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Soft algebraic structures embedded with soft members and soft elements: an abstract approach

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Abstract

As a new area of study in pure mathematics, the theory of soft sets is expanding by redefining fundamental ideas as algebraic structures, such as soft groups, soft rings, and soft fields. It also finds applications in other domains, regarding data analysis and decision-making. This study manipulates soft members and soft elements to explore soft structures from a traditional point of view, making it easier to comprehend soft algebraic structures. The soft inverse of a soft member and the soft identity member are generalized for any soft group, and a method to count the number of possible soft subgroups of a soft group is also provided.

Keywords: Soft member, mathematical model, soft group, soft ring, soft field, computational model.

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1. Introduction

Set theory was first developed by Cantor in the 1870s, with the assistance of Dedekind [14]. Numerous significant real-world applications demonstrated its significance [23]. Sets can be used to symbolize an assortment of boys or girls in a class or a group of odd-numbered book chapters. There are just a few membership variations in these sets, where a member can either belong to a set or not. In 1965, Zadeh [37] constructed fuzzy sets by extending this fundamental idea to handle cases when each element of a set has a partially specified membership. Fuzzy sets, for instance, are used to show the area that each piece of furniture takes up in a room. Molodstove [24] generalized fuzzy set as soft set (SSt). Through data analysis and decision-making computations, this generalization enables the handling of uncertainties in a variety of domains, including engineering, medical science, economics, social sciences, and environmental science [21, 30, 32]. Soft subset (SSbS) and soft superset, equivalent SSts, null and absolute SSts, as well as some set operations like union, intersection, complement, binary operations AND

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and OR, were defined by Maji et al. [20] in 2003. They also verified some results, such as De Morgan's laws for SSts. Later in 2009, Ali et al. [3] defined a few additional operations, such as restricted and extended intersection, restricted and extended union, and restricted difference. In 2010, a few important entities over SSts appeared as Çağman and Enginoğlu [12] introduced matrices, Babitha and Sunil [10, 11] introduced relations and functions over SSts, followed by anti-symmetric, ordering, transitive closure, and Yang and Guo [35] presented the closure and kernel of soft mappings and soft relations. More contributions towards the operations (extended version) and verification of some properties with respect to these operations were made by different mathematicians [7, 15, 17, 18, 25, 26, 33, 34, 38]. While set theory was developing by defining algebraic entities as operations, matrices, relations, and functions, as discussed above, on the other side, mathematicians were trying to develop its connection with algebraic structures as well. As a result of these efforts, in 2007, Aktaş and Çağman [2] made it possible to define a soft group (SGp). In 2010, Acar et al. [1] established the definition of soft ring (SRg). Later in 2012, Aslam and Qurashi [9] progressed the idea presented by Aktaş and Çağman and defined some sub-concepts of groups concerning SGp as normal, abelian, cyclic, factor, and maximal normal SGps as well as soft subgroup (SSbGp). Das and Samantha [13] introduced soft real numbers that motivated other researchers such as Goldar and Ray, and they discussed SSts, SGps [27], and soft topological axioms [16] with the help of soft elements (SEt). There is always scope for improvement, which helps researchers work toward new findings and the development of previous concepts. The contributions of scholars [4–6, 8, 22, 28, 29, 36] regarding the introduction of algebraic structures based on generalized hybrid set structures are worth noting and acknowledgeable. Saeed et al. [31] introduced a new way to discuss soft algebraic structures using soft members (SMb) and SEt. They have also defined the Cartesian soft product, SSt relation, and soft binary operations that played a vital role in observing the SGp as a classical algebraic group and made it easy to discuss its properties using Cayley's table. This approach gave rise to a new horizon and created a lot of space for new research, e.g., observing the soft algebraic structures in comparison to the classical algebraic structures, including SSbGp, SRg, and soft field, etc. It motivated us to present the SSbGp, SRg, and soft field in the form of a classical algebraic subgroup, ring, and field. Soft set theory demonstrated its value by solving numerous practical issues that conventional sets were unable to handle. Soft sets, together with their various hybrids and generalizations, offered up new possibilities and allowed for the modeling of various scenarios, particularly those involving uncertainty. The definition of several soft algebraic structures in an abstract manner led to the establishment of soft set theory. Soft elements and soft members could be used to analyze these soft structures traditionally. The contribution of Saeed et al. [31] have been extended in this paper. By using soft elements and soft members, we have developed soft subgroups, soft rings, and soft fields, which enable us to examine the features of these soft structures from a classical perspective. The salient contributions are as follows.

1. The existing works of literature on soft sets, soft elements, soft members, and soft groups have been modified.
2. To investigate soft structures from a conventional perspective, the soft members and soft elements have been adjusted, which facilitates the understanding of soft algebraic structures.
3. A refinement of the soft inverse of a soft member and the soft identity member for any soft group has been communicated, along with a technique to enumerate all potential soft subgroups of a given soft group.

The rest of the paper is organized as follows. Section 2 provides foundational definitions to support the key conclusions of this research. In the upcoming sections, SSbGp, SRg, and soft field are redefined using SMbs in light of classical algebraic structures. Constructing examples helps to clarify the ideas, and the new notations are used to verify some of the associated pre-defined theorems. A formula for determining the number of potential SSbGps in an SGp is given in Section 3. A theorem for SRg that is comparable to the one defined for classical rings and given in Section 4 can be defined because of the suggested definition. Soft algebraic structures parallel to the field are presented in Section 5. The last section wraps up the work and inspires more research.

2. Preliminaries

In order to comprehend the paper's primary findings, some basic definitions, mainly following Saeed et al. [31], are presented in this section. Let's define few notations that we have used for this paper. The set of alternatives and its power set are represented as $\mathcal{U}$ and $\mathcal{P}(\mathcal{U})$, respectively. $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are representing the subsets of $\mathcal{E}$, the collection of parameters or attributes. The mappings from subsets of $\mathcal{E}$ to $\mathcal{P}(\mathcal{U})$ are denoted by symbols θ, ϕ, and ψ, whereas $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ have been used to represent the mappings from subsets of $\mathcal{E}$ to $\mathcal{U}$.

In 1999, Molodtsov [24] put forward idea of SSt to make the existing structures dealing with vagueness compatible with parameterization.

Definition 2.1 ([24])**.** The pair $(\theta, \mathcal{E}) = \{\theta(\tilde{\rho}) \in \mathcal{P}(\mathcal{U}) : \tilde{\rho} \in \mathcal{E}\}$ is named as SSt, where $\theta : \mathcal{E} \to \mathcal{P}(\mathcal{U})$.

Definition 2.2 ([20])**.** The SSt $(\theta, \mathcal{E})$ is named as relative null SSt if $\theta(\tilde{\rho}) = \varnothing$, for all $\tilde{\rho} \in \mathcal{X}$.

Definition 2.3 ([20])**.** Let $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ are SSts over $\mathcal{U}$, then $(\theta, \mathcal{X})$ is SSbS of $(\phi, \mathcal{Y})$ if

1. $\mathcal{X} \subseteq \mathcal{Y}$;
2. $\theta(\tilde{\rho})$ and $\phi(\tilde{\rho})$ are identical approximations for all $\tilde{\rho} \in \mathcal{X}$.

Saeed et al. [31] conducted a thorough analysis of the idea of SEts and SMbs in SSts in 2020.

Definition 2.4 ([31])**.** Let $(\theta, \mathcal{X})$ be a SSt and $\tilde{\alpha} : \mathcal{X} \to \mathcal{U}$ be a mapping such that $\tilde{\alpha}(\tilde{\rho}) \in \theta(\tilde{\rho})$ where $\tilde{\rho} \in \mathcal{X}$. Then the SMb of $(\theta, \mathcal{X})$ be termed as $\tilde{q} = \{(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho})) : \tilde{\rho} \in \mathcal{X}, \tilde{\alpha}(\tilde{\rho}) \in \theta(\tilde{\rho}) \neq \varnothing\}$ and $(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho}))$, $\tilde{\rho} \in \mathcal{X}$ is named as SEt. Clearly, the cardinality of each SMb of $(\theta, \mathcal{X})$ equals the cardinality of $\mathcal{X}$ and by multiplying the cardinality of $\theta(\tilde{\rho})$, for all $\tilde{\rho} \in \mathcal{X}$, number of SMbs in $(\theta, \mathcal{X})$ is obtained.

