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The reverse order law for EP modular operators

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Abstract

In this paper, we present new conditions that reverse order law holds for EP modular operators.

Keywords: EP operator, reverse order law, Moore-Penrose inverse, Hilbert C^*-module, closed range.
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1. Introduction and preliminaries

It is a classical result of Greville [8], that $(TS)^\dagger = S^\dagger T^\dagger$ if and only if $\operatorname{ran}(T^*TS) \subset \operatorname{ran}(S)$ and $\operatorname{ran}(SS^*T^*) \subset \operatorname{ran}(T^*)$, in the case when T and S are complex (possibly rectangular) matrices. This result is extended for bounded linear operators on Hilbert spaces, by Bouldin [2, 3], and Izumino [9]. Then, in [5], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. Recently, Sharifi [13] and Mohammadzadeh Karizaki [11, 12] studied Moore -Penrose inverse of product of the operators with closed range in Hilbert C^*-modules.

In this paper, we state new conditions that reverse order law for the Moore-Penrose inverse holds. A bounded linear operator T with closed range on a complex Hilbert space H is called an EP operator if T and T^* have the same range. Djordjević [4] gave necessary and sufficient conditions for a product of two EP operators with closed ranges to be an EP operator with a closed range. In addition, we state new conditions that if $T, S \in \mathcal{L}(\mathcal{X})$ are EP operators with closed ranges and $TS = ST^* = S^*T$, then TS has closed range and $(TS)^\dagger = S^\dagger T^\dagger$.

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Hilbert C^*-modules are objects like Hilbert spaces, except that the inner product takes its values in a C^*-algebra, instead of being complex-valued. Throughout the paper, $\mathcal{A}$ is a C^*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$

for each $x, y, z \in \mathcal{X},\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example, every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, then, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$, the so-called adjoint of T such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [10, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ for the kernel and the range of operators, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where $\mathcal{Y}^\perp := \{y \in \mathcal{X} : \ \langle x, y\rangle = 0 \ \text{ for } \ \text{all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [6, 7, 10] and the references cited therein for more details.

Throughout this paper, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however, Lance [10] proved that certain submodules are orthogonally complemented as follows:

Theorem 1.1 ([10])**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\operatorname{ran}(T^*)$*;*
- $\operatorname{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$*;*
- *The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.*

Xu and Sheng [16] have shown that a bounded adjointable operator between two Hilbert C^*-modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range.

Definition 1.2. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T (if it exists) is an element in $X \in L(\mathcal{Y}, \mathcal{X})$ which satisfies:

1. $TXT = T$;
2. $XTX = X$;
3. $(TX)^* = TX$;
4. $(XT)^* = XT$.

If $\theta \subseteq \{1, 2, 3, 4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is a θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. If $\operatorname{ran}(T)$ is closed, then $T\{1, 2, 3, 4\} = \{T^\dagger\}$. An operator X is an inner inverse of T if $X \in T\{1\}$. In this case, T is inner invertible, or relatively regular. It is well-known that T is inner invertible if and only if $\operatorname{ran}(T)$ is closed.

Motivated by these conditions, $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$.

By Definition 1.2, we have

$$\begin{array}{ll} \operatorname{ran}(T) = \operatorname{ran}(TT^\dagger), & \operatorname{ran}(T^\dagger) = \operatorname{ran}(T^\dagger T) = \operatorname{ran}(T^*), \\ \ker(T) = \ker(T^\dagger T), & \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*), \end{array}$$

and by Theorem 1.1, we obtain

$$\mathcal{X} = \ker(T) \oplus \operatorname{ran}(T^\dagger) = \ker(T^\dagger T) \oplus \operatorname{ran}(T^\dagger T),$$
$$\mathcal{Y} = \ker(T^\dagger) \oplus \operatorname{ran}(T) = \ker(TT^\dagger) \oplus \operatorname{ran}(TT^\dagger).$$

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_{\mathcal{M}}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}},\ T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}}),\ T_3 = (1 - P_{\mathcal{N}}) T P_{\mathcal{M}},\ T_4 = (1 - P_{\mathcal{N}}) T (1 - P_{\mathcal{M}})$.

