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Kamal transform and Ulam stability of φ^{th} order linear differential equations

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Abstract

In this manuscript, we discusses the Kamal transform for homogeneous and non-homogeneous linear differential equations. Using this unique integral transform, we resolve higher-order linear differential equations. Alternatively, it can produce the conditions required for Hyers-Ulam stability by using the Kamal transform. This is the first attempt to use the Kamal transform to show that a linear differential equation is stable. The Kamal transform method is more useful for investigating the stability problem for differential equations with constant coefficients, as this study also shows. The discussion of applications follows to illustrate our approach. In other words, we establish sufficient with a constant coefficient by using the Kamal transform method. Moreover, this paper provides a new method to investigate the stability of differential equations. Further, this paper shows that the Kamal transform method is more convenient for investigating stability problems for linear differential equations with a constant coefficient.

Keywords: Linear differential equation (LDE), Hyers-Ulam stability (HUS), Kamal transform.

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1. Introduction

The Kamal transform is gotten from the traditional Fourier integral. In light of the numerical effortlessness of the Kamal transform and its essential properties, Kamal transform was acquainted by Adelilah Kamal [7] with work with the most common way of tackling normal and incomplete differential equations in the dime area. Normally, Fourier, Laplace, Samudu, Elzaki, Aboodh and Mahgoub transform

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are helpful numerical tools for settling DEs. Additionally, Kamal transform and a portion of its crucial properties are utilized to tackle DEs. Define a set A by

$$A = \left\{ h(\mathfrak{z}) : \exists\, M, K_1, K_2, \dots, K_\varphi > 0, |h(\mathfrak{z})| < M e^{\frac{|\mathfrak{z}|}{K_j}}, \text{ if } \mathfrak{z} \in (-1)^j \times [0, \infty) \right\},$$

where $K_1, K_2, \dots, K_\varphi$ may be finite or infinite and the constant M must be finite number. The Kamal transform of a function $h(\mathfrak{z})$ of exponential order is defined by the integral equation

$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z}) e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z}, \;\; \mathfrak{z} \geqslant 0, \; K_1 \leqslant \psi \leqslant K_\varphi, \tag{1.1}$$

where ν is used to factor the variable $\mathfrak{z}$ in the argument of the function with the Laplace, Elzaki, Aboodh, and Mahgoub transforms and $K(\cdot)$ is the Kamal transform operator. An essential component of the qualitative theory of LDEs is the theory of stability. When may we claim that the answers of an inequality are almost identical to one of the exact solutions of the associated equation? Ulam [12] raised this issue. Since then, numerous researchers have become interested in this issue. Take note that [6] provided the initial response to this puzzle. Following that, some scholars [2, 3, 10] enhanced the finding from [6]. Obloza [8, 9] was among the first contributions dealing with HUS of differential equations. Alsina [1] proved HUS of differential equation $h'(x) = h(x)$ and Takahasi [11] extended the results of [1] to the Banach space. For more details of the HUS, see [4, 5, 13].

Our main goal is to prove the HUS of the φ^{th} order homogeneous and non-homogeneous LDEs

$$h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1} h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}) = 0 \tag{1.2}$$

and

$$h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1} h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}) = m(\mathfrak{z}) \tag{1.3}$$

by using the Kamal integral transform method, where $\upsilon_{\varphi-1}, \dots, \upsilon_2, \upsilon_1, \upsilon_0$ are scalars and $h(\mathfrak{z})$ is a continuously differential function of exponential order.

2. Kamal transform

In this section, for any function $h(\mathfrak{z})$, we assume that the integral equation (1.1) exists. The sufficient conditions for the existence of Kamal transform are that $h(\mathfrak{z})$ $(\mathfrak{z} \geqslant 0)$ is piecewise continuous and of exponential order, otherwise, Kamal transform may or may not exist.

The Kamal transform of simple functions ae as follows.

1. Let $h(\mathfrak{z}) = 1$. By (1.1), we get
$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z}) e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \int_0^\infty e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \nu.$$
2. Let $h(\mathfrak{z}) = \mathfrak{z}$. Then
$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z}) e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \int_0^\infty \mathfrak{z} e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \psi^2.$$
Let $h(\mathfrak{z}) = \mathfrak{z}^\varphi$ for any nonnegative integer φ. Then
$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z}) e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \int_0^\infty \mathfrak{z}^\varphi e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \psi^2 \varphi(\varphi-1) \int_0^\infty e^{\frac{-\mathfrak{z}}{\psi}} \mathfrak{z}^{\varphi-2} d\mathfrak{z} = \cdots = \psi^\varphi \varphi!.$$
3. Let $h(\mathfrak{z}) = e^{at}$. Then
$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z}) e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \int_0^\infty e^{at} e^{\frac{-\mathfrak{z}}{\psi}} d\mathfrak{z} = \frac{-\nu}{a\nu - 1} = \frac{\psi}{1 - a\nu}.$$
This result will be useful to find Kamal transform of
$$K(\sin at) = \frac{a\psi^2}{1 + a^2\psi^2}, \;\; K(\cos at) = \frac{\psi}{1 + a^2\psi^2}, \;\; K(\sinh at) = \frac{a\psi^2}{1 - a^2\psi^2}, \;\; K(\cosh at) = \frac{\psi}{1 - a^2\psi^2}.$$

2.1. Properties of Kamal transforms

In this section, we discuss some basic properties of Kamal transform by using derivative methods of various substitutions.

Theorem 2.1. *Let* $H(\psi)$ *be the Kamal transform of* $h(\mathfrak{z})$*. Then*

(i) $K(h'(\mathfrak{z})) = \frac{1}{\psi}H(\psi) - h(0)$;

(ii) $K(h''(\mathfrak{z})) = \frac{1}{\psi^2}H(\psi) - \frac{1}{\psi}h(0) - h'(0)$;

(iii) $K(h'''(\mathfrak{z})) = \frac{1}{\psi^3}H(\psi) - \frac{1}{\psi^2}h(0) - \frac{1}{\psi}h'(0) - h''(0)$;

(iv) $K(h^{(\varphi)}(\mathfrak{z})) = \frac{1}{\psi^\varphi}H(\psi) - \sum_{k=0}^{\varphi-1}\frac{1}{\psi^{k-\varphi+1}}h^{(k)}(0)$.