Definition 2.5 ([31])**.** Let $(\theta, \mathcal{X})$ be a SSt, its SMb $\tilde{q}$ is given as $\tilde{q} = \{(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho})) : \tilde{\alpha}(\tilde{\rho}) \notin \theta(\tilde{\rho})$, for all $\tilde{\rho} \in \mathcal{X}\}$. In this definition, $\tilde{\alpha}(\tilde{\rho})$ and $\theta(\tilde{\rho})$ are same as discussed in Definition 2.4.

Definition 2.6 ([31])**.** Let $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ are SSts over $\mathcal{U}$ and $\tilde{q}$, $\tilde{y}$ be two SMbs of $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$, respectively. Then $\tilde{q}$ is referred to be a sub-SMb of $\tilde{y}$ if

1. $\mathcal{X} \subseteq \mathcal{Y}$;
2. for all $\tilde{\rho} \in \mathcal{X}$ each SEt $(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho}))$ of $\tilde{q}$ is a SEt of $\tilde{y}$.

Definition 2.7 ([31])**.** The SSt using SMbs takes the form $(\theta, \mathcal{X}) = \{\tilde{q} : \tilde{q} = \{(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho})), \tilde{\alpha} : \mathcal{X} \to \mathcal{U}, \tilde{\alpha}(\tilde{\rho}) \in \theta(\tilde{\rho}) \neq \varnothing$, for all $\tilde{\rho} \in \mathcal{X}\}\}$.

Definition 2.8 ([31])**.** A SSt $(\theta, \mathcal{X})$ is regarded as non-null SSt if it has non-empty support where $\mathrm{supp}((\theta, \mathcal{X})) = \{\tilde{\rho} \in \mathcal{X} : \theta(\tilde{\rho}) \neq \varnothing\}$.

Definition 2.9 ([31])**.** Let $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ are SSts over $\mathcal{U}$, then $(\theta, \mathcal{X})$ is SSbS of $(\phi, \mathcal{Y})$ if (i) $\mathcal{X} \subseteq \mathcal{Y}$; (ii) each SMb $\tilde{q}$ of $(\theta, \mathcal{X})$ is a sub-SMb of at least one SMb $\tilde{y}$ of $(\phi, \mathcal{Y})$.

Let $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ are SSts over $\mathcal{U}$, then following are some of the SSt operations recalled from Ali et al. [3].

Definition 2.10. $(\theta, \mathcal{X}) \tilde{\cup}_R (\phi, \mathcal{Y})$, the restricted union of $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ is termed as a SSt $(\psi, \mathcal{Z}) = \{\tilde{z} : \tilde{z} = (\tilde{\rho}, \tilde{\gamma}(\tilde{\rho})), \tilde{\gamma}(\tilde{\rho}) \in \tilde{\alpha}(\tilde{\rho}) \cup \tilde{\beta}(\tilde{\rho})$, for all $\tilde{\rho} \in \mathcal{Z}\}$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}$.

Definition 2.11. $(\theta, \mathcal{X}) \tilde{\cap}_R (\phi, \mathcal{Y})$, the restricted intersection of $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ is termed as a SSt $(\psi, \mathcal{Z}) = \{\tilde{z} : \tilde{z} = (\tilde{\rho}, \tilde{\gamma}(\tilde{\rho})), \tilde{\gamma}(\tilde{\rho}) \in \tilde{\alpha}(\tilde{\rho}) \cap \tilde{\beta}(\tilde{\rho})$, for all $\tilde{\rho} \in \mathcal{Z}\}$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}$.

Definition 2.12. $(\theta, \mathcal{X}) \tilde{\cap}_R (\phi, \mathcal{Y})$, the extended intersection of $(\theta, \mathcal{X})$ and $(\phi, \mathcal{Y})$ is termed as a SSt $(\psi, \mathcal{Z}) = \{\tilde{z} : \tilde{z} = (\tilde{\rho}, \tilde{\gamma}(\tilde{\rho}))$, for all $\tilde{\rho} \in \mathcal{Z}\}$, where $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ and

$$\tilde{\gamma}(\tilde{\rho}) \in \begin{cases} \theta(\tilde{\rho}), & \tilde{\rho} \in \mathcal{X}, \tilde{\rho} \notin \mathcal{Y}, \\ \phi(\tilde{\rho}), & \tilde{\rho} \in \mathcal{Y}, \tilde{\rho} \notin \mathcal{X}, \\ \theta(\tilde{\rho}) \cap \phi(\tilde{\rho}), & \tilde{\rho} \in \mathcal{X} \cap \mathcal{Y}. \end{cases}$$

The following definitions are recalled from Aslam et al. [9] and Goldar & Ray [27].

Definition 2.13. Let $(\theta, \mathcal{X})$ be a non-null SSt and $*$ be a binary operation from $\mathcal{U} \times \mathcal{U}$ to $\mathcal{U}$ then corresponding soft binary operation $\tilde{*}$ from $(\theta, \mathcal{X}) \times (\theta, \mathcal{X})$ to $(\theta, \mathcal{X})$ is termed as for any $(\tilde{q}_i, \tilde{q}_j) \in (\theta, \mathcal{X}) \times (\theta, \mathcal{X})$, $\tilde{q}_i \tilde{*} \tilde{q}_j = \{(\tilde{\rho}, \theta_k(\tilde{\rho}) * \theta_l(\tilde{\rho})), \tilde{\rho} \in \mathcal{X}\}$.

Definition 2.14. A SMb $\tilde{q}_e$ is named as identity element with reference to the soft binary operation $\tilde{*}$ if $\tilde{q}_e \tilde{*} \tilde{q} = \tilde{q} \tilde{*} \tilde{q}_e$, for all $\tilde{q} \in (\theta, \mathcal{X})$.

Definition 2.15. A soft binary operation $\tilde{*}$ from $(\theta, \mathcal{X}) \times (\theta, \mathcal{X})$ to $(\theta, \mathcal{X})$ is referred to be softly commutative if $\tilde{q}_i \tilde{*} \tilde{q}_j = \tilde{q}_j \tilde{*} \tilde{q}_i$, for all $(\tilde{q}_i, \tilde{q}_j) \in (\theta, \mathcal{X}) \times (\theta, \mathcal{X})$.

Definition 2.16. A non-null SSt $(\theta, \mathcal{X})$ is named as SGp if (i) the set of alternatives $\mathcal{U}$ is a group; (ii) for every $\tilde{\rho} \in \mathcal{X}, \theta(\tilde{\rho})$ is a subgroup of $\mathcal{U}$.

Definition 2.17. A non-null SSt $(\theta, \mathcal{X})$ is regarded as SGp denoted by $((\theta, \mathcal{X}), \tilde{*})$ if the set of alternatives $(\mathcal{U}, *)$ is a group, θ is mapping elements of $\mathcal{X}$ to its subgroups and following properties are satisfied.

1. For all $\tilde{q}_i, \tilde{q}_j \in (\theta, \mathcal{X}), \tilde{q}_i \tilde{*} \tilde{q}_j \in (\theta, \mathcal{X})$.
2. For all $\tilde{q}_i, \tilde{q}_j, \tilde{q}_k \in (\theta, \mathcal{X}), \tilde{q}_i \tilde{*} (\tilde{q}_j \tilde{*} \tilde{q}_k) = (\tilde{q}_i \tilde{*} \tilde{q}_j) \tilde{*} \tilde{q}_k$.
3. Soft identity element $\tilde{q}_e$ exists in $(\theta, \mathcal{X})$.
4. For each $\tilde{q}_i \in (\theta, \mathcal{X})$ there exist $\tilde{q}_i^{-1} \in (\theta, \mathcal{X})$ such that $\tilde{q}_i^{-1} \tilde{*} \tilde{q}_i = \tilde{q}_e = \tilde{q}_i \tilde{*} \tilde{q}_i^{-1}$.

