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Superlinear distributed deviating arguments to study second-order neutral differential equations

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Abstract

The main aim of this paper is to obtain new criteria for oscillating all solutions of second-order differential equations with distributed deviating arguments and superlinear neutral terms. Using the comparative and integral averaging techniques, we find new conditions for oscillation that generalize and add to some of the already found results. There are examples to show how important the main results are.

Keywords: Superlinear neutral term, distributed deviating argument, second-order, oscillation.

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1. Introduction

This article concerns the oscillatory properties of solutions of second-order differential equations with superlinear neutral terms and distributed deviating arguments

$$\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' + \int_a^b q(\zeta,s)y^{\delta}(\sigma(\zeta,s))ds = 0, \tag{E}$$

where $\zeta \in I = [\zeta_0, \infty)$, and

$$\psi(\zeta) = y(\zeta) + \int_c^d p(\zeta,s)y^{\alpha}(\tau(\zeta,s))ds.$$

We will make use of the following conditions:

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(H_1) α, β and δ are ratios of odd positive integers with $\alpha \geqslant 1$;
(H_1) $\eta \in C(I, (0, \infty)), p \in C(I \times [c, d], (0, \infty)), q \in C(I \times [a, b], (0, \infty))$, and

$$\int_{\zeta_0}^{\infty} \eta^{-\frac{1}{\beta}}(\zeta) d\zeta = \infty;$$

(H_1) $\tau, \sigma \in C(I \times I, \mathbb{R}), \tau(\zeta, s) \leqslant \zeta, \sigma(\zeta, s) \leqslant \zeta$ and $\lim_{\zeta\to\infty} \tau(\zeta, s) = \lim_{\zeta\to\infty} \sigma(\zeta, s) = \infty$.

By a solution of (E), we mean a function $y \in C^1([\zeta_y, \infty), \mathbb{R}), \zeta_y \geqslant \zeta_0$, which has the property $\eta(\zeta) (\psi'(\zeta))^{\beta} \in C^1([\zeta_0, \infty), \mathbb{R})$, and satisfies (E) on $[\zeta_y, \infty)$. We consider only those solutions y of (E) satisfying $\sup\{|y(\zeta)| : \zeta \geqslant \zeta_y\} > 0$ for all $\zeta \geqslant \zeta_y$. If such a solution has infinitely many zeros in I, it is called oscillatory; otherwise, it is nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

In dynamical models, deviation and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [11, 18, 19, 26]. The oscillatory behavior of solutions of various classes of neutral differential equations with or without distributed deviating arguments has been widely investigated by many of the authors using different methods, we refer the reader to [1–8, 12, 13, 15, 16, 20–22, 24, 25, 28, 29, 35] and the references therein. In recent years, the authors have considered the following equation in [9, 14, 33],

$$\left(\eta(\zeta) \left(\psi'(\zeta)\right)^{\beta}\right)' + f(\zeta) y^{\delta}(\sigma(\zeta)) = 0, \zeta \geqslant \zeta_0, \tag{E_1}$$

where $\psi(\zeta) = y(\zeta) + p(\zeta) y^{\alpha}(\tau(\zeta))$, and obtained criteria for the oscillation of (E_1) in the case $\alpha \geqslant 1$. Very recently, in [31], the authors considered equation (E) with $0 < \alpha \leqslant 1$ and established oscillation results.

However, no oscillation conditions are currently available for the equation (E) when $\alpha \geqslant 1$. Because of this, observation showed a renewed interest in investigating the oscillatory behavior of solutions to (E). As a consequence, the findings that were acquired via the use of this article are novel and add to those found in [9, 10, 13, 23, 27, 30, 31, 33–35, 37].

2. Main results

For easy reference, we consider the following notation:

$$G(\zeta) = \int_a^b q(\zeta, s) \left[1 - A^{\alpha-1}(\sigma(\zeta, s)) \rho^{\alpha-1}(\sigma(\zeta, s)) \int_c^d p(\sigma(\zeta, s), \nu) d\nu\right] ds,$$

where $\rho(\zeta)$ is a positive real decreasing function with $\rho(\zeta) \to 0$ as $\zeta \to \infty$,

$$A(\zeta) = \int_{\zeta_0}^{\zeta} \eta^{-\frac{1}{\beta}}(s) ds, \ \bar{A}(\zeta) = A(\zeta) + \frac{1}{\beta} \int_{\zeta_1}^{\zeta} A(u) A^{\beta}(\sigma(u, a)) G(u) du, \ B(\zeta) = \exp\left(-\beta \int_{\sigma(\zeta)}^{\zeta} \frac{du}{\bar{A}(u) \eta^{\frac{1}{\beta}}(u)}\right).$$

The following lemmas show that our main conclusions are correct.