Proof.

(i) By the definition, we have

$$K(h'(\mathfrak{z})) = \int_0^\infty h'(\mathfrak{z})e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z} = \frac{1}{\psi}H(\psi) - h(0).$$

(ii) By (i), we have

$$K(h''(\mathfrak{z})) = \int_0^\infty h''(\mathfrak{z})e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z} = -h'(0) + \frac{1}{\psi^2}H(\psi) - \frac{1}{\psi}h(0) = \frac{1}{\psi^2}H(\psi) - \frac{1}{\psi}h(0) - h'(0).$$

(iii) By (ii), we have

$$\begin{aligned} K(h'''(\mathfrak{z})) &= \int_0^\infty h'''(\mathfrak{z})e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z} = -h''(0) + \frac{1}{\psi^3}H(\psi) - \frac{1}{\psi^2}h(0) - \frac{1}{\psi}h'(0) = -h''(0) + \frac{1}{\psi^2}H(\psi) - \frac{1}{\psi}h(0) \\ &= \frac{1}{\psi^3}H(\psi) - \frac{1}{\psi^2}h(0) - \frac{1}{\psi}h'(0) - h''(0). \end{aligned}$$

(iv) By mathematical induction, we have

$$K(h^{(\varphi)}(\mathfrak{z})) = \frac{1}{\psi^\varphi}H(\psi) - \sum_{k=0}^{\varphi-1}\frac{1}{\psi^{k-\varphi+1}}h^{(k)}(0).$$

This completes the proof. □

3. Preliminaries

We offer some common notations and concepts in this part that will help to support our key findings.

Definition 3.1. The Kamal integral transform of function $h : [0, \infty) \to R$ is defined by

$$K(h(\mathfrak{z})) = H(\psi) = \int_0^\infty h(\mathfrak{z})e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z},$$

where K is the Kamal integral transform operator.

Definition 3.2. The convolution of the functions $h(\mathfrak{z})$ and $j(\mathfrak{z})$ is defined by

$$h(\mathfrak{z}) * j(\mathfrak{z}) = (h * j)(\mathfrak{z}) = \int_0^{\mathfrak{z}} h(\mathfrak{z} - s)j(s)ds = \int_0^{\mathfrak{z}} h(s)j(\mathfrak{z} - s)ds.$$

Theorem 3.3. *Assume that* $h(\mathfrak{z})$ *and* $j(\mathfrak{z})$ *are given functions defined on* $[0, \infty)$*. If* $K(h(\mathfrak{z})) = H(\psi)$ *and* $K(j(\mathfrak{z})) = J(\psi)$*, then* $K(h * j)(\mathfrak{z}) = H(\psi)J(\psi)$.

Proof. The convolution of the functions h and j of Kamal transform is given by

$$K(h*j)(\mathfrak{z}) = \int_0^\infty (h*j)\mathfrak{z}e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z} = \int_0^\infty\int_0^\infty h(s)j(\mathfrak{z}-s)e^{\frac{-\mathfrak{z}}{\psi}}dsd\mathfrak{z} = \int_0^\infty h(s)ds\int_0^\infty j(\mathfrak{z}-s)e^{\frac{-\mathfrak{z}}{\psi}}d\mathfrak{z}.$$

Letting $\mathfrak{z}-s=y$, we have

$$K(h*j)(\mathfrak{z}) = \int_0^\infty h(s)ds\int_0^\infty j(t)e^{\frac{-(t+s)}{\psi}}dt = \int_0^\infty h(s)e^{\frac{-(s)}{\psi}}ds\int_0^\infty j(t)e^{\frac{-(t)}{\psi}}dt = H(\psi)J(\psi).$$

This completes the proof. □

Definition 3.4. If $K(h(\mathfrak{z})) = H(\psi)$, then $h(\mathfrak{z})$ is called the inverse Kamal transform of $H(\psi)$ and mathematically is defined as $h(\mathfrak{z}) = K^{-1}(H(\psi))$.

Definition 3.5. The Mittag-Leffler function of one parameter is defined by

$$E_\beta(\mathfrak{z}) = \sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{\Gamma(\beta k+1)},$$

where $R(\beta) > 0$ is the real part of β and $\mathfrak{z},\beta \in \mathbb{C}$. If $\beta = 1$, then

$$E_1(\mathfrak{z}) = \sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{k!} = e^t.$$

Throughout the following sections, we consider

$$H = \{h : [0,\infty) \to K | h \text{ is a continuously differentiable function of exponential order}\}.$$

Also, let $\sigma : [0,\infty) \to (0,\infty)$ is an increasing function and $m : [0,\infty) \to \infty$ is a continuous function of exponential order.

Definition 3.6.

1. The LDE (1.2) has HUS (for class H) if there exists a function $h \in H$ and a constant $K > 0$ such that

$$|h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})| \leqslant \epsilon, \tag{3.1}$$

for any $\epsilon > 0$, for all $\mathfrak{z} \geqslant 0$. There exists a solution $\ell : [0,\infty) \to K$ of DE (1.2) such that

$$|h(\mathfrak{z}) - \ell(\mathfrak{z})| \leqslant K\epsilon, \quad \ell \in H,$$

for all $\mathfrak{z} \geqslant 0$.

2. Let $\sigma : [0,\infty) \to (0,\infty)$. We say that (1.2) is said to have the σHUS, if

$$|h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})| \leqslant \sigma(\mathfrak{z})\epsilon, \tag{3.2}$$

and $|h(\mathfrak{z}) - \ell(\mathfrak{z})| \leqslant K\sigma(\mathfrak{z})\epsilon$, $\ell \in H$, for any $\epsilon > 0$, for all $\mathfrak{z} \geqslant 0$.

3. Let $E_\beta(\mathfrak{z})$ be the Mittag-Leffler function. The LDE (1.2) is said to have the Mittag-Leffler-HUS, if

$$|h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})| \leqslant E_\beta(\mathfrak{z})\epsilon, \tag{3.3}$$

and $|h(\mathfrak{z}) - \ell(\mathfrak{z})| \leqslant KE_\beta(\mathfrak{z})\epsilon$, for any $\epsilon > 0$, for all $\mathfrak{z} \geqslant 0$.