Moreover, $((\theta, \mathcal{X}), \tilde{*})$ is named as abelian SGp if the soft binary operation $\tilde{*}$ is softly commutative.

In the following sections, the soft structures are defined using SMbs to present a ground work for detailed discussion of soft algebraic structures parallel to classical algebraic structures.

3. Soft algebraic structure parallel to subgroup

Let $(\theta, \mathcal{X})$ as well as $(\phi, \mathcal{Y})$ be the SGps over $(\mathcal{U}, *)$, a group over the binary operation $*$ then $(\theta, \mathcal{X})$ is named as a SSbGp of $(\phi, \mathcal{Y})$ if (i) $\mathcal{X} \subseteq \mathcal{Y}$; (ii) $\theta(\tilde{\rho}) \subseteq \phi(\tilde{\rho})$ for all $\tilde{\rho} \in \mathcal{X}$. The following section explains the SSbGp using SMbs and SEts.

Definition 3.1. Let $((\theta, \mathcal{X}), \tilde{*})$ as well as $((\phi, \mathcal{Y}), \tilde{*})$ be the SGps over an algebraic group $(\mathcal{U}, *)$, then $((\theta, \mathcal{X}), \tilde{*})$ is named as SSbGp of $((\phi, \mathcal{Y}), \tilde{*})$ if (i) $\mathcal{X} \subseteq \mathcal{Y}$; (ii) for each SMb $\tilde{q}$ there exists atleast one SMb $\tilde{y}$ such that for all $\tilde{\rho} \in \mathcal{X}$, each SEt $(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho}))$ of $\tilde{q}$ is a SEt of $\tilde{y}$, that is each SMb of $(\theta, \mathcal{X})$ is a sub-SMb of at least one SMb of $(\phi, \mathcal{Y})$.

Example 3.2. Let the set of alternatives $(\mathcal{U}, \cdot)$ be the Klein's four group that is $\mathcal{U} = \{e, a, b, c\}$ with the Cayley's table given in Table 1, and $\mathcal{E} = \{\tilde{\rho}_i, i = 1, 2, \ldots, 10\}$ be a set of attributes.

Table 1: Cayley's table for Klein's four group.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $\mathcal{X} = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$ be a subset of $\mathcal{E}$ and $\theta : \mathcal{X} \to \mathcal{P}(\mathcal{U})$ such that $\theta(\tilde{\rho}_1) = \{e, a\}$, $\theta(\tilde{\rho}_2) = \{e, b\}$ and $\theta(\tilde{\rho}_5) = \{e, c\}$. Clearly $((\theta, \mathcal{X}), \tilde{\cdot})$ is SGp with following SMbs,

$\tilde{q}_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$, $\tilde{q}_2 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$, $\tilde{q}_3 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$,
$\tilde{q}_4 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, c)\}$, $\tilde{q}_5 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$, $\tilde{q}_6 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$,
$\tilde{q}_7 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$, $\tilde{q}_8 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, b), (\tilde{\rho}_5, c)\}$.

and Cayley's table as Table 2 (Table is completed using soft binary operation definition as $\tilde{q}_2 \tilde{\cdot} \tilde{q}_3 = \{(\tilde{\rho}_1, e \cdot e), (\tilde{\rho}_2, e \cdot b), (\tilde{\rho}_5, c \cdot e)\} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, c)\} = \tilde{q}_4$).

Table 2: Cayley's table for SGp $((\theta, \mathcal{X}), \tilde{\cdot})$.

$\tilde{\cdot}$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_1$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_2$	$\tilde{q}_2$	$\tilde{q}_1$	$\tilde{q}_4$	$\tilde{q}_3$	$\tilde{q}_6$	$\tilde{q}_5$	$\tilde{q}_8$	$\tilde{q}_7$
$\tilde{q}_3$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_5$	$\tilde{q}_6$
$\tilde{q}_4$	$\tilde{q}_4$	$\tilde{q}_3$	$\tilde{q}_2$	$\tilde{q}_1$	$\tilde{q}_8$	$\tilde{q}_7$	$\tilde{q}_6$	$\tilde{q}_5$
$\tilde{q}_5$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$
$\tilde{q}_6$	$\tilde{q}_6$	$\tilde{q}_5$	$\tilde{q}_8$	$\tilde{q}_7$	$\tilde{q}_2$	$\tilde{q}_1$	$\tilde{q}_4$	$\tilde{q}_3$
$\tilde{q}_7$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_1$	$\tilde{q}_2$
$\tilde{q}_8$	$\tilde{q}_8$	$\tilde{q}_7$	$\tilde{q}_6$	$\tilde{q}_5$	$\tilde{q}_4$	$\tilde{q}_3$	$\tilde{q}_2$	$\tilde{q}_1$

Now, we will discuss all possible SSbGps of this SGp. As we know that there are 6 non-trivial subsets of $\mathcal{X}$ that along with $\mathcal{X}$ itself, will serve as set of attributes for the SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$. All possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$ are defined below:

1. Over the same set of alternatives, let $((\phi_1, \mathcal{Y}_1), \tilde{\cdot})$ be another SGp with $\mathcal{Y}_1 = \{\tilde{\rho}_1, \tilde{\rho}_5\}$ be a subset of $\mathcal{X}$ and $\phi_1 : \mathcal{Y}_1 \to \mathcal{P}(\mathcal{U})$ such that $\phi_1(\tilde{\rho}_1) = \{e, a\}$, $\phi_1(\tilde{\rho}_5) = \{e\}$ and SMbs, $\tilde{y}^1_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$, $\tilde{y}^1_2 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_5, e)\}$. The axioms of SGp for $((\phi_1, \mathcal{Y}_1), \tilde{\cdot})$ are satisfied as shown by Cayley's table in Table 3. Here $\mathcal{Y}_1 \subseteq \mathcal{X}$. Now consider $\tilde{y}_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$, $\tilde{q}_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ and

Table 3: Cayley's table for SGp $((\phi_1, \mathcal{Y}_1), \tilde{\cdot})$.

$\tilde{\cdot}$	$\tilde{y}^1_1$	$\tilde{y}^1_2$
$\tilde{y}^1_1$	$\tilde{y}^1_1$	$\tilde{y}^1_2$
$\tilde{y}^1_2$	$\tilde{y}^1_2$	$\tilde{y}^1_1$

$\tilde{q}_3 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$. Clearly, each SEt of $\tilde{y}_1$ is a SEt of $\tilde{q}_1$ as well as $\tilde{q}_3$ so, $\tilde{y}_1$ is a sub-SMb of $\tilde{q}_1$ and $\tilde{q}_3$. Similarly, $\tilde{y}_2$ is a sub SMb of $\tilde{q}_5$ and $\tilde{q}_7$. Hence by Definition 3.1, $((\phi_1, \mathcal{Y}_1), \tilde{\cdot})$ is a SSbGp of $((\theta, \mathcal{X}), \tilde{\cdot})$.

2. Over the same set of alternatives, let $((\phi_2, \mathcal{Y}_1), \tilde{\cdot})$ be a SGp with $\mathcal{Y}_1 = \{\tilde{\rho}_1, \tilde{\rho}_5\}$, $\phi_2 : \mathcal{Y}_1 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_2(\tilde{\rho}_1) = \{e\}$, $\phi_2(\tilde{\rho}_5) = \{e, c\}$ and SMbs, $\tilde{y}^2_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$, $\tilde{y}^2_2 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, c)\}$. The axioms of SGp for $((\phi_2, \mathcal{Y}_1), \tilde{\cdot})$ are satisfied as shown by Cayley's table in Table 4.

Table 4: Cayley's table for SGp $((\phi_2, \mathcal{Y}_1), \tilde{\cdot})$.