Definition 1.3 ([14])**.** Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-modules. An operator $T \in \mathcal{L}(\mathcal{X})$ is called EP if $\operatorname{ran}(T)$ and $\operatorname{ran}(T^*)$ have the same closure.

Lemma 1.4 ([14])**.** *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X})$ with closed range. Then T is EP if and only if it is of the matrix form*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T) \end{bmatrix}$$

for some invertible operator $T_1 \in \mathcal{L}(\operatorname{ran}(T), \operatorname{ran}(T))$.

2. The reverse order law

In this section, we state some new conditions that reverse order law holds for EP modular operators.

Theorem 2.1. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $T, S \in \mathcal{L}(\mathcal{X})$ are EP operators with closed ranges and $TS = ST^*$. Then TS has closed range and $(TS)^\dagger TS = S^\dagger T^\dagger TS$.*

Moreover, if $P_{\mathrm{ran(S)}} T = P_{\mathrm{ran(S)}} T P_{\mathrm{ran(S)}}$, then

(i) $TS(TS)^\dagger = TSS^\dagger T^\dagger$,

(ii) $(TS)^\dagger = S^\dagger T^\dagger$.

Proof. Since T is EP operator with closed range, then by Lemma 1.4, operators S and T have the following matrix representations

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T) \end{bmatrix}, \tag{2.1}$$

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T) \end{bmatrix}. \tag{2.2}$$

Since $TS = ST^*$, then

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} T_1S_1 & T_1S_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1 & 0 \\ S_3T_1 & 0 \end{bmatrix}. \tag{2.3}$$

Equation (2.3) shows that $T_1S_2 = S_3T_1 = 0$, by invertibility of T_1 we conclude that $S_2 = S_3 = 0$. Since $\mathrm{ran}(S)$ is closed, then $\mathrm{ran}(S_1)$ and $\mathrm{ran}(S_4)$ are closed. We let $\{y_n = z_n \oplus x_n\}$ be a sequence chosen in $\mathrm{ran}(T) \oplus \ker(T)$, $\{z_n\}, \{x_n\}$ be sequences chosen in $\mathrm{ran}(T)$ and $\ker(T)$, respectively, such that $S(z_n \oplus x_n) \to y$. Since ran(S) is assumed to be closed, then $y \in$ ran(S). On the other hand, $y = z + x$ for some $z \in \mathrm{ran}(T)$ and $x \in \ker(T)$. By direct sum property, $S(z_n \oplus x_n) = (S_1 \oplus S_4)(z_n \oplus x_n) = S_1(z_n) \oplus S_4(x_n) \to z + x$, that is $S_1(z_n) \to z$ and $S_4(x_n) \to x$. Since $S = S_1 \oplus S_4$, then $z \in \mathrm{ran}(S_1)$ and $x \in \mathrm{ran}(S_4)$. This is implies that $\mathrm{ran}(S_1)$ and $\mathrm{ran}(S_4)$ are closed.

Therefore, obviously $S^\dagger = \begin{bmatrix} S_1^\dagger & 0 \\ 0 & S_4^\dagger \end{bmatrix}$ is Moore-Penrose inverse of $S = \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix}$. Hence, we have

$$TS = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix} = \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.4}$$

$$S^\dagger T^\dagger = \begin{bmatrix} S_1^\dagger & 0 \\ 0 & S_4^\dagger \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^\dagger T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.5}$$

Since

$$TSS^\dagger T^\dagger TS = \begin{bmatrix} (T_1S_1)S_1^\dagger T_1^{-1}(T_1S_1) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$S^\dagger T^\dagger TSS^\dagger T^\dagger = \begin{bmatrix} S_1^\dagger T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1^\dagger T^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^\dagger T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$(S^\dagger T^\dagger TS)^* = \begin{bmatrix} S_1^\dagger T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^\dagger S_1 & 0 \\ 0 & 0 \end{bmatrix} = S^\dagger T^\dagger TS.$$

Then $S^\dagger T^\dagger \in (TS)\{1,2,4\}$. By using [1, Lemma 2.1.] we conclude that TS has closed range and [15, Theorem 2.2] implies that $(TS)^\dagger TS = S^\dagger T^\dagger TS$.