4. The LDE (1.2) is said to have the Mittag-Leffler σHUS, if

$$|h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})| \leqslant \sigma(\mathfrak{z})E_\beta(\mathfrak{z})\epsilon, \tag{3.4}$$

and $|h(\mathfrak{z}) - \ell(\mathfrak{z})| \leqslant K\sigma(\mathfrak{z})E_\beta(\mathfrak{z})\epsilon$, for any $\epsilon > 0$, for all $\mathfrak{z} \geqslant 0$.

Similarly, we can define various stabilities of (1.3).

4. Stability results of (1.2)

In this section, we prove several types of HUS of homogeneous LDE (1.2) of order φ by using the Kamal transform.

Theorem 4.1. *Let* $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)>0$*, where* $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ *is a constant. Then* (1.2) *has HUS in the class* H.

Proof. Assume that $h \in H$ satisfies (3.1). Define a function $i:[0,\infty)\to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2 h''(\mathfrak{z})+\upsilon_1 h'(\mathfrak{z})+\upsilon_0 h(\mathfrak{z}),\quad \forall\, \mathfrak{z}\geqslant 0.$$

By (3.1) $\Rightarrow$, the inequality $|i(\mathfrak{z})|\leqslant \epsilon,\ \mathfrak{z}\geqslant 0$ holds. Kamal transform of $i(\mathfrak{z})$ gives the result:

$$\begin{aligned}I(\psi)=K(i(\mathfrak{z})) &= K[h^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2 h''(\mathfrak{z})+\upsilon_1 h'(\mathfrak{z})+\upsilon_0 h(\mathfrak{z})]\\ &= K[h^{(\varphi)}(\mathfrak{z})]+\upsilon_{\varphi-1}K[h^{(\varphi-1)}(\mathfrak{z})]+\cdots+\upsilon_2 K[h''(\mathfrak{z})]+\upsilon_1 K[h'(\mathfrak{z})]+\upsilon_0 K[h(\mathfrak{z})].\end{aligned}$$

Since $h(\psi)=K[h(\mathfrak{z})]$ and

$$\begin{aligned}K(h'(\mathfrak{z})) &= \frac{1}{\psi}h(\psi)-h(0),\\ K(h''(\mathfrak{z})) &= \frac{1}{\psi^2}h(\psi)-\frac{1}{\psi}h(0)-h'(0),\\ K(h'''(\mathfrak{z})) &= \frac{1}{\psi^3}h(\psi)-\frac{1}{\psi^2}h(0)-\frac{1}{\psi}h'(0)-h''(0),\\ &\ \ \vdots\\ K(h^{(\varphi-1)}(\mathfrak{z})) &= \frac{1}{\psi^{\varphi-1}}h(\psi)-\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}h^{(k)}(0),\\ K(h^{(\varphi)}(\mathfrak{z})) &= \frac{1}{\psi^{\varphi}}h(\psi)-\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}h^{(k)}(0).\end{aligned}$$

Also,

$$\begin{aligned}I(\psi) &= \frac{1}{\psi^{\varphi}}h(\psi)-\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}h^{(k)}(0)+\upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}h(\psi)-\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}h^{(k)}(0)\right)\\ &\quad +\cdots+\upsilon_2\left(\frac{1}{\psi^2}h(\psi)-\frac{1}{\psi}h(0)-h'(0)\right)+\upsilon_1\left(\frac{1}{\psi^2}h(\psi)-\frac{1}{\psi}h(0)-h'(0)\right)+\upsilon_0 h(\psi), \qquad (4.1)\\ h(\psi) &= \frac{I(\psi)+\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}h^{(k)}(0)+\upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}h^{(k)}(0)+\cdots+\upsilon_2\frac{1}{\psi}h(0)\upsilon_2 h'(0)+\upsilon_1 h(0)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}.\end{aligned}$$

If we put $\mathfrak{z}=0$ in $\ell(\mathfrak{z})=e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}h(0)\in H$, then $\ell(0)=h(0)$. Takeing Kamal transform of $\ell(\mathfrak{z})$, we obtain

$$K(\ell(\mathfrak{z}))=X(\psi)=\frac{\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0)+\upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)+\cdots+\upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0)+\upsilon_1\ell(0)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}. \qquad (4.2)$$

Thus

$$K(\ell(\mathfrak{z}))=K[\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})]$$

$$= \frac{1}{\psi^{\varphi}}X(\psi) - \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}X(\psi) - \sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)\right)$$
$$+ \cdots + \upsilon_2\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_1\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_0\ell(\psi).$$

So

$$0 = \frac{1}{\psi^{\varphi}}X(\psi) - \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}X(\psi) - \sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)\right)$$
$$+ \cdots + \upsilon_2\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_1\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_0\ell(\psi).$$

Hence

$$\frac{1}{\psi^{\varphi}}X(\psi) + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}X(\psi) + \cdots + \upsilon_2\frac{1}{\psi^2}X(\psi) + \upsilon_1\frac{1}{\psi}X(\psi) + \upsilon_0 X(\psi)$$
$$= \sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0) + \cdots + \upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0) + \upsilon_1\ell(0).$$

Therefore,

$$X(\psi)\left[\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0\right]$$
$$= \sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0) + \cdots + \upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0) + \upsilon_1\ell(0),$$
$$X(\psi) = \frac{\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0) + \cdots + \upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0) + \upsilon_1\ell(0)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0}.$$

By (4.2), we have

$$K[\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2\ell''(\mathfrak{z}) + \upsilon_1\ell'(\mathfrak{z}) + \upsilon_0\ell(\mathfrak{z})] = 0.$$

Since K is an injective operator, then

$$\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2\ell''(\mathfrak{z}) + \upsilon_1\ell'(\mathfrak{z}) + \upsilon_0\ell(\mathfrak{z}) = 0.$$

So $\ell(\mathfrak{z})$ is a solution of (1.2). By (4.1) and (4.2), we obtain

$$K(h(\mathfrak{z})) - K(\ell(\mathfrak{z})) = h(\psi) - X(\psi) = \frac{I(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0} = I(\psi)L(\psi) = K(i(\mathfrak{z}) * l(\mathfrak{z})),$$

where

$$L(\psi) = \frac{1}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0}, \quad K(l(\mathfrak{z})) = \frac{1}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0}.$$