$\tilde{\cdot}$	$\tilde{y}^2_1$	$\tilde{y}^2_2$
$\tilde{y}^2_1$	$\tilde{y}^2_1$	$\tilde{y}^2_2$
$\tilde{y}^2_2$	$\tilde{y}^2_2$	$\tilde{y}^2_1$

Clearly, $((\phi_2, \mathcal{Y}_1), \tilde{\cdot})$ is a SSbGp of $((\theta, \mathcal{X}), \tilde{\cdot})$.

3. Over the same set of alternatives, let $((\phi_3, \mathcal{Y}_1), \tilde{\cdot})$ be a SGp with $\mathcal{Y}_1 = \{\tilde{\rho}_1, \tilde{\rho}_5\}$, $\phi_3 : \mathcal{Y}_1 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_3(\tilde{\rho}_1) = \{e, a\}$, $\phi_3(\tilde{\rho}_5) = \{e, c\}$ and SMbs, $\tilde{y}^3_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$, $\tilde{y}^3_2 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, c)\}$, $\tilde{y}^3_3 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_5, e)\}$, $\tilde{y}^3_4 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_5, c)\}$. The axioms of SGp for $((\phi_3, \mathcal{Y}_1), \tilde{\cdot})$ are satisfied as shown by Cayley's table in Table 5.

Table 5: Cayley's table for SGp $((\phi_3, \mathcal{Y}_1), \tilde{\cdot})$.

$\tilde{\cdot}$	$\tilde{y}^3_1$	$\tilde{y}^3_2$	$\tilde{y}^3_3$	$\tilde{y}^3_4$
$\tilde{y}^3_1$	$\tilde{y}^3_1$	$\tilde{y}^3_2$	$\tilde{y}^3_3$	$\tilde{y}^3_4$
$\tilde{y}^3_2$	$\tilde{y}^3_2$	$\tilde{y}^3_1$	$\tilde{y}^3_4$	$\tilde{y}^3_3$
$\tilde{y}^3_3$	$\tilde{y}^3_3$	$\tilde{y}^3_4$	$\tilde{y}^3_1$	$\tilde{y}^3_2$
$\tilde{y}^3_4$	$\tilde{y}^3_4$	$\tilde{y}^3_3$	$\tilde{y}^3_2$	$\tilde{y}^3_1$

Clearly, $((\phi_3, \mathcal{Y}_1), \tilde{\cdot})$ is a SSbGp of $((\theta, \mathcal{X}), \tilde{\cdot})$.

4. $((\phi_4, \mathcal{Y}_1), \tilde{\cdot})$ is another subgroup of $((\theta, \mathcal{X}), \tilde{\cdot})$ where $\mathcal{Y}_1 = \{\tilde{\rho}_1, \tilde{\rho}_5\}$, $\phi_4 : \mathcal{Y}_1 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_4(\tilde{\rho}_1) = \{e\}$, $\phi_4(\tilde{\rho}_5) = \{e\}$. It is a singleton set consisting of following soft identity element, $\tilde{y}^4 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$.

The remaining possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$ are shown in the Tables 6, 7, 8, 9, 10, and 11.

Table 6: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-I.

Remaining possibilities	Subsets of $\mathcal{X}$	Function	SMbs
V: $((\phi_5, \mathcal{Y}_2), \tilde{\cdot})$	$\mathcal{Y}_2 = \{\tilde{\rho}_1, \tilde{\rho}_2\}$	$\phi_5 : \mathcal{Y}_2 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_5(\tilde{\rho}_1) = \{e, a\}$, $\phi_5(\tilde{\rho}_2) = \{e\}$	$\tilde{y}^5_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e)\}$; $\tilde{y}^5_2 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e)\}$
VI: $((\phi_6, \mathcal{Y}_2), \tilde{\cdot})$	$\mathcal{Y}_2 = \{\tilde{\rho}_1, \tilde{\rho}_2\}$	$\phi_6 : \mathcal{Y}_2 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_6(\tilde{\rho}_1) = \{e\}$, $\phi_6(\tilde{\rho}_2) = \{e, b\}$	$\tilde{y}^6_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e)\}$; $\tilde{y}^6_2 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b)\}$
VII: $((\phi_7, \mathcal{Y}_2), \tilde{\cdot})$	$\mathcal{Y}_2 = \{\tilde{\rho}_1, \tilde{\rho}_2\}$	$\phi_7 : \mathcal{Y}_2 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_7(\tilde{\rho}_1) = \{e, a\}$, $\phi_7(\tilde{\rho}_2) = \{e, b\}$	$\tilde{y}^7_1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e)\}$; $\tilde{y}^7_2 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b)\}$; $\tilde{y}^7_3 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e)\}$; $\tilde{y}^7_4 = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, b)\}$

Table 7: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-II.

Remaining possibilities	Subsets of $\mathcal{X}$	Function	SMbs
VIII: $((\phi_8, \mathcal{Y}_2), \tilde{\cdot})$	$\mathcal{Y}_2 = \{\tilde{\rho}_1, \tilde{\rho}_2\}$	$\phi_8 : \mathcal{Y}_2 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_8(\tilde{\rho}_1) = \{e\}$, $\phi_8(\tilde{\rho}_2) = \{e\}$	$\tilde{y}^8 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e)\}$
IX: $((\phi_9, \mathcal{Y}_3), \tilde{\cdot})$	$\mathcal{Y}_3 = \{\tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_9 : \mathcal{Y}_3 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_9(\tilde{\rho}_2) = \{e, b\}$, $\phi_9(\tilde{\rho}_5) = \{e\}$	$\tilde{y}^9_1 = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$; $\tilde{y}^9_2 = \{(\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$
X: $((\phi_{10}, \mathcal{Y}_3), \tilde{\cdot})$	$\mathcal{Y}_3 = \{\tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{10} : \mathcal{Y}_3 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_9(\tilde{\rho}_2) = \{e\}$, $\phi_9(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}^{10}_1 = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$; $\tilde{y}^{10}_2 = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$
XI: $((\phi_{11}, \mathcal{Y}_3), \tilde{\cdot})$	$\mathcal{Y}_3 = \{\tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{11} : \mathcal{Y}_3 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{11}(\tilde{\rho}_2) = \{e\}$, $\phi_{11}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}^{11} = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$

Table 8: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-III.

Remaining possibilities	Subsets of $\mathcal{X}$	Function	SMbs
XII: $((\phi_{12}, \mathcal{Y}_3), \tilde{\cdot})$	$\mathcal{Y}_3 = \{\tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{12} : \mathcal{Y}_3 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{12}(\tilde{\rho}_2) = \{e, b\}$, $\phi_{12}(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}^{12}_1 = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$; $\tilde{y}^{12}_2 = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$; $\tilde{y}^{12}_3 = \{(\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$; $\tilde{y}^{12}_4 = \{(\tilde{\rho}_2, b), (\tilde{\rho}_5, c)\}$
XIII: $((\phi_{13}, \mathcal{Y}_4), \tilde{\cdot})$	$\mathcal{Y}_4 = \{\tilde{\rho}_1\}$	$\phi_{13} : \mathcal{Y}_4 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{13}(\tilde{\rho}_1) = \{e, a\}$	$\tilde{y}^{13}_1 = \{(\tilde{\rho}_1, e)\}$; $\tilde{y}^{13}_2 = \{(\tilde{\rho}_1, a)\}$
XIV: $((\phi_{14}, \mathcal{Y}_4), \tilde{\cdot})$	$\mathcal{Y}_4 = \{\tilde{\rho}_1\}$	$\phi_{14} : \mathcal{Y}_4 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{14}(\tilde{\rho}_1) = \{e\}$	$\tilde{y}^{14} = \{(\tilde{\rho}_1, e)\}$
XV: $((\phi_{15}, \mathcal{Y}_5), \tilde{\cdot})$	$\mathcal{Y}_5 = \{\tilde{\rho}_2\}$	$\phi_{15} : \mathcal{Y}_5 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{15}(\tilde{\rho}_2) = \{e, b\}$	$\tilde{y}^{15}_1 = \{(\tilde{\rho}_2, e)\}$; $\tilde{y}^{15}_2 = \{(\tilde{\rho}_2, b)\}$

Table 9: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-IV.