(i) Since S is EP operator with closed range, then by Lemma 1.4 operators S and T have the following matrix representations

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(S) \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(S) \\ \ker(S) \end{bmatrix}, \tag{2.6}$$

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(S) \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(S) \\ \ker(S) \end{bmatrix}. \tag{2.7}$$

Since $TS = ST^*$, then

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} T_1S_1 & 0 \\ T_3S_1 & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^* & S_1^*T_3^* \\ 0 & 0 \end{bmatrix}. \tag{2.8}$$

By using (2.8) we have $T_3S_1 = 0$, therefore by the invertibility of S_1, we conclude that $T_3 = 0$. On the other hand, since $P_{\mathrm{ran(S)}}T = P_{\mathrm{ran(S)}}TP_{\mathrm{ran(S)}}$, then $P_{\mathrm{ran(S)}}T - P_{\mathrm{ran(S)}}TP_{\mathrm{ran(S)}} = P_{\mathrm{ran(S)}}T(1- P_{\mathrm{ran(S)}}) = 0$, that is, $T_2 = 0$. Hence

$$TS = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

By using a similar argument for the closedness of the range of S_1 and S_4 in this proof we imply that, since T has closed range, then T_1 and T_4 have closed ranges and $T^\dagger = \begin{bmatrix} T_1^\dagger & 0 \\ 0 & T_4^\dagger \end{bmatrix}$. On the other hand, since $(T_1S_1)S_1^{-1}T_1^\dagger(T_1S_1) = T_1S_1$ and $S_1^{-1}T_1^\dagger(T_1S_1)S_1^{-1}T_1^\dagger = S_1^{-1}T_1^\dagger$ and $((T_1S_1)S_1^{-1}T_1^\dagger)^* = (T_1T_1^\dagger)^* = T_1T_1^\dagger$, then $S^\dagger T^\dagger \in (TS)\{1,2,3\}$. Therefore, [15, Theorem 2.1] implies that $TS(TS)^\dagger = TSS^\dagger T^\dagger$.

(ii) Since $TS(TS)^\dagger = TSS^\dagger T^\dagger$ and $(TS)^\dagger TS = S^\dagger T^\dagger TS$, then by [15, Corollary 2.3] we have $(TS)^\dagger = S^\dagger T^\dagger$.

□

Theorem 2.2. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $T, S \in \mathcal{L}(\mathcal{X})$ are EP operators with closed ranges and $TS = S^*T$. Then TS has closed range and $TS(TS)^\dagger = TSS^\dagger T^\dagger$.*

Moreover, if $P_{\mathrm{ran(S)}}T = P_{\mathrm{ran(S)}}TP_{\mathrm{ran(S)}}$, then

(i) $(TS)^\dagger TS = S^\dagger T^\dagger TS$;

(ii) $(TS)^\dagger = S^\dagger T^\dagger$.

Proof. Since S is EP operator with closed range, then operators T, S have the same matrix representations (2.6) and (2.7), respectively. Since $TS = S^*T$, then

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} T_1S_1 & 0 \\ T_3S_1 & 0 \end{bmatrix} = \begin{bmatrix} S_1^*T_1 & S_1^*T_2 \\ 0 & 0 \end{bmatrix}.$$

Then $S_1^*T_2 = T_3S_1 = 0$. By the invertibility of S_1 we conclude that $T_2 = T_3 = 0$. Hence

$$TS = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

A similar argument as in the proof of the previous theorem shows that we have

$$S^\dagger T^\dagger = \begin{bmatrix} S_1^{-1}T_1^\dagger & 0 \\ 0 & 0 \end{bmatrix} \in (TS)\{1,2,3\}.$$

By using [1, Lemma 2.1], we conclude that TS has closed range and [15, Theorem 2.1] implies that $TS(TS)^\dagger = TSS^\dagger T^\dagger$.

