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Improved lower bounds for numerical radius via Cartesian decomposition

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Abstract

In this article, we derive various lower bounds for the numerical radius of operators that refine the well-known inequality $w^2(A) \geqslant \frac{1}{4}\|A^*A + AA^*\|$.

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1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$. Every $A \in \mathbb{B}(\mathbb{H})$ can be written as $A = \text{Re}(A) + i\text{Im}(A)$, where $\text{Re}(A) = \frac{A+A^*}{2}$ and $\text{Im}(A) = \frac{A-A^*}{2i}$, which is called the Cartesian decomposition of A. We denote by $|A| = (A^*A)^{\frac{1}{2}}$ the positive square root of A^*A. The numerical range of A, denoted by $W(A)$, is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

The classical numerical radius $w(A)$, is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\},$$

and the Crawford number $c(A)$, is defined by

$$c(A) = \inf\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

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The usual operator norm of an operator A is defined to be

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Moreover, for every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|. \tag{1.1}$$

Many refinements of the inequality (1.1) has been obtained, we refer the reader to [1, 6, 7, 12, 15–17], and references therein. In particular, Kittaneh [14] established the following improvement:

$$\frac{1}{4}\|A^*A + AA^*\| \leqslant w^2(A) \leqslant \frac{1}{2}\|A^*A + AA^*\|. \tag{1.2}$$

For more generalizations, counterparts, and recent related results of the numerical radius $w(\cdot)$, the reader may refer to [2, 4, 5, 10, 13, 18].

In this paper we obtain several refinements of the first inequality in (1.2) in terms of $\|\mathrm{Re}(A) + \mathrm{Im}(A)\|$ and $\|\mathrm{Re}(A) - \mathrm{Im}(A)\|$.

2. Main results

To give our first improvement of the first inequality of the inequality (1.2), we need the following lemma (see [3]).

Lemma 2.1. *Let* $a, b \in [0, \infty)$. *We have*

$$(a+b)^r \leqslant a^r + b^r + (2^r - 2)\min\{a^r, b^r\},$$

where $0 < r \leqslant 1$, *and*

$$a^r + b^r + (2^r - 2)\min\{a^r, b^r\} \leqslant (a+b)^r,$$

where $r \geqslant 1$.

Our first result in this section can be stated as follows.

Theorem 2.2. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then, for* $r \geqslant 2$, *we have*

$$\begin{aligned}
\frac{1}{2^r}\|A^*A + AA^*\|^{\frac{r}{2}} &\leqslant \frac{1}{2^{\frac{r}{2}+1}}\left[\|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r\right] \\
&\leqslant \frac{1}{2^{\frac{r}{2}+1}}\left(\begin{array}{c} \|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A) + \mathrm{Im}(A)) + c^r(\mathrm{Re}(A) - \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\} \end{array}\right) \leqslant w^r(A).
\end{aligned}$$

Proof. For every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\begin{aligned}
\frac{1}{2^r}\|A^*A + AA^*\|^{\frac{r}{2}} &= \frac{1}{2^r}\left\|(\mathrm{Re}(A) + \mathrm{Im}(A))^2 + (\mathrm{Re}(A) - \mathrm{Im}(A))^2\right\|^{\frac{r}{2}} \\
&\leqslant \frac{1}{2^r}\left[\|\mathrm{Re}(A) + \mathrm{Im}(A)\|^2 + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^2\right]^{\frac{r}{2}} \\
&\leqslant \frac{1}{2^{\frac{r}{2}+1}}\left[\|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r\right],
\end{aligned}$$

this proves the first inequality. The second inequality follows directly.