Thus

$$l(\mathfrak{z}) = K^{-1}\left\{\frac{1}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0}\right\}$$

$$= K^{-1}\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}$$
$$= K^{-1}\left\{\frac{\psi^{\varphi}}{1+(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\psi+\cdots+\psi^{\varphi-2}+\psi^{\varphi-1}+\psi^{\varphi})}\right\},$$

using inverse Kamal transform method. We know that

$$K(e^{at}) = \frac{\psi}{1+av}, \qquad e^{at} = K^{-1}\left(\frac{\psi}{1+av}\right).$$

So

$$l(\mathfrak{z}) = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z})}.$$

Consequently,

$$K(h(\mathfrak{z})) - K(\ell(\mathfrak{z})) = K(i(\mathfrak{z}) * l(\mathfrak{z}))$$

and thus

$$h(\mathfrak{z}) - K(\ell(\mathfrak{z}) = i(\mathfrak{z}) * l(\mathfrak{z}).$$

Taking modulus on each sides, we have

$$|h(\mathfrak{z}) - \ell(\mathfrak{z})| = |i(\mathfrak{z}) * l(\mathfrak{z})| = \left|\int_0^{\mathfrak{z}} i(s)l(\mathfrak{z}-s)ds\right| \leqslant \int_0^{\mathfrak{z}} |i(s)||l(\mathfrak{z}-s)|ds \leqslant \epsilon \int_0^{\mathfrak{z}} |l(\mathfrak{z}-s)|ds.$$

Since $l(\mathfrak{z}) = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z})}$ or $l(\mathfrak{z}) = e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z})}$, we have

$$\begin{aligned}
|h(\mathfrak{z}) - \ell(\mathfrak{z})| &\leqslant \epsilon \int_0^{\mathfrak{z}} e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}ds \\
&\leqslant \epsilon \int_0^{\mathfrak{z}} e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)t+R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s}ds \\
&\leqslant \epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} \int_0^{\mathfrak{z}} e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s}ds \\
&\leqslant \epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} \left[\frac{e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}-1}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\right] \\
&\leqslant \left[\frac{e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)} - \frac{e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\right] \\
&\leqslant \frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1-e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right) \leqslant K\epsilon,
\end{aligned}$$

where $K = \frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0-x)}$. Hence, (1.2) has HUS for the class H. □

Note: If $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0) < 0$, then

$$\frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1-e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right)$$

diverges to infinity as $\mathfrak{z} \to \infty$. In this case, we cannot prove the HUS by applying the Kamal transform method.

Theorem 4.2. *Let $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ is a constant with $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0) > 0$ and $\sigma : [0,\infty) \to (0,\infty)$. Then (1.2) has σHUS for the class H.*

Proof. Let $h \in H$ and $\sigma : [0,\infty) \to (0,\infty)$ satisfying (3.4). Define $i : [0,\infty) \to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})$$

such that $|i(\mathfrak{z})| \leqslant \sigma(\mathfrak{z})\epsilon$, $\forall\, \mathfrak{z} \geqslant 0$. By Theorem 4.1, we can get

$$\ell(\mathfrak{z}) = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} h(0) \in H$$

is a solution of (1.2). On the other hand,

$$L(a) = \left\{ \frac{\psi^{\varphi}}{(1 + \upsilon_{\varphi-1}\psi + \cdots + \upsilon_2\psi^{\varphi-2} + \upsilon_1\psi^{\varphi-1} + \upsilon_0\psi^{\varphi})} \right\},$$

which gives

$$l(\mathfrak{z}) = K^{-1} \left\{ \frac{\psi^{\varphi}}{(1 + \upsilon_{\varphi-1}\psi + \cdots + \upsilon_2\psi^{\varphi-2} + \upsilon_1\psi^{\varphi-1} + \upsilon_0\psi^{\varphi})} \right\} = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Moreover, it follows from (4.1) and (4.2) that

$$\begin{aligned} K[h(\mathfrak{z})] - K[\ell(\mathfrak{z})] = h(\psi) - X(\psi) &= \frac{I(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0} \\ &= \frac{I(\psi)\psi^{\varphi}}{(1 + \upsilon_{\varphi-1}\psi + \cdots + \upsilon_2\psi^{\varphi-2} + \upsilon_1\psi^{\varphi-1} + \upsilon_0\psi^{\varphi})} \\ &= I(\psi)L(\psi) = K[i(\mathfrak{z}) * l(\mathfrak{z})], \end{aligned}$$

which yields that

$$K[h(\mathfrak{z}) - \ell(\mathfrak{z})] = K[i(\mathfrak{z}) * l(\mathfrak{z})].$$

Therefore, $h(\mathfrak{z}) - \ell(\mathfrak{z}) = i(\mathfrak{z}) * l(\mathfrak{z}) = i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}$. By Theorem 4.1, we can get

$$\begin{aligned} |h(\mathfrak{z}) - \ell(\mathfrak{z})| &= \left| i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} \right| \\ &= \left| \int_0^{\mathfrak{z}} i(s) e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)} ds \right| \\ &\leqslant \sigma(\mathfrak{z})\epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)} \int_0^{\mathfrak{z}} e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s} ds \leqslant K\sigma(\mathfrak{z})\epsilon, \end{aligned}$$

for all $\mathfrak{z} \geqslant 0$, where $K = \frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. □

Theorem 4.3. *Let $\upsilon_{\varphi-1} + \cdots + \upsilon_2 + \upsilon_1 + \upsilon_0$ and $\beta > 0$ are constants with $R(\upsilon_{\varphi-1} + \cdots + \upsilon_2 + \upsilon_1 + \upsilon_0) > 0$. Then (1.2) has Mittag-Leffler-HUS for the class H.*

Proof. Let $h \in H$ satisfies (3.3) and define $i : [0,\infty) \to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}), \quad \forall\, \mathfrak{z} \geqslant 0.$$