Lemma 2.1 ([6]). *If y is a positive solution of (E) on $[\zeta_0, \infty)$, then $\exists\ \zeta_1 \in [\zeta_0, \infty)$ such that*

$$\psi(\zeta) > 0, \psi'(\zeta) > 0, \left(\eta(\zeta) \left(\psi'(\zeta)\right)^{\beta}\right)' \leqslant 0 \tag{2.1}$$

on $[\zeta_1, \infty)$

Lemma 2.2. *If y is a positive solution of (E) on $[t_0, \infty)$, then*

$$\frac{\psi(\zeta)}{A(\zeta)} \text{ is decreasing for } \zeta \geqslant \zeta_1 \geqslant \zeta_0.$$

Proof. Let y be a positive solution of (E) on $[\zeta_0, \infty)$, then by Lemma 2.1, we see that (2.1) holds. From (2.1), we conclude that $\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)$ is positive and decreasing, $\forall\ \zeta \geqslant \zeta_1$. So

$$\psi(\zeta) = \psi(\zeta_1) + \int_{\zeta_1}^{\zeta} \frac{\eta^{\frac{1}{\beta}}(s)\psi'(s)}{\eta^{\frac{1}{\beta}}(s)} ds \geqslant A(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta). \tag{2.2}$$

From this, it is not difficult to deduce that $\frac{\psi(\zeta)}{A(\zeta)}$ is decreasing, $\forall\ \zeta \geqslant \zeta_1$. □

Lemma 2.3. *Let y be a positive solution of* (E)*. Then the function ψ satisfies*

$$\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' \leqslant -G(\zeta)\psi^{\delta}(\sigma(\zeta, a)), \tag{2.3}$$

$$\psi(\zeta) \geqslant \bar{A}(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta), \quad \textit{if } \delta = \beta, \tag{2.4}$$

and

$$\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' \leqslant -G(\zeta)B(\zeta)\psi^{\beta}(\zeta), \quad \textit{if } \delta = \beta. \tag{2.5}$$

Proof. Let y be a positive solution of (E) on $[\zeta_0, \infty)$. Then $\exists$ a $\zeta_1 \geqslant \zeta_0$ such that $y(\tau(\zeta, \nu)) > 0$ and $y(\sigma(\zeta, s)) > 0$ for $\zeta \geqslant \zeta_1, \nu \in [c, d]$ and $s \in [a, b]$. From Lemma 2.1, we have (2.1). Hence, by the definition of $\psi(\zeta)$, we have

$$y(\zeta) = \psi(\zeta) - \int_c^d p(\zeta, \nu) y^{\alpha}(\tau(\zeta, \nu)) d\nu \geqslant \psi(\zeta) - \int_c^d p(\zeta, \nu)\psi^{\alpha}(\tau(\zeta, \nu)) d\nu. \tag{2.6}$$

Now using Lemma 2.2 in (2.6), one obtains

$$y(\zeta) \geqslant \left[\psi(\zeta) - \psi^{\alpha}(\zeta)\int_c^d p(\zeta, \nu) d\nu\right] \geqslant \psi(\zeta)\left[1 - \psi^{\alpha-1}(\zeta)\int_c^d p(\zeta, \nu) d\nu\right]. \tag{2.7}$$

Since $\frac{\psi(\zeta)}{A(\zeta)}$ is decreasing and $\rho(\zeta)$ is increasing and tending to infinity, we have $\frac{\psi(\zeta)}{A(\zeta)} \leqslant \rho(\zeta), \forall\ \zeta \geqslant \zeta_2 \geqslant \zeta_1$. Using this in (2.7) yields

$$y(\zeta) \geqslant \psi(\zeta)\left[1 - A^{\alpha-1}(\zeta)\rho^{\alpha-1}(\zeta)\int_c^d p(\zeta, s) d\nu\right],$$

which, with equation (E), implies that

$$\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' \leqslant -\int_a^b q(\zeta, s)\psi^{\delta}(\sigma(\zeta, s))\left[1 - \left(A(\sigma(\zeta, s))\rho(\sigma(\zeta, s))\right)^{\alpha-1}\int_c^d p(\sigma(\zeta, s), \nu) d\nu\right]^{\delta} ds.$$