For the third inequality, take $x \in \mathbb{H}$ with $\|x\| = 1$. Using the Cartesian decomposition of A and applying Lemma 2.1, we have

$$\begin{aligned}
|\langle Ax, x\rangle|^r &= \left(|\langle Ax, x\rangle|^2\right)^{\frac{r}{2}}\\
&= \left(|\langle \mathrm{Re}(A)x, x\rangle|^2 + |\langle \mathrm{Im}(A)x, x\rangle|^2\right)^{\frac{r}{2}}\\
&= \frac{1}{2^{\frac{r}{2}}}\left[|\langle \mathrm{Re}(A)x, x\rangle + \langle \mathrm{Im}(A)x, x\rangle|^2 + |\langle \mathrm{Re}(A)x, x\rangle - \langle \mathrm{Im}(A)x, x\rangle|^2\right]^{\frac{r}{2}}\\
&= \frac{1}{2^{\frac{r}{2}}}\left(|\langle (\mathrm{Re}(A) + \mathrm{Im}(A))\, x, x\rangle|^2 + |\langle (\mathrm{Re}(A) - \mathrm{Im}(A))\, x, x\rangle|^2\right)^{\frac{r}{2}}\\
&\geqslant \frac{1}{2^{\frac{r}{2}}}\begin{pmatrix} |\langle (\mathrm{Re}(A) + \mathrm{Im}(A))\, x, x\rangle|^r + |\langle (\mathrm{Re}(A) - \mathrm{Im}(A))\, x, x\rangle|^r \\ +(2^{\frac{r}{2}} - 2)\min\{|\langle (\mathrm{Re}(A) + \mathrm{Im}(A))\, x, x\rangle|^r, |\langle (\mathrm{Re}(A) - \mathrm{Im}(A))\, x, x\rangle|^r\}\end{pmatrix}.
\end{aligned}$$

Therefore, we have the following two inequalities:

$$w^r(A) \geqslant \frac{1}{2^{\frac{r}{2}}}\begin{pmatrix} \|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A) - \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\}\end{pmatrix}, \tag{2.1}$$

and

$$w^r(A) \geqslant \frac{1}{2^{\frac{r}{2}}}\begin{pmatrix} \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A) + \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\}\end{pmatrix}. \tag{2.2}$$

It follows from the inequalities (2.1) and (2.2) that

$$w^r(A) \geqslant \frac{1}{2^{\frac{r}{2}+1}}\begin{pmatrix} \|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A) + \mathrm{Im}(A)) + c^r(\mathrm{Re}(A) - \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\}\end{pmatrix}.$$

□

Application of Theorem 2.2 can be seen in the following corollaries. In the first corollary we give a generalization of the recently obtained inequalities in [8, Theorem 2.1].

Corollary 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $r \geqslant 2$, we have*

$$\begin{aligned}
\frac{1}{4}\|A^*A + AA^*\| &\leqslant \frac{1}{2^{\frac{2}{r}+1}}\left[\|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r\right]^{\frac{2}{r}}\\
&\leqslant \frac{1}{2^{\frac{2}{r}+1}}\begin{pmatrix} \|\mathrm{Re}(A) + \mathrm{Im}(A)\|^r + \|\mathrm{Re}(A) - \mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A) + \mathrm{Im}(A)) + c^r(\mathrm{Re}(A) - \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\}\end{pmatrix}^{\frac{2}{r}} \leqslant w^2(A).
\end{aligned}$$

Corollary 2.4. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $r \geqslant 2$, we have*

$$w^r(A) \geqslant \left[\begin{matrix} \frac{1}{2^r}\|A^*A + AA^*\|^{\frac{r}{2}} \\ +\frac{1}{2^{\frac{r}{2}+1}}\begin{pmatrix} c^r(\mathrm{Re}(A) + \mathrm{Im}(A)) + c^r(\mathrm{Re}(A) - \mathrm{Im}(A)) \\ +(2^{\frac{r}{2}} - 2)\min\{c^r(\mathrm{Re}(A) + \mathrm{Im}(A)), c^r(\mathrm{Re}(A) - \mathrm{Im}(A))\}\end{pmatrix}\end{matrix}\right].$$

In particular,

$$w^2(A) \geqslant \frac{1}{4}\|A^*A + AA^*\| + \frac{c^2(\mathrm{Re}(A) + \mathrm{Im}(A)) + c^2(\mathrm{Re}(A) - \mathrm{Im}(A))}{4}.$$