By (3.3), we have $|i(\mathfrak{z})| \leqslant \epsilon$, $\forall\, \mathfrak{z} \geqslant 0$. Kamal transform of $i(\mathfrak{z})$ gives the result:

$$\begin{aligned} I(\psi) = K(i(\mathfrak{z})) &= K[h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z})] \\ &= \frac{1}{\psi^{\varphi}} h(\psi) - \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} h^{(k)}(0) + \upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}} h(\psi) - \sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} h^{(k)}(0) \right) \\ &\quad + \cdots + \upsilon_2 \left(\frac{1}{\psi^2} h(\psi) - \frac{1}{\psi} h(0) - h'(0) \right) + \upsilon_1 \left(\frac{1}{\psi^2} h(\psi) - \frac{1}{\psi} h(0) - h'(0) \right) + \upsilon_0 h(\psi). \end{aligned}$$

Thus we get

$$h(\psi)=\frac{I(\psi)+\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}h^{(k)}(0)+\upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}h^{(k)}(0)+\cdots+\upsilon_2\frac{1}{\psi}h(0)\upsilon_2h'(0)+\upsilon_1h(0)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}. \tag{4.3}$$

If we put $\ell(\mathfrak{z})=e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}h(0)\in H$, then $\ell(0)=h(0)$. Kamal transform of $\ell(\mathfrak{z})$ gives that

$$X(\psi)=\frac{\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0)+\upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)+\cdots+\upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0)+\upsilon_1\ell(0)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}. \tag{4.4}$$

Thus it follows from (4.4) that

$$\begin{aligned}K(\ell(\mathfrak{z}))&=K[\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})]\\&=\frac{1}{\psi^{\varphi}}X(\psi)-\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0)+\upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}X(\psi)-\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)\right)\\&\quad+\cdots+\upsilon_2\left(\frac{1}{\psi^2}X(\psi)-\frac{1}{\psi}\ell(0)-\ell'(0)\right)+\upsilon_1\left(\frac{1}{\psi^2}X(\psi)-\frac{1}{\psi}\ell(0)-\ell'(0)\right)+\upsilon_0\ell(\psi).\end{aligned}$$

Since K is a injective operator, we have

$$\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})=0.$$

If we set

$$L(\psi)=\left\{\frac{\psi^{\varphi}}{1+(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\psi+\cdots+\psi^{\varphi-2}+\psi^{\varphi-1}+\psi^{\varphi})}\right\},$$

then we get

$$l(\mathfrak{z})=K^{-1}\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}=e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z})}.$$

By (4.3) and (4.4), we have

$$\begin{aligned}K(h(\mathfrak{z}))-K(\ell(\mathfrak{z}))=h(\psi)-X(\psi)&=\frac{I(\psi)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}\\&=I(\psi)\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}\\&=I(\psi)L(\psi)=K(i(\mathfrak{z})*l(\mathfrak{z})).\end{aligned}$$

This gives $h(\mathfrak{z})-\ell(\mathfrak{z})=i(\mathfrak{z})*l(\mathfrak{z})=i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z})}$. Since $|i(\mathfrak{z})|\leqslant\epsilon E_\beta(\mathfrak{z})$, $\mathfrak{z}\geqslant 0$ and $E_\beta(\mathfrak{z})$ is increasing, then

$$\begin{aligned}|h(\mathfrak{z})-\ell(\mathfrak{z})|&=\left|i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right|\\&=\left|\int_0^{\mathfrak{z}}i(s)e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}ds\right|\\&\leqslant\int_0^{\mathfrak{z}}|i(s)||e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}|ds\\&\leqslant E_\beta(\mathfrak{z})\epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\int_0^{\mathfrak{z}}e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s}ds\end{aligned}$$

$$\leqslant \frac{E_\beta(\mathfrak{z})\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1-e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0(\mathfrak{z}))\mathfrak{z}}\right)$$
$$\leqslant KE_\beta(\mathfrak{z})\epsilon, \quad \forall\, \mathfrak{z} \geqslant 0,$$

where $K = \frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. □

Theorem 4.4. *Let* $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ *and* $\beta > 0$ *are constants which* $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0) > 0$ *and* $\sigma : [0,\infty) \to (0,\infty)$. *Then* (1.2) *has the Mittag-Leffler* σ*HUS for the class* H.

Proof. Let $\sigma : [0,\infty) \to (0,\infty)$ and $h(\mathfrak{z}), \ell(\mathfrak{z}) \in H$ satisfies (3.1). To prove: There exist a solution $\ell : [0,\infty) \to K$ of (1.2) and $K > 0 \in \mathbb{Z}$ (independent of ϵ) such that

$$|h(\mathfrak{z}) - \ell(\mathfrak{z})| \leqslant K\sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z}), \quad \forall \mathfrak{z} \geqslant 0.$$

Define $i : [0,\infty) \to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}), \ \forall \mathfrak{z} \geqslant 0.$$

Then $|i(\mathfrak{z})| \leqslant \sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z})$, $\forall\, \mathfrak{z} \geqslant 0$. By the proof of Theorem 4.3, we can get

$$\begin{aligned}
|h(\mathfrak{z}) - \ell(\mathfrak{z})| &= \left|i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right| \\
&= \left|\int_0^{\mathfrak{z}} i(s) e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)} ds\right| \\
&\leqslant \int_0^{\mathfrak{z}} |i(s)||e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}| ds \\
&\leqslant \sigma(\mathfrak{z})E_\beta(\mathfrak{z})\epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)} \int_0^{\mathfrak{z}} e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s} ds \\
&\leqslant \frac{\sigma(\mathfrak{z})E_\beta(\mathfrak{z})\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1 - e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right) \\
&\leqslant K\sigma(\mathfrak{z})E_\beta(\mathfrak{z})\epsilon, \quad \forall \mathfrak{z} \geqslant 0,
\end{aligned}$$

where $K = \frac{\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. □

5. Stability results of (1.3)

In this section, we prove several types of HUS of non-homogeneous LDE (1.3) of order φ by using the Kamal transform.

Theorem 5.1. *Assume that* $m : [0,\infty) \to \infty$ *is a continuous function and* $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ *is a constant with* $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0) > 0$. *Then the LDE* (1.3) *has the HUS for the class* H.