Remaining Possibilities	Subsets of $\mathcal{X}$	Function	SMbs
XVI: $((\phi_{16}, \mathcal{Y}_5), \tilde{\cdot})$	$\mathcal{Y}_5 = \{\tilde{\rho}_2\}$	$\phi_{16} : \mathcal{Y}_5 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{16}(\tilde{\rho}_2) = \{e\}$	$\tilde{y}^{16} = \{(\tilde{\rho}_2, e)\}$
XVII: $((\phi_{17}, \mathcal{Y}_6), \tilde{\cdot})$	$\mathcal{Y}_6 = \{\tilde{\rho}_5\}$	$\phi_{17} : \mathcal{Y}_6 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{17}(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}_1^{17} = \{(\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{17} = \{(\tilde{\rho}_5, c)\}$
XVIII: $((\phi_{18}, \mathcal{Y}_6), \tilde{\cdot})$	$\mathcal{Y}_6 = \{\tilde{\rho}_5\}$	$\phi_{18} : \mathcal{Y}_6 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{18}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}^{18} = \{(\tilde{\rho}_5, e)\}$
XIX: $((\phi_{19}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{19} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{19}(\tilde{\rho}_1) = \{e\}$, $\phi_{19}(\tilde{\rho}_2) = \{e, b\}$ $\phi_{19}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}_1^{19} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{19} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$

Table 10: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-V.

Remaining Possibilities	Subsets of $\mathcal{X}$	Function	SMbs
XX: $((\phi_{20}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{20} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{20}(\tilde{\rho}_1) = \{e, a\}$, $\phi_{20}(\tilde{\rho}_2) = \{e\}$ $\phi_{20}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}_1^{20} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{20} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$
XXI: $((\phi_{21}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{21} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{21}(\tilde{\rho}_1) = \{e, a\}$, $\phi_{21}(\tilde{\rho}_2) = \{e\}$ $\phi_{21}(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}_1^{21} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{21} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$ $\tilde{y}_3^{21} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_4^{21} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$
XXII: $((\phi_{22}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{22} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{22}(\tilde{\rho}_1) = \{e, a\}$, $\phi_{22}(\tilde{\rho}_2) = \{e, b\}$ $\phi_{22}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}_1^{22} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{22} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$ $\tilde{y}_3^{22} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_4^{22} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$
XXIII: $((\phi_{23}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{23} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{23}(\tilde{\rho}_1) = \{e\}$, $\phi_{23}(\tilde{\rho}_2) = \{e\}$ $\phi_{23}(\tilde{\rho}_5) = \{e\}$	$\tilde{y}^{23} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$

Table 11: Possible SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$-VI.

Remaining possibilities	Subsets of $\mathcal{X}$	Function	SMbs
XXIV: $((\phi_{24}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{24} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{24}(\tilde{\rho}_1) = \{e\}$, $\phi_{24}(\tilde{\rho}_2) = \{e\}$ $\phi_{24}(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}_1^{24} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{24} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$
XXV: $((\phi_{25}, \mathcal{Y}_7), \tilde{\cdot})$	$\mathcal{Y}_7 = \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_5\}$	$\phi_{25} : \mathcal{Y}_7 \to \mathcal{P}(\mathcal{U})$ defined by $\phi_{25}(\tilde{\rho}_1) = \{e\}$, $\phi_{25}(\tilde{\rho}_2) = \{e, b\}$ $\phi_{25}(\tilde{\rho}_5) = \{e, c\}$	$\tilde{y}_1^{25} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ $\tilde{y}_2^{25} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, e), (\tilde{\rho}_5, c)\}$ $\tilde{y}_3^{25} = \{(\tilde{\rho}_1, e), (\tilde{\rho}_2, b), (\tilde{\rho}_5, e)\}$ $\tilde{y}_4^{25} = \{(\tilde{\rho}_1, a), (\tilde{\rho}_2, b), (\tilde{\rho}_5, c)\}$

3.1. SGp entities parallel to classical group

In the light of above example, we are able to generalize following few important SMbs in a SGp parallel to the classical group. Let $((\theta, \mathcal{X}), \tilde{*})$ be a SGp over $(\mathcal{U}, *)$ and $\tilde{q}$ be its SMb, then by definition $\tilde{q} = \{(\tilde{\rho}, \tilde{\alpha}(\tilde{\rho})) : \tilde{\rho} \in \mathcal{X}, \tilde{\alpha}(\tilde{\rho}) \in \theta(\tilde{\rho}) \neq \varnothing\}$, where $\theta(\tilde{\rho})$ is subgroup of $(\mathcal{U}, *)$.

Definition 3.3. The soft identity member $\tilde{q}_e$ of a SGp is given as $\tilde{q}_e = \{(\tilde{\rho}, e) : \tilde{\rho} \in \mathcal{X}, e$ is the corresponding identity in group $(\mathcal{U}, *)\}$.

Definition 3.4. Let $a = \tilde{\alpha}(\tilde{\rho})$, then as defined above, the SMb of $((\theta, \mathcal{X}), \tilde{*})$ take the form, $\tilde{q} = \{(\tilde{\rho}, a) : \tilde{\rho} \in \mathcal{X}, a \in \theta(\tilde{\rho})\}$ then its soft inverse will be $\tilde{q}^{-1} = \{(\tilde{\rho}, a^{-1}) : \tilde{\rho} \in \mathcal{X}, a^{-1}$ is the inverse of a in group $(\mathcal{U}, *)\}$. The existence of soft inverse member is justified by the presence of inverse element a^{-1} for each a in $\theta(\tilde{\rho})$ as $\theta(\tilde{\rho})$ is a subgroup of $(\mathcal{U}, *)$. More precisely, the soft additive inverse of $\tilde{q}$ is termed as $-\tilde{q} = \{(\tilde{\rho}, -a) : \tilde{\rho} \in A, -a$ is the additive inverse of a in group $(\mathcal{U}, *)\}$.

In above example, each SSbGp is containing an identity element, e.g., $\tilde{y}_1^1 = \{(\tilde{\rho}_1, e), (\tilde{\rho}_5, e)\}$, $\tilde{y}_1^{12} = \{(\tilde{\rho}_2, e), (\tilde{\rho}_5, e)\}$ are the identity elements in $((\phi_1, \mathcal{Y}_1), \tilde{\cdot})$ and $((\phi_{12}, \mathcal{Y}_3), \tilde{\cdot})$, respectively.

3.2. Formula for evaluating the number of SSbGps

We are able to generalize the number of SSbGps of a considered SGp as follows. Let $((\theta, \mathcal{X}), \tilde{\cdot})$ be SGp over set of alternatives $(\mathcal{U}, *)$ which is a group itself. Let the cardinality of $\mathcal{X}$ is n and for each $\tilde{\rho}_s$ in $\mathcal{X}$, the number of subgroups of $\theta(\tilde{\rho}_s)$ is m_s, for all $s = 1, 2, \ldots, n$. Then, the total number of SSbGps of $((\theta, \mathcal{X}), \tilde{\cdot})$ is

$$\sum_{s=1}^{n} m_s + \sum_{s=1}^{n} \sum_{t=s+1}^{n} m_s m_t + \sum_{s=1}^{n} \sum_{t=s+1}^{n} \sum_{l=t+1}^{n} m_s m_t m_l + \cdots + m_1 m_2 \cdots m_n - 1.$$

We know that the number of non-null subsets of $\mathcal{X}$ is $2^n - 1$ and there are ${}^nC_k, k = 1, 2, \ldots, n$ subsets of cardinality k, i.e., there are ${}^nC_1 = n$ singleton subsets, ${}^nC_2$ subsets of cardinality 2 and ${}^nC_n = 1$ subset of cardinality n. Consider the singleton subsets of $\mathcal{X}$ having m_s possibilities for each $\tilde{\rho}_s, s = 1, 2, \ldots, n$. Hence the number of SSbGps in this case is $\sum_{s=1}^{n} m_s = m_1 + m_2 + \cdots + m_n$. A detailed description to the number of SSbGps for all possibilities is presented in Table 12.

