b_3=\mathfrak{G}_1^{\phi}(t)l_2+\mathfrak{G}_2^{\phi}(t)l_1^2. \tag{5.8}$$

From (5.5) and (5.7)

$$\begin{aligned}&s_1=-l_1,\\&b_2^2=\frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(2-\beta)^2},\qquad s_1^2+l_1^2=\frac{2(2-\beta)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}.\end{aligned} \tag{5.9}$$

Summation of (5.6) and (5.8) gives

$$2(1+(\beta-1)^2)b_2^2=\mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\mathfrak{G}_2^{\phi}(t)(s_1^2+l_1^2)=\mathfrak{G}_1^{\phi}(t)(s_2+l_2)+\mathfrak{G}_2^{\phi}(t)\left[\frac{2(2-\beta)^2b_2^2}{[\mathfrak{G}_1^{\phi}(t)]^2}\right]. \tag{5.10}$$

Applying (5.9) in (5.10), yields

$$[(1+(\beta-1)^2)[\mathfrak{G}_1^{\phi}(t)]^2-2(2-\beta)^2\mathfrak{G}_2^{\phi}(t)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2), \tag{5.11}$$

$$[8\phi^2x^2(1+(\beta-1)^2)-2(2-\beta)^2\phi(2(1+\phi)t^2-1)]b_2^2=[\mathfrak{G}_1^{\phi}(t)]^3(s_2+l_2),$$

which gives

$$|b_2|\leqslant 2|\phi|t\sqrt{\frac{2\phi t}{4\phi^2t^2(1+(\beta-1)^2)-(2-\beta)^2\phi(2(1+\phi)t^2-1)}}.$$

Hence, (5.8) minus (5.6) gives us

$$4(3-2\beta)(b_3-b_2^2)=\mathfrak{G}_1^{\phi}(t)(s_2-l_2)+\mathfrak{G}_2^{\phi}(t)(s_1^2-l_1^2). \tag{5.12}$$

Then, using (1.3), (5.9), and (5.12), we get

$$b_3=b_2^2+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\beta)} \tag{5.13}$$

or

$$b_3=\frac{[\mathfrak{G}_1^{\phi}(t)]^2(s_1^2+l_1^2)}{2(2-\beta)^2}+\frac{\mathfrak{G}_1^{\phi}(t)(s_2-l_2)}{4(3-2\beta)}.$$

Applying (1.3), yields

$$|b_3| \leqslant \frac{4\phi^2 t^2}{(2-\beta)^2} + \frac{\phi t}{3-2\beta}.$$

From (5.13), for $\eta \in \mathcal{R}$, we have

$$b_3 - \eta b_2^2 = (1-\eta) b_2^2 + \frac{\mathfrak{G}_1^{\phi}(t)(s_2 - l_2)}{4(3-2\beta)}. \tag{5.14}$$

By substituting (5.11) in (5.14), we have

$$\begin{aligned} b_3 - \eta b_2^2 &= \frac{(1-\eta)[\mathfrak{G}_1^{\phi}(t)]^3 (s_2 + l_2)}{2(1+(\beta-1)^2)[\mathfrak{G}_1^{\phi}(t)]^2 - 2(2-\beta)^2 \mathfrak{G}_2^{\phi}(t)} + \frac{\mathfrak{G}_1^{\phi}(t)(s_2 - l_2)}{4(3-2\beta)} \\ &= \mathfrak{G}_1^{\phi}(t) \left\{ \left(G(\eta) + \frac{1}{4(3-2\beta)} \right) s_2 + \left(G(\eta) - \frac{1}{4(3-2\beta)} \right) l_2 \right\}, \end{aligned}$$

where

$$G(\eta) = \frac{(1-\eta)[\mathfrak{G}_1^{\phi}(t)]^2}{2(1+(\beta-1)^2)[\mathfrak{G}_1^{\phi}(t)]^2 - 2(2-\beta)^2 \mathfrak{G}_2^{\phi}(t)}.$$

Thus, according to (1.3), we have

$$\left| b_3 - \eta b_2^2 \right| \leqslant \begin{cases} \frac{|\mathfrak{G}_1^{\phi}(t)|}{2(3-2\beta)}, & 0 \leqq |G(\eta)| \leqq \frac{1}{4(3-2\beta)}, \\ 2|G(\eta)||\mathfrak{G}_1^{\phi}(t)|, & |G(\eta)| \geqq \frac{1}{4(3-2\beta)}, \end{cases}$$

hence, after some calculations, we have

$$\left| b_3 - \eta b_2^2 \right| \leqslant \begin{cases} \frac{|\phi| t}{|3-2\beta|}, & |\eta - 1| \leqq |W|, \\ \frac{8\phi^3 t^3 |1-\eta|}{4\phi^2 x^2 (1+(\beta-1)^2) - (2-\beta)^2 \phi (2(1+\phi)t^2 - 1)}, & |\eta - 1| \geqq |W|. \end{cases}$$

□

6. Conclusion

Recently, there are many researchers in the world, who have been investigating bi-univalent functions connecting with orthogonal polynomials. Since, there is not much research in the literature on bi-univalent functions for the Gegenbauer polynomial.

In the present work, we have first defined certain new subclasses of analytic and bi-univalent functions linked with Gegenbauer polynomial. Then, we have determined some useful results like estimation for first two Taylor-Maclaurin coefficients and the Fekete-Szegö functional problems for every one of our defined function classes.

Moreover, we draw the attention of the interested readers to the potential for examining the q-generalizations of findings in this article, which were influenced by a recently published survey-cum-expository review article by Srivastava [21]. Furthermore, according to the proposed extension, the $(\mathfrak{p}, q)$-extension will only be minor and inconsequently change, as the additional parameter $\mathfrak{p}$ is redundant (see, for details, Srivastava [21, p.340]). Furthermore, the reader's curiosity is drawn to future research into the (k, s)-extension of the Riemann-Liouville fractional integral in light of Srivastava's recent work [22].

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Authors contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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