Proof. Let $h \in H$ satisfies HUS, for all $\mathfrak{z} \geqslant 0$. Define $i : [0,\infty) \to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}) - m(\mathfrak{z}), \quad \forall \mathfrak{z} \geqslant 0.$$

Then $|i(\mathfrak{z})| \leqslant \epsilon$, $\forall \mathfrak{z} \geqslant 0$ holds. Kamal transform of $i(\mathfrak{z})$ gives that

$$I(\psi) = K(i(\mathfrak{z})) = K[h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}) - m(\mathfrak{z})].$$

This implies that

$$h(\psi) = \frac{I(\psi) + \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} h^{(k)}(0) + \upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} h^{(k)}(0) + \cdots + \upsilon_2 \frac{1}{\psi} h(0)\upsilon_2 h'(0) + \upsilon_1 h(0) + M(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2 \frac{1}{\psi^2} + \upsilon_1 \frac{1}{\psi} + \upsilon_0}. \tag{5.1}$$

If we put $\mathfrak{z} = 0$ in $\ell(\mathfrak{z}) = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}h(0) + \left(m(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right)$, then $\ell(0) = h(0)$ and $\ell \in H$. Kamal transform of $\ell(\mathfrak{z})$ gives that

$$K(\ell(\mathfrak{z})) = X(\psi) = \frac{\sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0) + \cdots + \upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0) + \upsilon_1\ell(0) + M(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0}. \tag{5.2}$$

On the other hand,

$$\begin{aligned} K(\ell(\mathfrak{z})) &= K[\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2\ell''(\mathfrak{z}) + \upsilon_1\ell'(\mathfrak{z}) + \upsilon_0\ell(\mathfrak{z})] \\ &= \frac{1}{\psi^{\varphi}}X(\psi) - \sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0) + \upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}X(\psi) - \sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)\right) \\ &\quad + \cdots + \upsilon_2\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_1\left(\frac{1}{\psi^2}X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0)\right) + \upsilon_0\ell(\psi). \end{aligned}$$

By (5.2), we have

$$K[\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2\ell''(\mathfrak{z}) + \upsilon_1\ell'(\mathfrak{z}) + \upsilon_0\ell(\mathfrak{z})] = M(\psi) = K[m(\mathfrak{z})]$$

and thus

$$\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2\ell''(\mathfrak{z}) + \upsilon_1\ell'(\mathfrak{z}) + \upsilon_0\ell(\mathfrak{z}) = m(\mathfrak{z}).$$

Hence $\ell(\mathfrak{z})$ is a solution of (1.3). By (5.1) and (5.2), we obtain

$$\begin{aligned} K(h(\mathfrak{z})) - K(\ell(\mathfrak{z})) = h(\psi) - X(\psi) &= I(\psi)\frac{1}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2\frac{1}{\psi^2} + \upsilon_1\frac{1}{\psi} + \upsilon_0} \\ &= I(\psi)\left\{\frac{\psi^{\varphi}}{(1 + \upsilon_{\varphi-1}\psi + \cdots + \upsilon_2\psi^{\varphi-2} + \upsilon_1\psi^{\varphi-1} + \upsilon_0\psi^{\varphi})}\right\} \\ &= I(\psi)L(\psi) = K(i(\mathfrak{z}) * l(\mathfrak{z})), \end{aligned}$$

where $L(a) = \left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}$. This implies

$$l(\mathfrak{z}) = K^{-1}\left\{\frac{\psi^{\varphi}}{(1 + \upsilon_{\varphi-1}\psi + \cdots + \upsilon_2\psi^{\varphi-2} + \upsilon_1\psi^{\varphi-1} + \upsilon_0\psi^{\varphi})}\right\} = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Therefore, we have

$$K[h(\mathfrak{z}) - \ell(\mathfrak{z})] = K[i(\mathfrak{z}) * l(\mathfrak{z})] = K[i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}],$$

which yields

$$h(\mathfrak{z}) - \ell(\mathfrak{z}) = i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Furthermore,

$$\begin{aligned} |h(\mathfrak{z}) - \ell(\mathfrak{z})| &= \left|i(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right| \\ &= \left|\int_0^{\mathfrak{z}} i(s)e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}ds\right| \\ &\leqslant \int_0^{\mathfrak{z}} |i(s)||e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}|ds \\ &\leqslant \epsilon e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\int_0^{\mathfrak{z}} e^{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)s}ds \\ &\leqslant \frac{\epsilon}{R(\upsilon_{\varphi-1} + \cdots + \upsilon_2 + \upsilon_1 + \upsilon_0)}\left(1 - e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right) \leqslant K\epsilon, \quad \forall \mathfrak{z} \geqslant 0, \end{aligned}$$

where $K = \frac{1}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. □

Theorem 5.2. *Let* $m : [0, \infty) \to (0, \infty)$ *is a continuous function,* $\sigma : [0, \infty) \to (0, \infty)$*, and* $\upsilon_{\varphi-1} + \cdots + \upsilon_2 + \upsilon_1 + \upsilon_0$ *is a constant with* $R(\upsilon_{\varphi-1} + \cdots + \upsilon_2 + \upsilon_1 + \upsilon_0) > 0$*. Then* (1.3) *has σHUS for the class* H.

Proof. Let $h \in H$ that satisfies σHUS. Define $i : [0, \infty) \to K$ by

$$i(\mathfrak{z}) = h^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 h''(\mathfrak{z}) + \upsilon_1 h'(\mathfrak{z}) + \upsilon_0 h(\mathfrak{z}) - m(\mathfrak{z}),$$

for all each $\mathfrak{z} \geqslant 0$. Then $|i(\mathfrak{z})| \leqslant \sigma(\mathfrak{z})\epsilon$, $\forall\, \mathfrak{z} \geqslant 0$. It is not difficult to show that

$$\begin{aligned} &K(h(\mathfrak{z}))\\ &= h(\psi) \qquad (5.3)\\ &= \frac{I(\psi) + \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} h^{(k)}(0) + \upsilon_{\varphi-1} \sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} h^{(k)}(0) + \cdots + \upsilon_2 \frac{1}{\psi} h(0)\upsilon_2 h'(0) + \upsilon_1 h(0) + M(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2 \frac{1}{\psi^2} + \upsilon_1 \frac{1}{\psi} + \upsilon_0}. \end{aligned}$$