Table 12: Number of possible SSbGps of a SGp.

Type of subsets of $\mathcal{X}$	Number of subsets of $\mathcal{X}$	Number of possibilities for image of each element in subset	The number of SSbGps
Singleton set	${}^nC_1 = n$	$\tilde{\rho}_s \to m_s$, for all $s = 1, 2, \ldots, n$	$\sum_{s=1}^{n} m_s$
Subsets with cardinality 2	${}^nC_2 = \frac{n(n-1)}{2}$	$\tilde{\rho}_s \to m_s, \tilde{\rho}_t \to m_t$, for all $s = 1, 2, \ldots, n; t = s + 1, \ldots, n$	$\sum_{s=1}^{n} \sum_{t=s+1}^{n} m_s m_t$
Subsets with cardinality 3	${}^nC_3 = \frac{n(n-1)(n-2)}{6}$	$\tilde{\rho}_s \to m_s, \tilde{\rho}_t \to m_t, \tilde{\rho}_l \to m_l$ for all $s = 1, 2, \ldots, n; t = s + 1, \ldots, n, l = s + 2, \ldots, n$	$\sum_{s=1}^{n} \sum_{t=s+1}^{n} \sum_{l=t+1}^{n} m_s m_t m_l$
⋮	⋮	⋮	⋮
Subsets with cardinality n	${}^nC_n = 1$	$\tilde{\rho}_s \to m_s$ for all $s = 1, 2, \ldots, n$,	$m_1 m_2 \cdots m_n - 1$

Hence total number of SSbGps of SGp $((\theta,\mathcal{X}),\tilde{*})$ is

$$\sum_{s=1}^{n} m_s + \sum_{s=1}^{n}\sum_{t=s+1}^{n} m_s m_t + \sum_{s=1}^{n}\sum_{t=s+1}^{n}\sum_{l=s+1}^{n} m_s m_t m_l + \cdots + m_1 m_2 \cdots m_n - 1,$$

which could be verified using above example for which $n = 3, m_1 = m_2 = m_3 = 2$ then by using the proposed formula, the number of SSbGps of considered SGp will be,

$$\sum_{s=1}^{3} m_s + \sum_{s=1}^{3}\sum_{t=s+1}^{3} m_s m_t + m_1 m_2 m_3 - 1 = m_1 + m_2 + m_3 + m_1 m_2 + m_1 m_3 + m_2 m_3 + m_1 m_2 m_3 - 1 = 25.$$

As we have presented the SSbGp in an innovative way, so in the following subsections, we have proved pre-defined theorems using the presented notations.

3.3. *Justifying the existence of SSbGps*

To validate the existence of SSbGps, the basic theorems from classical group theory are presented that are satisfied by SSbGps.

Theorem 3.5 ([9]). *Let $((\theta,\mathcal{X}),\tilde{*})$ be a SGp over $(\mathcal{U},*)$ and $\{((\phi_i,\mathcal{Y}_i),\tilde{*}) : i \in I\}$, where I is representing the set of indices, is representing collection of SSbGps of $((\theta,\mathcal{X}),\tilde{*})$. Then their restricted intersection is also a SSbGp of $((\theta,\mathcal{X}),\tilde{*})$ over $(\mathcal{U},*)$.*

Proof. Let $\{((\phi_i,\mathcal{Y}_i),\tilde{*}) : i \in I\}$ be a collection of SSbGps of $((\theta,\mathcal{X}),\tilde{*})$. Therefore, by definition, for each $i \in I$, (i) $\mathcal{Y}_i \subseteq \mathcal{X}$; (ii) each SMb of $((\phi_i,\mathcal{Y}_i),\tilde{*})$ is a sub-SMb of $((\theta,\mathcal{X}),\tilde{*})$. Let $(\psi,\mathcal{Z})$ be the restricted intersection of SSbGps $\{((\phi_i,\mathcal{Y}_i),\tilde{*}) : i \in I\} = \{\{(\tilde{\rho},\tilde{\beta}_i(\tilde{\rho})) : \tilde{\beta}_i : \mathcal{Y}_i \to \mathcal{U}, \tilde{\beta}_i(\tilde{\rho}) \in \phi_i(\tilde{\rho}), \text{for all } \tilde{\rho} \in \mathcal{Y}_i\}, i \in I\}$, then by definition, $\mathcal{Z} = \cap_i \mathcal{Y}_i$ and $(\psi,\mathcal{Z}) = \{\tilde{z} : \tilde{z} = (\tilde{\rho},\tilde{\gamma}(\tilde{\rho})), \tilde{\gamma}(\tilde{\rho}) \in \cap_i \tilde{\beta}_i(\tilde{\rho}), \text{for all } \tilde{\rho} \in \mathcal{Z}\}$. Clearly, (i) $\mathcal{Z} \subseteq \mathcal{Y}_i$, for all $i \in I$ and $\mathcal{Y}_i \subseteq \mathcal{X}$, for all $i \in I$ that leads to $\mathcal{Z} \subseteq \mathcal{X}$; (ii) for all $\tilde{\rho} \in \mathcal{Z}, \tilde{\gamma}(\tilde{\rho}) \in \cap_i \tilde{\beta}_i(\tilde{\rho}) \implies \tilde{\gamma}(\tilde{\rho}) \in \tilde{\beta}_i(\tilde{\rho})$, for all $i \in I, \tilde{\rho} \in \mathcal{Z}$, which shows that each SMb of $(\psi,\mathcal{Z})$ is sub SMb of at least one of the SMbs of each of $(\phi_i,\mathcal{Y}_i)$, for all $i \in I$ and hence a sub-SMb of at least one of the SMbs of $(\theta,\mathcal{X})$ which shows that $((\psi,\mathcal{Z}),\tilde{*})$ is a SSbGp of $((\theta,\mathcal{X}),\tilde{*})$. Hence the statement is proved. □

In the following subsection, the theorem stated above is verified through example.

Example 3.6.

(i) In Example 3.2, consider the SSbGps $((\phi_1,\mathcal{Y}_1),\tilde{*})$, $((\phi_2,\mathcal{Y}_1),\tilde{*})$ and $((\phi_{12},\mathcal{Y}_3),\tilde{*})$ of $((\theta,\mathcal{X}),\tilde{*})$. Their restricted intersection is a SSbGp of $((\theta,\mathcal{X}),\tilde{*})$ namely $((\phi_{18},\mathcal{Y}_6),\tilde{*})$.

(ii) Consider the collection of subgroups $\{((\phi_i,\mathcal{Y}_3),\tilde{*}) : i = 9, 10, 11, 12\}$ of $((\theta,\mathcal{X}),\tilde{*})$. The restricted intersection of this collection is again a SSbGp of $((\theta,\mathcal{X}),\tilde{*})$ namely $((\phi_{12},\mathcal{Y}_3),\tilde{*})$. Hence the theorem is verified.