Since $\psi'(\zeta) > 0$ and $\frac{\partial}{\partial s}\sigma(\zeta, s) > 0$, we see that $\psi(\zeta, s) \geqslant \psi(\zeta, a)$ and so

$$\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' \leqslant -G(\zeta)\psi^{\delta}(\sigma(\zeta, a)),$$

which proves (2.3). Using the chain rule and basic math, it's clear that

$$A(\zeta)\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)' = -\beta\left(\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)\right)^{\beta-1}\frac{d}{d\zeta}\left(\psi(\zeta) - A(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)\right). \tag{2.8}$$

Combining (2.3) and (2.8), we obtain

$$\frac{d}{d\zeta}\left(\psi(\zeta) - A(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)\right) \geqslant \frac{1}{\beta}A(\zeta)\left(\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)\right)^{1-\beta}G(\zeta)\psi^{\delta}(\sigma(\zeta, a)).$$

Integrating from ζ_1 to ζ, we have

$$\psi(\zeta) \geqslant A(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta) + \frac{1}{\beta}\int_{t_1}^{\zeta} A(u)G(u)\left(\eta^{\frac{1}{\beta}}(u)\psi'(u)\right)^{1-\beta}\psi^{\delta}(\sigma(u,a))du. \tag{2.9}$$

Using (2.2) in (2.9) yields

$$\begin{aligned}\psi(\zeta) &\geqslant A(\zeta)\eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta) + \frac{1}{\beta}\int_{\zeta_1}^{\zeta} A(u)G(u)\left(\eta^{\frac{1}{\beta}}(u)\psi'(u)\right)^{1-\beta}A^{\delta}(\sigma(u,a))\left(\eta^{\frac{1}{\beta}}(\sigma(u,a))\psi'(\sigma(u,a))\right)^{\delta}du\\ &\geqslant \eta^{\frac{1}{\beta}}(\zeta)\psi'(\zeta)\left[A(\zeta) + \frac{1}{\beta}\int_{\zeta_1}^{\zeta} A(u)A^{\delta}(\sigma(u,a))G(u)\left(\eta^{\frac{1}{\beta}}(u)\psi'(u)\right)^{\delta-\beta}\right]du.\end{aligned}$$

Setting $\beta = \delta$, then we obtain (2.4). From (2.4), we have

$$\frac{\psi'(\zeta)}{\psi(\zeta)} \geqslant \frac{1}{\bar{A}(\zeta)\eta^{\frac{1}{\beta}}(\zeta)}.$$

Integrating from $\sigma(\zeta,a)$ to ζ, yields

$$\frac{\psi(\sigma(\zeta,a))}{\psi(\zeta)} \geqslant \exp\left(-\int_{\sigma(\zeta,a)}^{\zeta}\frac{du}{\bar{A}(u)\eta^{\frac{1}{\beta}}(u)}\right),$$

which with (2.3) for $\delta = \beta$ gives

$$\frac{\left(\eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}\right)'}{\psi^{\beta}(\zeta)} \leqslant -G(\zeta)\left(\frac{\psi(\sigma(\zeta,a))}{\psi(\zeta)}\right)^{\beta} \leqslant -G(\zeta)B(\zeta),$$

which proves (2.5). Hence proved. □

Now, we are going to study oscillation criterion for (E).

Theorem 2.4. *Let $\delta = \beta$ and $\exists$ a positive increasing function $\rho(\zeta)$ such that $\rho(\zeta) \to \infty$ as $\zeta \to \infty$ and $G(\zeta) > 0$ for $\zeta \geqslant \zeta_1$. If the first-order delay differential equation*

$$w'(\zeta) + \bar{A}^{\beta}(\zeta)(\sigma(\zeta,a))G(\zeta)w(\sigma(\zeta,a)) = 0 \tag{2.10}$$

is oscillatory, then (E) is oscillatory.