Corollary 2.5. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then, for* $r \geqslant 2$, *we have* $w^r(A) \geqslant \max\{\alpha, \beta\}$, *where*

$$\alpha = \frac{1}{2^{\frac{r}{2}}} \begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) \\ +(2^{\frac{r}{2}}-2)\min\{c^r(\mathrm{Re}(A)+\mathrm{Im}(A)), c^r(\mathrm{Re}(A)-\mathrm{Im}(A))\} \end{pmatrix},$$

$$\beta = \frac{1}{2^{\frac{r}{2}}} \begin{pmatrix} \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \\ +(2^{\frac{r}{2}}-2)\min\{c^r(\mathrm{Re}(A)+\mathrm{Im}(A)), c^r(\mathrm{Re}(A)-\mathrm{Im}(A))\} \end{pmatrix}.$$

Corollary 2.5 generalizes Corollary 2.3 given in [8]. Also, it refines Theorem 2.3 given in [9], when $r = 2$.

Remark 2.6. We have

$$\begin{aligned} \max\{\alpha,\beta\} &= \frac{1}{2^{\frac{r}{2}+1}} \begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) \\ +\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r + c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \\ +2(2^{\frac{r}{2}}-2)\min\{c^r(\mathrm{Re}(A)+\mathrm{Im}(A)), c^r(\mathrm{Re}(A)-\mathrm{Im}(A))\} \end{pmatrix} \\ &\quad + \frac{1}{2^{\frac{r}{2}+1}} \left(\left| \begin{array}{c} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r - \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) - c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \end{array} \right| \right) \\ &\geqslant \frac{1}{2^{\frac{r}{2}+1}} \begin{pmatrix} c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) + c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \\ +2(2^{\frac{r}{2}}-2)\min\{c^r(\mathrm{Re}(A)+\mathrm{Im}(A)), c^r(\mathrm{Re}(A)-\mathrm{Im}(A))\} \end{pmatrix} \\ &\quad + \frac{1}{2^r} \left\| (\mathrm{Re}(A)+\mathrm{Im}(A))^2 + (\mathrm{Re}(A)-\mathrm{Im}(A))^2 \right\|^{\frac{r}{2}} \\ &\quad + \frac{1}{2^{\frac{r}{2}+1}} \left| \begin{array}{c} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r - \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) - c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \end{array} \right|. \end{aligned}$$

Thus,

$$\begin{aligned} w^r(A) &\geqslant \frac{1}{2^r}\|A^*A+AA^*\|^{\frac{r}{2}} + \frac{1}{2^{\frac{r}{2}+1}} \begin{pmatrix} c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) + c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \\ +(2^{\frac{r}{2}+1}-4)\min\{c^r(\mathrm{Re}(A)-\mathrm{Im}(A)), c^r(\mathrm{Re}(A)-\mathrm{Im}(A))\} \end{pmatrix} \\ &\quad + \frac{1}{2^{\frac{r}{2}+1}} \left| \begin{array}{c} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r - \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ +c^r(\mathrm{Re}(A)-\mathrm{Im}(A)) - c^r(\mathrm{Re}(A)+\mathrm{Im}(A)) \end{array} \right|. \end{aligned}$$

For the following lemma, we refer to [6].

Lemma 2.7. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$\|A+B\|^2 \leqslant 2\max\{\|A^*A+B^*B\|, \|AA^*+BB^*\|\}.$$

Based on the above lemma, we obtain our second refinement of the first inequality of the inequality (1.2) as follows.

Theorem 2.8. *Let* $A \in \mathbb{B}(\mathbb{H})$ *and* $0 < r \leqslant 4$. *Then*

$$\frac{1}{2^r}\|A^*A+AA^*\|^{\frac{r}{2}} \leqslant \frac{1}{2^{\frac{3r}{4}}} \begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r + \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ -(2-2^{\frac{r}{4}})\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \end{pmatrix} \leqslant w^r(A).$$