Theorem 3.7 ([9]). *Let $((\theta,\mathcal{X}),\tilde{*})$ be a SGp over $(\mathcal{U},*)$ and $\{(\phi_i,\mathcal{Y}_i) : i \in I)\}$, where I is representing the set of indices, be the class of SSbGps of $((\theta,\mathcal{X}),\tilde{*})$. Then the SSt obtained by their extended intersection is also a SSbGp of $((\theta,\mathcal{X}),\tilde{*})$.*

Proof. Could be proved on similar lines. □

Definition 3.8. A non-null SSt $(\theta,\mathcal{X})$ is regarded as soft semi group denoted by $((\theta,\mathcal{X}),\tilde{*})$ if the set of alternatives $(\mathcal{U},*)$ is a group, θ is mapping elements of $\mathcal{X}$ to its subgroups and (i) for all $\tilde{q}_i, \tilde{q}_j \in (\theta,\mathcal{X}), \tilde{q}_i \tilde{*} \tilde{q}_j \in (\theta,\mathcal{X})$; (ii) for all $\tilde{q}_i, \tilde{q}_j, \tilde{q}_k \in (\theta,\mathcal{X}), \tilde{q}_i \tilde{*} (\tilde{q}_j \tilde{*} \tilde{q}_k) = (\tilde{q}_i \tilde{*} \tilde{q}_j) \tilde{*} \tilde{q}_k$, that is the soft binary operation $\tilde{*}$ satisfies only closure and associative property.

4. Soft algebraic structure parallel to ring

The SSt $(\theta, \mathcal{X})$ is named as SRg if the set of alternatives $\mathcal{U}$ is a ring itself and θ maps the attributes in $\mathcal{X}$ to the subrings of $\mathcal{U}$ [1]. The following definition enables us to represent SRg as classical algebraic ring.

Definition 4.1. Let $(\theta, \mathcal{X})$ be a SSt over a ring $(\mathcal{U}, +, \cdot)$ and θ is mapping the attributes in $\mathcal{X}$ to the subrings of $(\mathcal{U}, +, \cdot)$. Assuming the soft binary operations $\tilde{+}$ and $\tilde{\cdot}$ are defined on $(\theta, \mathcal{X})$, then $(\theta, \mathcal{X})$ is named as SRg, denoted by $((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$, if

1. $((\theta, \mathcal{X}), \tilde{+})$ is an abelian SGp;
2. $((\theta, \mathcal{X}), \tilde{\cdot})$ is soft semi group;
3. distributive property of $\tilde{\cdot}$ over $\tilde{+}$ holds, i.e., for any SMbs $\tilde{q}_i, \tilde{q}_j, \tilde{q}_k$ of $(\theta, \mathcal{X})$,

$$\tilde{q}_i \tilde{\cdot} (\tilde{q}_j \tilde{+} \tilde{q}_k) = (\tilde{q}_i \tilde{\cdot} \tilde{q}_j) \tilde{+} (\tilde{q}_i \tilde{\cdot} \tilde{q}_k) \quad \text{(left distributive property)},$$
$$(\tilde{q}_i \tilde{+} \tilde{q}_j) \tilde{\cdot} \tilde{q}_k = (\tilde{q}_i \tilde{\cdot} \tilde{q}_k) \tilde{+} (\tilde{q}_j \tilde{\cdot} \tilde{q}_k) \quad \text{(right distributive property)}.$$

In the following section, an example is presented to show that how proposed definition of SRg enables one to deal with SRg in the similar manner as classical ring.

Example 4.2. Let the set of alternatives $(\mathcal{U}, +, \cdot)$ be the ring Z_6 under modulo addition and multiplication, that is $\mathcal{U} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $\mathcal{E} = \{\tilde{\rho}_i, i = 1, 2, \ldots, 5\}$ be a set of attributes. Let $\mathcal{X} = \{\tilde{\rho}_1, \tilde{\rho}_2\}$ be a subset of $\mathcal{E}$ and $\theta : \mathcal{X} \to \mathcal{P}(\mathcal{U})$ such that $\theta(\tilde{\rho}_1) = \{\bar{0}, \bar{2}, \bar{4}\}$ and $\theta(\tilde{\rho}_2) = \{\bar{0}, \bar{3}\}$. Clearly, $((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$ is SRg with the SMbs, $\tilde{q}_o = \{(\tilde{\rho}_1, \bar{0}), (\tilde{\rho}_2, \bar{0})\}, \tilde{q}_1 = \{(\tilde{\rho}_1, \bar{0}), (\tilde{\rho}_2, \bar{3})\}, \tilde{q}_2 = \{(\tilde{\rho}_1, \bar{2}), (\tilde{\rho}_2, \bar{0})\}, \tilde{q}_3 = \{(\tilde{\rho}_1, \bar{2}), (\tilde{\rho}_2, \bar{3})\}, \tilde{q}_4 = \{(\tilde{\rho}_1, \bar{4}), (\tilde{\rho}_2, \bar{0})\}, \tilde{q}_5 = \{(\tilde{\rho}_1, \bar{4}), (\tilde{\rho}_2, \bar{3})\}$, and cayley's table as in Table 13.

Table 13: Cayley's table for SRg $((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$ over soft addition $\tilde{+}$.

$\tilde{+}$	$\tilde{q}_o$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$
$\tilde{q}_o$	$\tilde{q}_o$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$
$\tilde{q}_1$	$\tilde{q}_1$	$\tilde{q}_o$	$\tilde{q}_3$	$\tilde{q}_2$	$\tilde{q}_5$	$\tilde{q}_4$
$\tilde{q}_2$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_o$	$\tilde{q}_1$
$\tilde{q}_3$	$\tilde{q}_3$	$\tilde{q}_2$	$\tilde{q}_5$	$\tilde{q}_4$	$\tilde{q}_1$	$\tilde{q}_o$
$\tilde{q}_4$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_o$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$
$\tilde{q}_5$	$\tilde{q}_5$	$\tilde{q}_4$	$\tilde{q}_1$	$\tilde{q}_o$	$\tilde{q}_3$	$\tilde{q}_2$

Associative and distributive properties could be easily verified, e.g., for $\tilde{q}_1, \tilde{q}_3$, and $\tilde{q}_4$ in $((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$,

$$\tilde{q}_1 \tilde{\cdot} (\tilde{q}_3 \tilde{\cdot} \tilde{q}_4) = \tilde{q}_o = (\tilde{q}_1 \tilde{\cdot} \tilde{q}_3) \tilde{\cdot} \tilde{q}_4 \quad \text{(associative property)},$$
$$\tilde{q}_1 \tilde{\cdot} (\tilde{q}_3 \tilde{+} \tilde{q}_4) = \tilde{q}_1 = (\tilde{q}_1 \tilde{\cdot} \tilde{q}_3) \tilde{+} (\tilde{q}_1 \tilde{\cdot} \tilde{q}_4) \quad \text{(left distributive property)},$$
$$(\tilde{q}_1 \tilde{+} \tilde{q}_3) \tilde{\cdot} \tilde{q}_4 = \tilde{q}_2 = (\tilde{q}_1 \tilde{\cdot} \tilde{q}_4) \tilde{+} (\tilde{q}_3 \tilde{\cdot} \tilde{q}_4) \quad \text{(right distributive property)}.$$

Consider the following theorem defined for elements of a ring. For ring $(R, +, \cdot)$ with additive identity 0, $a, b \in R$, (i) $a \cdot 0 = 0 = 0 \cdot a$; (ii) $a(-b) = (-a)b = -(ab)$; and (iii) $(-a)(-b) = ab$. With the reference of above definition, we are able to define similar theorem using the SMbs of a SRg.

Theorem 4.3. *Let $((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$ be a SRg with soft additive identity $\tilde{q}_o$ and $-\tilde{q}$ being soft additive inverse of $\tilde{q}$. Then for any $\tilde{q}_1, \tilde{q}_2 \in ((\theta, \mathcal{X}), \tilde{+}, \tilde{\cdot})$,*

1. $\tilde{q}_o \tilde{\cdot} \tilde{q}_1 = \tilde{q}_o = \tilde{q}_1 \tilde{\cdot} \tilde{q}_o$;
2. $\tilde{q}_1 \tilde{\cdot} (-\tilde{q}_2) = (-\tilde{q}_1) \tilde{\cdot} \tilde{q}_2 = -(\tilde{q}_1 \tilde{\cdot} \tilde{q}_2)$;
3. $(-\tilde{q}_1) \tilde{\cdot} (-\tilde{q}_2) = (\tilde{q}_1) \tilde{\cdot} (\tilde{q}_2)$.