If we set

$$\ell(\mathfrak{z}) = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} h(0) + \left(m(\mathfrak{z}) * e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}} \right),$$

then $\ell \in H$. Further, applying the Kamal transform on both sides, we get

$$X(\psi) = \frac{\sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} \ell^{(k)}(0) + \upsilon_{\varphi-1} \sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} \ell^{(k)}(0) + \cdots + \upsilon_2 \frac{1}{\psi} \ell(0)\upsilon_2 \ell'(0) + \upsilon_1 \ell(0) + M(\psi)}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2 \frac{1}{\psi^2} + \upsilon_1 \frac{1}{\psi} + \upsilon_0}. \qquad (5.4)$$

On the other hand,

$$\begin{aligned} K(\ell(\mathfrak{z})) &= K[\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 \ell''(\mathfrak{z}) + \upsilon_1 \ell'(\mathfrak{z}) + \upsilon_0 \ell(\mathfrak{z})]\\ &= \frac{1}{\psi^{\varphi}} X(\psi) - \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} \ell^{(k)}(0) + \upsilon_{\varphi-1} \left(\frac{1}{\psi^{\varphi-1}} X(\psi) - \sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} \ell^{(k)}(0) \right)\\ &\quad + \cdots + \upsilon_2 \left(\frac{1}{\psi^2} X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0) \right) + \upsilon_1 \left(\frac{1}{\psi^2} X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0) \right) + \upsilon_0 \ell(\psi),\\ 0 &= \frac{1}{\psi^{\varphi}} X(\psi) - \sum_{k=0}^{\varphi-1} \frac{1}{\psi^{-(k-\varphi+1)}} \ell^{(k)}(0) + \upsilon_{\varphi-1} \left(\frac{1}{\psi^{\varphi-1}} X(\psi) - \sum_{k=0}^{\varphi-2} \frac{1}{\psi^{-(k-\varphi+2)}} \ell^{(k)}(0) \right)\\ &\quad + \cdots + \upsilon_2 \left(\frac{1}{\psi^2} X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0) \right) + \upsilon_1 \left(\frac{1}{\psi^2} X(\psi) - \frac{1}{\psi}\ell(0) - \ell'(0) \right) + \upsilon_0 \ell(\psi). \end{aligned}$$

The relation (5.4) implies that

$$K[\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 \ell''(\mathfrak{z}) + \upsilon_1 \ell'(\mathfrak{z}) + \upsilon_0 \ell(\mathfrak{z})] = M(\psi) = K[m(\mathfrak{z})]$$

and thus

$$\ell^{(\varphi)}(\mathfrak{z}) + \upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z}) + \cdots + \upsilon_2 \ell''(\mathfrak{z}) + \upsilon_1 \ell'(\mathfrak{z}) + \upsilon_0 \ell(\mathfrak{z}) = m(\mathfrak{z}).$$

That is, $\ell(\mathfrak{z})$ is a solution of (1.3). Using (5.3) and (5.4), we obtain

$$\begin{aligned} K(h(\mathfrak{z})) - K(\ell(\mathfrak{z})) &= h(\psi) - X(\psi)\\ &= I(\psi) \frac{1}{\frac{1}{\psi^{\varphi}} + \upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}} + \cdots + \upsilon_2 \frac{1}{\psi^2} + \upsilon_1 \frac{1}{\psi} + \upsilon_0} = I(\psi)L(\psi) = K(i(\mathfrak{z}) * l(\mathfrak{z})), \end{aligned}$$

where $L(a) = \left\{ \frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})} \right\}$. This gives

$$l(\mathfrak{z}) = K^{-1} \left\{ \frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})} \right\} = e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Therefore, we have

$$K[h(\mathfrak{z})-\ell(\mathfrak{z})]=K[i(\mathfrak{z})*l(\mathfrak{z})]=K[i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}],$$

which yields

$$h(\mathfrak{z})-\ell(\mathfrak{z})=i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Furthermore,

$$\begin{aligned}|h(\mathfrak{z})-\ell(\mathfrak{z})|&=\left|i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right|\\&=\left|\int_0^{\mathfrak{z}}i(s)e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}ds\right|\\&\leqslant\int_0^{\mathfrak{z}}|i(s)||e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}|ds\\&\leqslant\frac{\sigma(\mathfrak{z})\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1-e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right)\leqslant K\sigma(\mathfrak{z})\epsilon,\end{aligned}$$

for all $\mathfrak{z}\geqslant 0$, where $K=\frac{1}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. □

Theorem 5.3. *Let* $m:[0,\infty)\to(0,\infty)$ *is a continuous function and* $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ *and* $\beta>0$ *are constants with* $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)>0$. *Then* (1.3) *has Mittag-Leffler-HUS for the class* H.

Proof. Suppose that $h\in H$ satisfies the Mittag-Leffler-HUS and a function $i:[0,\infty)\to K$ is defined by

$$i(\mathfrak{z})=h^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2h''(\mathfrak{z})+\upsilon_1h'(\mathfrak{z})+\upsilon_0h(\mathfrak{z})-m(\mathfrak{z}),$$

for all $\mathfrak{z}\geqslant 0$. It follows from the Mittag-Leffler-HUS that $|i(\mathfrak{z})|\leqslant E_\beta(\mathfrak{z})\epsilon$, $\forall\,\mathfrak{z}\geqslant 0$. Kamal transform of $i(\mathfrak{z})$ yields that

$$I(V)=K(i(\mathfrak{z}))=K[h^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2h''(\mathfrak{z})+\upsilon_1h'(\mathfrak{z})+\upsilon_0h(\mathfrak{z})-m(\mathfrak{z})].$$

If we set

$$\ell(\mathfrak{z})=e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}h(0)+\left(m(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right)\in H,$$

then $\ell(0)=h(0)$. Kamal transform of $\ell(\mathfrak{z})$ gives that

$$K(\ell(\mathfrak{z}))=X(\psi)=\frac{\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0)+\upsilon_{\varphi-1}\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)+\cdots+\upsilon_2\frac{1}{\psi}\ell(0)\upsilon_2\ell'(0)+\upsilon_1\ell(0)+M(\psi)}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}.\tag{5.5}$$