Proof. Assume the contrary that (E) has a nonoscillatory solution y on $[\zeta_0,\infty)$. Without loss of generality, we assume that $y(t) > 0$, $y(\tau(\zeta,\nu)) > 0$, and $y(\sigma(\zeta,s)) > 0$ for $\zeta \geqslant \zeta_1 \geqslant \zeta_0, \nu \in [c,d]$ and $s \in [a,b]$. From Lemma 2.3, we have (2.3) and (2.4). Combining (2.3) and (2.4), we see that $w(t) = \eta(\zeta)\left(\psi'(\zeta)\right)^{\beta}$ is a positive solution of the first-order delay differential inequality

$$w'(\zeta) + \bar{A}^{\beta}(\zeta)(\sigma(\zeta,a))G(\zeta)w(\sigma(\zeta,a)) \leqslant 0.$$

Contrary to Theorem 1 of [37], a positive solution exists to the equivalent delay differential equation (2.10). Hence proved. □

Corollary 2.5. *Let $\delta = \beta$ and $\exists$ a positive increasing function $\rho(\zeta)$ such that $\rho(\zeta) \to \infty$ as $\zeta \to \infty$ and $G(\zeta) > 0$, $\forall\zeta \geqslant \zeta_1$. If*

$$\limsup_{\zeta\to\infty}\int_{\sigma(\zeta,a)}^{\zeta}\bar{A}^{\beta}(\sigma(u,a))G(u)du > 1, \frac{\partial}{\partial\zeta}\sigma(\zeta,s) \geqslant 0 \tag{2.11}$$

or

$$\liminf_{\zeta\to\infty}\int_{\sigma(\zeta,a)}^{\zeta}\bar{A}^{\beta}(\sigma(u,a))G(u)du > \frac{1}{e}, \tag{2.12}$$

then (E) is oscillatory.

Proof. It's well knowledge that (2.11) or (2.12) ensures the oscillation of (2.10) (Theorem 2.1.1 of [15]). Now the conclusion follows from Theorem 2.4. Hence proved, then (E) is oscillatory. □

Theorem 2.6. *Assume that ∃ a positive increasing function* $\rho(\zeta)$ *such that* $\rho(\zeta) \to \infty$ *as* $\zeta \to \infty$ *and* $G(\zeta) > 0$ *for all* $\zeta \geqslant \zeta_1$. *If the first-order delay differential equation*

$$w'(\zeta) + G(\zeta)A^{\delta}(\sigma(\zeta, a))w^{\frac{\delta}{\beta}}(\sigma(\zeta, a)) = 0 \tag{2.13}$$

is oscillatory, then (E) *is oscillatory.*

Proof. Assume the contrary that (E) has a nonoscillatory solution x on $[\zeta_0, \infty)$. Without loss of generality, we assume that ∃ a $\zeta_1 \geqslant \zeta_0$ such that $y(\zeta) > 0$, $y(\tau(\zeta, \nu)) > 0$, and $y(\sigma(\zeta, s)) > 0$ for $\zeta \geqslant \zeta_1, \nu \in [c, d]$ and $s \in [a, b]$. Combining (2.2) and (2.3), we obtain that $w(\zeta) = \eta(\zeta)\,(\psi'(\zeta))^{\beta}$ is a positive solution of the first-order delay differential inequality

$$w'(\zeta) + G(\zeta)A^{\delta}(\sigma(\zeta, a))w^{\frac{\delta}{\beta}}(\sigma(\zeta, a)) \leqslant 0.$$

According to Theorem 2.1 of [36], the corresponding delay differential equation (2.13) has a positive solution, which is a contradiction. □

Corollary 2.7. *Let* $\delta < \beta$ *and ∃ a positive increasing continuous function* $\rho(\zeta)$ *such that* $\rho(\zeta) \to \infty$ *as* $\zeta \to \infty$ *and* $G(\zeta) > 0$, $\forall \zeta \geqslant \zeta_1$. *If*

$$\int_{\zeta_0}^{\infty} G(\zeta)A^{\delta}(\sigma(\zeta, a))d\zeta = \infty, \tag{2.14}$$

then (E) *is oscillatory.*

Proof. It's well knowledge that condition (2.14) ensures oscillation of (2.14) (Theorem 2 of [17]). The conclusion now follows from Theorem 2.6. □

Corollary 2.8. *Let* $\delta > \beta$ *and ∃ a positive increasing continuous function* $\rho(\zeta)$ *such that* $\rho(\zeta) \to 0$ *as* $\zeta \to \infty$ *and* $G(\zeta) > 0$ *for all* $\zeta \geqslant \zeta_1$. *If* $\sigma(\zeta, a) = \zeta - k$, *where* $k > 0$ *is a constant, and ∃* λ *such that* $\lambda > \frac{1}{k} \ln \frac{\delta}{\beta}$ *and*

$$\liminf_{\zeta \to \infty} \left(G(\zeta)A^{\delta}(\sigma(\zeta, a)) \exp\left(-e^{\lambda\zeta}\right) \right) > 0, \tag{2.15}$$

then (E) *is oscillatory.*

Proof. It's well knowledge that condition (2.15) ensures oscillation of (2.13) (Theorem 3 of [32]). The conclusion now follows from Theorem 2.6. □