(i) Since T is EP operator with closed range, therefore the operators T and S have the matrix representations (2.1) and (2.2), respectively. Since $TS = S^*T$, then

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} = \begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} T_1S_1 & T_1S_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^*T_1 & 0 \\ S_2^*T_1^* & 0 \end{bmatrix}.$$

Hence $T_1S_2 = 0$, by invertibility of T_1 we conclude that $S_2 = 0$. On the other hand, since $SP_{\mathrm{ran}}(T) = P_{\mathrm{ran(T)}}SP_{\mathrm{ran}}(T)$, then $S_3 = 0$. Similar argument for (2.4) and (2.5) in the previous theorem, implies that $S^\dagger T^\dagger = \begin{bmatrix} S_1^\dagger T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in (TS)\{1,2,4\}$. Therefore [15, Theorem 2.2] implies that $(TS)^\dagger TS = S^\dagger T^\dagger TS$.

(ii) Since $TS(TS)^\dagger = TSS^\dagger T^\dagger$ and $(TS)^\dagger TS = S^\dagger T^\dagger TS$, then by [15, Corollary 2.3] we have $(TS)^\dagger = S^\dagger T^\dagger$.

□

Theorem 2.3. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $T, S \in \mathcal{L}(\mathcal{X})$ are EP operators with closed ranges and $TS = ST^* = S^*T$. Then TS has closed range and $(TS)^\dagger = S^\dagger T^\dagger$.*

Proof. Since $TS = ST^*$, then Theorem 2.1 implies that TS has closed range and $(TS)^\dagger TS = S^\dagger T^\dagger TS$. On the other hand, since $TS = S^*T$, then Theorem 2.2 shows that $TS(TS)^\dagger = TSS^\dagger T^\dagger$. Therefore [15, Corollary 2.3] implies that $(TS)^\dagger = S^\dagger T^\dagger$. □

References

[1] T. Aghasizadeh, S. Hejazian, *Maps preserving semi-Fredholm operators on Hilbert C^*-modules*, J. Math. Anal. Appl., **354** (2009), 625–629. 2, 2

[2] R. Bouldin, *The product of operators with closed range*, Tôhoku Math. J., **25** (1973), 359–363. 1

[3] R. Bouldin, *Closed range and relative regularity for products*, J. Math. Anal. Appl., **61** (1977), 397–403. 1

[4] D. S. Djordjević, *Products of EP operators on Hilbert spaces*, Proc. Amer. Math. Soc., **129** (2001), 1727–1731. 1

[5] D. S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl., **29** (2007), 1242–1246. 1

[6] M. Frank, *Self-duality and C^*-reflexivity of Hilbert C^*-moduli*, Z. Anal. Anwendungen, **9** (1990), 165–176. 1

[7] M. Frank, *Geometrical aspects of Hilbert C^*-modules*, Positivity, **3** (1999), 215–243. 1

[8] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev., **8** (1966), 518–521. 1

[9] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tôhoku Math. J., **34** (1982), 43–52. 1

[10] E. C. Lance, *Hilbert C^*-modules*, A toolkit for operator algebraists, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, (1995). 1, 1.1

[11] M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari, *Moore-Penrose inverse of product operators in Hilbert C^*-modules*, Filomat, (To appear). 1

[12] M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari, M. Khosravi, *Operator matrix of Moore-Penrose inverse operators on Hilbert C^*-modules*, Colloq. Math., **140** (2015), 171–182 1

[13] K. Sharifi, *The product of operators with closed range in Hilbert C^*-modules*, Linear Algebra Appl., **435** (2011), 1122–1130. 1

[14] K. Sharifi, *EP modular operators and their products*, J. Math. Anal. Appl., **419** (2014), 870–877. 1.3, 1.4

[15] K. Sharifi, B. A. Bonakdar, *The reverse order law for Moore-Penrose inverses of operators on Hilbert* C**-modules*, Bull. Iranian Math. Soc., **42** (2016), 53–60. 2, 2, 2, 2

[16] Q. Xu, L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules*, Linear Algebra Appl., **428** (2008), 992–1000. 1