Proof. For every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\begin{aligned} \frac{1}{2^r}\|A^*A+AA^*\|^{\frac{r}{2}} &= \frac{1}{2^r}\left\|(\mathrm{Re}(A)+\mathrm{Im}(A))^2 + (\mathrm{Re}(A)-\mathrm{Im}(A))^2\right\|^{\frac{r}{2}} \\ &\leqslant \frac{1}{2^{\frac{3r}{4}}}\left\|(\mathrm{Re}(A)+\mathrm{Im}(A))^4 + (\mathrm{Re}(A)-\mathrm{Im}(A))^4\right\|^{\frac{r}{4}} \quad \text{(by Lemma 2.7)} \\ &\leqslant \frac{1}{2^{\frac{3r}{4}}}\left(\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^4 + \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^4\right)^{\frac{r}{4}} \\ &\leqslant \frac{1}{2^{\frac{3r}{4}}} \begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r + \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ -(2-2^{\frac{r}{4}})\min\left\{ \begin{array}{c} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r \\ ,\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \end{array} \right\} \end{pmatrix} \text{(by Lemma 2.1).} \end{aligned} \tag{2.3}$$

Now, when $\|\mathrm{Re}(A)-\mathrm{Im}(A)\| \leqslant \|\mathrm{Re}(A)+\mathrm{Im}(A)\|$ and using the inequalities (2.1) and (2.2), we have

$$\begin{aligned}
&\frac{1}{2^{\frac{3r}{4}}}\begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ -(2-2^{\frac{r}{4}})\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \end{pmatrix}\\
&\quad= \frac{1}{2^{\frac{3r}{4}}}\left(\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r+\left(2^{\frac{r}{4}}-1\right)\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r\right)\\
&\quad\leqslant \frac{1}{2^{\frac{3r}{4}}}\left(2^{\frac{r}{2}}w^r(A)+2^{\frac{r}{2}}\left(2^{\frac{r}{4}}-1\right)w^r(A)\right)=w^r(A).
\end{aligned} \tag{2.4}$$

Similarly, when $\|\mathrm{Re}(A)+\mathrm{Im}(A)\| \leqslant \|\mathrm{Re}(A)-\mathrm{Im}(A)\|$, we have

$$\begin{aligned}
&\frac{1}{2^{\frac{3r}{4}}}\begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \\ -(2-2^{\frac{r}{4}})\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r \end{pmatrix}\\
&\quad= \frac{1}{2^{\frac{3r}{4}}}\left(\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r+\left(2^{\frac{r}{4}}-1\right)\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r\right)\\
&\quad\leqslant \frac{1}{2^{\frac{3r}{4}}}\left(2^{\frac{r}{2}}w^r(A)+2^{\frac{r}{2}}\left(2^{\frac{r}{4}}-1\right)w^r(A)\right)=w^r(A).
\end{aligned} \tag{2.5}$$

The inequality (2.3), together with the inequalities (2.4) and (2.5), gives the result. □

The following corollary gives a refinement of the second inequality of Theorem 2.8 given in [8].

Corollary 2.9. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then*

$$\frac{1}{4}\|A^*A+AA^*\| \leqslant \frac{1}{2\sqrt{2}}\begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2 \\ -\left(2-\sqrt{2}\right)\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2 \end{pmatrix} \leqslant w^2(A).$$

Corollary 2.10. *Let* $A \in \mathbb{B}(\mathbb{H})$ *and* $0 < r \leqslant 4$. *Then*

$$\frac{1}{2^r}\|A^*A+AA^*\|^{\frac{r}{2}} \leqslant \max\{\alpha,\beta\} \leqslant w^r(A),$$

where

$$\alpha = \frac{1}{2^{\frac{3r}{4}}}\left(\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r+\left(2^{\frac{r}{4}}-1\right)\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r\right)$$

and

$$\beta = \frac{1}{2^{\frac{3r}{4}}}\left(\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^r+\left(2^{\frac{r}{4}}-1\right)\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^r\right).$$

We need the following lemma (see [11]) to complete our work.

Lemma 2.11. *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *be positive. Then*

$$\|A+B\| \leqslant \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}.$$

Now, we have our third refinement of the first inequality of the inequality (1.2). In this result, we give a refinement of the second inequality of Theorem 2.12 given in [8].