Corollary 2.9. *Let* $\delta > \beta$ *and ∃ a positive increasing continuous function* $\rho(\zeta)$ *such that* $\rho(\zeta) \to \infty$ *as* $t \to \infty$ *and* $G(\zeta) > 0$ *for all* $\zeta \geqslant \zeta_1$. *If* $\sigma(\zeta, a) = \theta\zeta$, $\theta \in (0, 1)$, *and ∃* $\mu > \frac{-\ln \frac{\delta}{\beta}}{\ln \theta}$ *such that*

$$\liminf_{\zeta \to \infty} \left[G(\zeta)A^{\delta}(\sigma(\zeta, a)) \exp\left(-\zeta^{\mu}\right)\right] > 0, \tag{2.16}$$

then (E) *is oscillatory.*

Proof. It's good knowing that condition (2.16) ensures oscillation of (2.13) (Theorem 4 of [32]). The conclusion now follows from Theorem 2.6. □

3. Examples

In this part, we illustrate the relevance of our primary findings with two cases.

Example 3.1. Consider the equation

$$\left(y(\zeta)+\frac{1}{\zeta^3}y^3(\zeta-2)\right)''+\int_1^2 \frac{\zeta+s}{\zeta}y\left(\frac{\zeta s}{2}\right)ds=0, \quad \zeta \geqslant 1. \tag{3.1}$$

Here $\eta(\zeta)=1$, $p(\zeta,s)=\frac{1}{\zeta^3}$, $\tau(\zeta,s)=\zeta-2$, $q(\zeta,s)=\frac{\zeta+s}{\zeta}$, $\sigma(\zeta,s)=\frac{\zeta s}{2}$, $a=1$, $b=2$, $c=1$, $d=2$, $\alpha=\frac{1}{3}$, and $\beta=\delta=1$. A simple calculation shows that $A(\zeta)\approx\zeta$ and conditions (H_1)-(H_3) hold. By $\rho(\zeta)=\zeta^{1/2}$, we see that $G(\zeta)\approx\frac{1}{2\zeta}$, $\bar{A}(\zeta)\approx\frac{\zeta^2}{8}$. Now condition (2.11) becomes

$$\limsup_{\zeta\to\infty}\int_{\zeta/2}^{\zeta}\left(\frac{u^2}{32}\right)\left(\frac{1}{2u}\right)du=\infty>1 \quad \text{and} \quad \frac{\partial}{\partial\zeta}\sigma(\zeta,s)=\frac{1}{2}>0,$$

that is, condition (2.11) holds. Also, the condition (2.12) holds. Hence by Corollary 2.5, equation (3.1) is oscillatory.

Example 3.2. Consider the equation

$$\left(y(\zeta)+\int_1^2\frac{1}{\zeta^3}y^3\left(\frac{\zeta s}{3}\right)ds\right)''+\int_0^1 \zeta s y^{\frac{1}{3}}\left(\frac{\zeta s}{2}\right)ds=0, \quad \zeta \geqslant 1. \tag{3.2}$$

Here $\eta(\zeta)=1$, $p(\zeta,s)=\frac{1}{\zeta^3}$, $\tau(\zeta,s)=\frac{\zeta s}{3}$, $q(\zeta,s)=\zeta s$, $\sigma(\zeta,s)=\frac{\zeta s}{2}$, $\alpha=3$, $\beta=1$, $\delta=\frac{1}{3}$, $c=1$, $d=2$, $a=0$, and $b=1$. A simple calculation yields that $A(\zeta)\approx\zeta$ and conditions (H_1)-(H_3) hold. By $\rho(\zeta)=\zeta^{1/2}$, we have $G(\zeta)\approx\frac{\zeta}{4}$. The condition (2.14) becomes

$$\int_1^\infty \frac{\zeta^{4/3}}{(4)2^{1/3}}d\zeta=\infty.$$

Hence by Corollary 2.7, (3.2) is oscillatory.

4. Conclusion

In the present investigation, we have determined the criteria for oscillating all solutions to (E) using the comparison approach. Our findings add to those published in the literature for neutral differential equations of the second order with or without distributed deviating arguments and superlinear neutral terms.

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