On the other hand,

$$\begin{aligned}K(\ell(\mathfrak{z}))&=K[\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})]\\&=\frac{1}{\psi^{\varphi}}X(\psi)-\sum_{k=0}^{\varphi-1}\frac{1}{\psi^{-(k-\varphi+1)}}\ell^{(k)}(0)+\upsilon_{\varphi-1}\left(\frac{1}{\psi^{\varphi-1}}X(\psi)-\sum_{k=0}^{\varphi-2}\frac{1}{\psi^{-(k-\varphi+2)}}\ell^{(k)}(0)\right)\\&\quad+\cdots+\upsilon_2\left(\frac{1}{\psi^2}X(\psi)-\frac{1}{\psi}\ell(0)-\ell'(0)\right)+\upsilon_1\left(\frac{1}{\psi^2}X(\psi)-\frac{1}{\psi}\ell(0)-\ell'(0)\right)+\upsilon_0\ell(\psi).\end{aligned}$$

By (5.2), we have

$$K[\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})]=M(\psi)=K[m(\mathfrak{z})]$$

and thus
$$\ell^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}\ell^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2\ell''(\mathfrak{z})+\upsilon_1\ell'(\mathfrak{z})+\upsilon_0\ell(\mathfrak{z})=m(\mathfrak{z}).$$
Hence $\ell(\mathfrak{z})$ is a solution of (1.3). By (5.5), we get

$$\begin{aligned}K(h(\mathfrak{z}))-K(\ell(\mathfrak{z}))=h(\psi)-X(\psi)&=I(\psi)\frac{1}{\frac{1}{\psi^{\varphi}}+\upsilon_{\varphi-1}\frac{1}{\psi^{\varphi-1}}+\cdots+\upsilon_2\frac{1}{\psi^2}+\upsilon_1\frac{1}{\psi}+\upsilon_0}\\&=I(\psi)\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}\\&=I(\psi)L(\psi)=K(i(\mathfrak{z})*l(\mathfrak{z})),\end{aligned}$$

where $L(a)=\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}$. This gives

$$l(\mathfrak{z})=K^{-1}\left\{\frac{\psi^{\varphi}}{(1+\upsilon_{\varphi-1}\psi+\cdots+\upsilon_2\psi^{\varphi-2}+\upsilon_1\psi^{\varphi-1}+\upsilon_0\psi^{\varphi})}\right\}=e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Therefore, we have

$$K[h(\mathfrak{z})-\ell(\mathfrak{z})]=K[i(\mathfrak{z})*l(\mathfrak{z})]=K[i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}],$$

which yields
$$h(\mathfrak{z})-\ell(\mathfrak{z})=i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}.$$

Furthermore,

$$\begin{aligned}|h(\mathfrak{z})-\ell(\mathfrak{z})|&=\left|i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right|\\&=\left|\int_0^{\mathfrak{z}}i(s)e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}ds\right|\\&\leqslant\int_0^{\mathfrak{z}}|i(s)||e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}|ds\\&\leqslant\frac{E_\beta(\mathfrak{z})\epsilon}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}\left(1-e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right)\leqslant KE_\beta(\mathfrak{z})\epsilon\end{aligned}$$

for all $\mathfrak{z}\geqslant 0$, where $K=\frac{1}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. This completes the proof. □

Theorem 5.4. *Let* $m:[0,\infty)\to(0,\infty)$ *is an continuous function,* $\sigma:[0,\infty)\to(0,\infty)$, *and* $\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0$ *and* $\beta>0$ *are constant with* $R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)>0$. *Then* (1.3) *has the Mittag-Liffler* σ*HUS for the class* H.

Proof. Let $h\in H$ and satisfy the σHUS. It is to be proved that there exist a solution $\ell:[0,\infty)\to K$ of (1.3) and a constant $K>0$ (independent of ϵ) such that

$$|h(\mathfrak{z})-\ell(\mathfrak{z})|\leqslant K\sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z}),\quad\forall\,\mathfrak{z}\geqslant 0.$$

Define $i:[0,\infty)\to K$ by

$$i(\mathfrak{z})=h^{(\varphi)}(\mathfrak{z})+\upsilon_{\varphi-1}h^{(\varphi-1)}(\mathfrak{z})+\cdots+\upsilon_2h''(\mathfrak{z})+\upsilon_1h'(\mathfrak{z})+\upsilon_0h(\mathfrak{z})-m(\mathfrak{z}),\quad\forall\,\mathfrak{z}\geqslant 0,$$

then $|i(\mathfrak{z})|\leqslant\sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z})$, $\forall\,\mathfrak{z}\geqslant 0$. By the proof of Theorem 5.3, there exists a solution $\ell:[0,\infty)\to K$ of (1.3) such that

$$|h(\mathfrak{z})-\ell(\mathfrak{z})|=\left|i(\mathfrak{z})*e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right|$$

$$
\begin{aligned}
&= \left| \int_0^{\mathfrak{z}} \mathfrak{i}(s) e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)} ds \right| \\
&\leqslant \int_0^{\mathfrak{z}} |\mathfrak{i}(s)| |e^{-(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)(\mathfrak{z}-s)}| ds \\
&\leqslant \frac{\sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z})}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)} \left(1 - e^{-R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)\mathfrak{z}}\right) \\
&\leqslant K\sigma(\mathfrak{z})\epsilon E_\beta(\mathfrak{z}), \quad \forall\, \mathfrak{z} \geqslant 0,
\end{aligned}
$$

where $K = \frac{1}{R(\upsilon_{\varphi-1}+\cdots+\upsilon_2+\upsilon_1+\upsilon_0)}$. This completes the proof. □

6. Conclusion

This manuscript discussed the Kamal transform for non-homogeneous and homogeneous linear differential equations. Using this unique integral transform, we resolve higher-order linear differential equations. Alternatively, it produced the conditions required for HUS by using the Kamal transform to show that a linear differential equation is stable. The Kamal transform method is more useful for investigating the stability problem for LDEs with constant coefficients, as this study also showed. The discussion of applications follows to illustrate our approach. Moreover, this paper provided a new method to investigate the HUS of linear differential equations.

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Author contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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