Theorem 2.12. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then*

$$\frac{1}{4}\|A^*A+AA^*\| \leqslant \frac{1}{4}\begin{pmatrix} \|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2 \\ +\frac{\left|\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2-\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\right|}{2} \end{pmatrix} \leqslant w^2(A).$$

Proof. For every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\begin{aligned}\frac{1}{4}\|A^*A + AA^*\| &= \frac{1}{4}\left\|(\mathrm{Re}(A)+\mathrm{Im}(A))^2 + (\mathrm{Re}(A)-\mathrm{Im}(A))^2\right\| \\ &\leqslant \frac{1}{4}\begin{pmatrix}\max\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\} \\ +\left\|(\mathrm{Re}(A)+\mathrm{Im}(A))^2(\mathrm{Re}(A)-\mathrm{Im}(A))^2\right\|^{\frac{1}{2}}\end{pmatrix} \quad \text{(by Lemma 2.11)} \\ &\leqslant \frac{1}{4}\begin{pmatrix}\max\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\} \\ +\|\mathrm{Re}(A)+\mathrm{Im}(A)\|\,\|\mathrm{Re}(A)-\mathrm{Im}(A)\|\end{pmatrix} \\ &\leqslant \frac{1}{4}\begin{pmatrix}\max\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\} \\ +\frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{2}\end{pmatrix}.\end{aligned} \tag{2.6}$$

Now, when $\|\mathrm{Re}(A)-\mathrm{Im}(A)\| \leqslant \|\mathrm{Re}(A)+\mathrm{Im}(A)\|$ and using the inequalities (2.1) and (2.2), when $r = 2$, we have

$$\begin{aligned}&\frac{1}{4}\begin{pmatrix}\max\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\} \\ +\frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{2}\end{pmatrix} \\ &= \frac{1}{4}\left(\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2 + \frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{2}\right) \leqslant w^2(A).\end{aligned} \tag{2.7}$$

Similarly, when $\|\mathrm{Re}(A)+\mathrm{Im}(A)\| \leqslant \|\mathrm{Re}(A)-\mathrm{Im}(A)\|$, we have

$$\begin{aligned}&\frac{1}{4}\begin{pmatrix}\max\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\} \\ +\frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{2}\end{pmatrix} \\ &= \frac{1}{4}\left(\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2 + \frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{2}\right) \leqslant w^2(A).\end{aligned} \tag{2.8}$$

The inequality (2.6), together with the inequalities (2.7) and (2.8), then applying the fact that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$, gives the result. ☐

Corollary 2.13. *Let* $A \in \mathbb{B}(\mathbb{H})$. *Then, we have*

$$\frac{1}{4}\|A^*A + AA^*\| \leqslant \max\{\alpha, \beta\} \leqslant w^2(A),$$

where

$$\alpha = \frac{3\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2 + \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{8}$$

and

$$\beta = \frac{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2 + 3\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2}{8}.$$

Remark 2.14. The following two examples show that the bounds in Theorem 2.8 when $0 < r \leqslant 4$ and the inequality given in Theorem 2.12 are not generally comparable.

(a) Let $A = \begin{bmatrix} 2+2i & 0 \\ 0 & 0 \end{bmatrix}$. Then $\mathrm{Re}(A) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathrm{Im}(A) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $\|\mathrm{Re}(A)+\mathrm{Im}(A)\| = 4$ and $\|\mathrm{Re}(A)-\mathrm{Im}(A)\| = 0$. The bound given in Theorem 2.8 when $r = 2$ will be

$$w^2(A) \geqslant \frac{1}{2^{\frac{3}{2}}}\begin{pmatrix}\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2 + \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2 \\ -\left(2-2^{\frac{1}{2}}\right)\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2, \|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\end{pmatrix} \approx 5.6568,$$

while the bound given in Theorem 2.12 is

$$w^2(A) \geqslant \frac{1}{4}\left(\begin{array}{c}\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\\ +\frac{\left|\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2-\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\right|}{2}\end{array}\right)=6.$$

(b) Let $A=\begin{bmatrix}3+2i & 0\\ 0 & 0\end{bmatrix}$. Then $\mathrm{Re}(A)=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}$ and $\mathrm{Im}(A)=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$. Clearly, $\|\mathrm{Re}(A)+\mathrm{Im}(A)\|=5$ and $\|\mathrm{Re}(A)-\mathrm{Im}(A)\|=1$. The bound given in Theorem 2.8 when $r=1$ will be

$$w(A) \geqslant \frac{1}{2^{\frac{3}{4}}}\left(\begin{array}{c}\|\mathrm{Re}(A)+\mathrm{Im}(A)\|+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|\\ -\left(2-2^{\frac{1}{4}}\right)\min\{\|\mathrm{Re}(A)+\mathrm{Im}(A)\|,\|\mathrm{Re}(A)-\mathrm{Im}(A)\|\end{array}\right)\approx 3.0855,$$

while the bound given in Theorem 2.12 is

$$w(A) \geqslant \sqrt{\frac{1}{4}\left(\begin{array}{c}\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2+\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\\ +\frac{\left|\|\mathrm{Re}(A)+\mathrm{Im}(A)\|^2-\|\mathrm{Re}(A)-\mathrm{Im}(A)\|^2\right|}{2}\end{array}\right)}\approx 3.0822.$$

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References

[1] A. Abu-Omar, F. Kittaneh, *A generalization of the numerical radius*, Linear Algebra Appl., **569** (2019), 323–334. 1
[2] M. W. Alomari, *Numerical radius inequalities for Hilbert space operators*, Complex Anal. Oper. Theory, **15** (2021), 19 pages. 1
[3] F. Alrimawi, O. Hirzallah, F. Kittaneh, *Norm inequalities related to Clarkson inequalities*, Electron. J. Linear Algebra, **34** (2018) 163–169. 2
[4] F. Alrimawi, O. Hirzallah, F. Kittaneh, *Norm inequalities involving the weighted numerical radii of operators*, Linear Algebra Appl., **657** (2023), 127–146. 1
[5] F. Alrimawi, H. Kawariq, F. A. Abushaheen, *Generalized-weighted numerical radius inequalities for Schatten p-norms*, Int. J. Math. Comput. Sci., **17** (2022), 1463–2022. 1
[6] P. Bhunia, S. Bag, K. Paul, *Bounds for zeros of a polynomial using numerical radius of Hilbert space operators*, Ann. Funct. Anal., **12** (2021), 14 pages. 1, 2
[7] P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on numerical radius inequalities*, In: Infosys Science Foundation Series in Mathematical Sciences, Springer, Cham, (2022). 1
[8] P. Bhunia, S. Jana, M. S. Moslehian, K. Paul, *Improved inequalities for numerical radius via Cartesian decomposition*, Funct. Anal. Appl., **57** (2023), 18–28. 2, 2, 2, 2
[9] P. Bhunia, S. Jana, K. Paul, *Refined inequalities for the numerical radius of Hilbert space operators*, arXiv preprint arXiv:2106.13949, (2021) 1–12. 2
[10] P. Bhunia, K. Paul, *Some improvements of numerical radius inequalities of operators and operator matrices*, Linear Multilinear Algebra, **70** (2020), 1995–2013. 1
[11] K. R. Davidson, S. C. Power, *Best approximation in C^*-algebras*, J. Reine Angew. Math., **368** (1986), 43–62. 2
[12] K. Feki, T. Yamazaki, *Joint numerical radius of spherical Aluthge transforms of tuples of Hilbert space operators*, Math. Inequal. Appl., **24** (2021), 405–420. 1
[13] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158** (2003), 11–17. 1
[14] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168** (2005), 73–80. 1
[15] M. S. Moslehian, Q. Xu, A. Zamani, *Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces*, Linear Algebra Appl., **591** (2020), 299–321. 1
[16] M. E. Omidvar, H. R. Moradi, K. Shebrawi, *Sharpening some classical numerical radius inequalities*, Oper. Matrices, **12** (2018), 407–416.
[17] S. Sahoo, N. C. Rout, M. Sababheh, *Some extended numerical radius inequalities*, Linear Multilinear Algebra, **69** (2021), 907–920. 1
[18] M. Sattari, M. S. Moslehian, T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl., **470** (2015), 216–227. 1