Proof. Now, let $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ be a SRg over set of alternatives $(\mathcal{U},+,\cdot)$ and $\tilde{q}_1=\{(\tilde{\rho},a):\tilde{\rho}\in\mathcal{X},a\in\tilde{\alpha}(\tilde{\rho})\}$, $\tilde{q}_2=\{(\tilde{\rho},b):\tilde{\rho}\in\mathcal{X},b\in\tilde{\alpha}(\tilde{\rho})\}$ be two SMbs and $\tilde{q}_o=\{(\tilde{\rho},0):\tilde{\rho}\in\mathcal{X}$, 0 is representing additive identity of $(\mathcal{U},+,\cdot)\}$ be the soft additive identity. Then their additive inverses will be $-\tilde{q}_1=\{(\tilde{\rho},-a):\tilde{\rho}\in\mathcal{X},a\in\tilde{\alpha}(\tilde{\rho})\}$ and $-\tilde{q}_2=\{(\tilde{\rho},-b):\tilde{\rho}\in\mathcal{X},b\in\tilde{\alpha}(\tilde{\rho})\}$, respectively.

1. $\tilde{q}_o\tilde{\cdot}\tilde{q}_1=\{(\tilde{\rho},0\cdot a):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},0):\tilde{\rho}\in\mathcal{X}\}=\tilde{q}_o$, and $\tilde{q}_1\tilde{\cdot}\tilde{q}_o=\{(\tilde{\rho},a\cdot 0):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},0):\tilde{\rho}\in\mathcal{X}\}=\tilde{q}_o$. Hence $\tilde{q}_o\tilde{\cdot}\tilde{q}_1=\tilde{q}_o=\tilde{q}_1\tilde{\cdot}\tilde{q}_o$.
2. $\tilde{q}_1\tilde{\cdot}(-\tilde{q}_2)=\{(\tilde{\rho},a\cdot(-b)):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},-(ab)):\tilde{\rho}\in\mathcal{X}\}$ $-(\tilde{q}_1)\tilde{\cdot}\tilde{q}_2=\{(\tilde{\rho},(-a)\cdot b):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},-(ab)):\tilde{\rho}\in\mathcal{X}\}$ $\tilde{q}_1\tilde{\cdot}\tilde{q}_2=\{(\tilde{\rho},a\cdot b):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},ab):\tilde{\rho}\in\mathcal{X}\}$. $\implies -(\tilde{q}_1\tilde{\cdot}\tilde{q}_2)=\{(\tilde{\rho},-(ab)):\tilde{\rho}\in\mathcal{X}\}$. Hence $\tilde{q}_1\tilde{\cdot}(-\tilde{q}_2)=(-\tilde{q}_1)\tilde{\cdot}\tilde{q}_2=-(\tilde{q}_1\tilde{\cdot}\tilde{q}_2)$.
3. $(-\tilde{q}_1)\tilde{\cdot}(-\tilde{q}_2)=\{(\tilde{\rho},(-a)\cdot(-b)):\tilde{\rho}\in\mathcal{X}\}=\{(\tilde{\rho},a\cdot b):\tilde{\rho}\in\mathcal{X}\}=(\tilde{q}_1)\tilde{\cdot}(\tilde{q}_2)$, which completes the proof.

□

In Example 4.2, $\tilde{q}_o$ is the soft additive identity member with soft additive inverses, $-\tilde{q}_1=\tilde{q}_1$, $-\tilde{q}_2=\tilde{q}_4$, $-\tilde{q}_3=\tilde{q}_5$, $-\tilde{q}_4=\tilde{q}_2$, and $-\tilde{q}_5=\tilde{q}_3$.

1. $\tilde{q}_o\tilde{\cdot}\tilde{q}_1=\tilde{q}_o$, $\tilde{q}_o\tilde{\cdot}\tilde{q}_2=\tilde{q}_o$.
2. $(-\tilde{q}_2)\tilde{\cdot}(\tilde{q}_1)=\tilde{q}_o=(\tilde{q}_2)\tilde{\cdot}(-\tilde{q}_1)$, $(-\tilde{q}_1)\tilde{\cdot}(\tilde{q}_5)=\tilde{q}_1=(\tilde{q}_1)\tilde{\cdot}(-\tilde{q}_5)$.
3. $(-\tilde{q}_2)\tilde{\cdot}(-\tilde{q}_3)=\tilde{q}_4=\tilde{q}_2\tilde{\cdot}\tilde{q}_3$, $(-\tilde{q}_3)\tilde{\cdot}(-\tilde{q}_5)=\tilde{q}_3=\tilde{q}_3\tilde{\cdot}\tilde{q}_5$.

Theorem 4.4. *Let* $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ *and* $((\phi,\mathcal{Y}),\tilde{+},\tilde{\cdot})$ *be two non-null SRgs. Then,*

1. $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})\,\tilde{\wedge}\,((\phi,\mathcal{Y}),\tilde{+},\tilde{\cdot})$ *is a SRg over* $\mathcal{E}$*;*
2. $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})\,\tilde{\cap}_R\,((\phi,\mathcal{Y}),\tilde{+},\tilde{\cdot})$ *is a SRg over* $\mathcal{E}$*.*

Proof. The proof is straightforward. □

5. Soft algebraic structures parallel to field

The SSt $(\theta,\mathcal{X})$ is named as soft field if the set of alternatives $\mathcal{U}$ is a field itself and θ maps the attributes in $\mathcal{X}$ to the subfields of $\mathcal{U}$ [19]. The following definition enables us to represent soft field as classical algebraic field.

Definition 5.1. Let $(\theta,\mathcal{X})$ be a SSt over a field $(\mathcal{U},+,\cdot)$ and θ is mapping the attributes in $\mathcal{X}$ to the subfields of $(\mathcal{U},+,\cdot)$. Assuming the soft binary operations defined on $(\theta,\mathcal{X})$ are $\tilde{+}$ and $\tilde{\cdot}$, it is named as soft field, denoted by $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ if

1. $((\theta,\mathcal{X}),\tilde{+})$ is an abelian SGp;
2. $((\phi,\mathcal{X}),\tilde{\cdot})$ is an abelian group, where $\phi:\mathcal{X}\to\mathcal{P}(\mathcal{U})$ such that $\phi(\tilde{\rho})=\theta(\tilde{\rho})-\{0\}$, for all $\tilde{\rho}\in\mathcal{X}$ and 0 is the additive identity of the field $(\mathcal{U},+,\cdot)$;
3. distributive property of $\tilde{\cdot}$ over $\tilde{+}$ holds, i.e., for any SMbs $\tilde{q}_i,\tilde{q}_j,\tilde{q}_k$ of $(\theta,\mathcal{X})$,

$$\tilde{q}_i\tilde{\cdot}(\tilde{q}_j\tilde{+}\tilde{q}_k)=(\tilde{q}_i\cdot\tilde{q}_j)\tilde{+}(\tilde{q}_i\tilde{\cdot}\tilde{q}_k)\quad\text{(left distributive property)},$$
$$(\tilde{q}_i\tilde{+}\tilde{q}_j)\tilde{\cdot}\tilde{q}_k=(\tilde{q}_i\tilde{\cdot}\tilde{q}_k)\tilde{+}(\tilde{q}_j\tilde{\cdot}\tilde{q}_k)\quad\text{(right distributive property)}.$$

An example is provided in the next part to help the reader understand how the definition of soft field as it is presented permits them to discuss the attributes of soft fields as classical fields.

Example 5.2. Let the set of alternatives $(\mathcal{U},+,\cdot)$ be the field Z_3 under modulo addition and multiplication, that is $\mathcal{U}=\{\bar{0},\bar{1},\bar{2}\}$ and $\mathcal{E}=\{\tilde{\rho}_i,i=1,2,\ldots,5\}$ be a set of attributes. Let $\mathcal{X}=\{\tilde{\rho}_1,\tilde{\rho}_2\}$ be a subset of $\mathcal{E}$ and $\theta:\mathcal{X}\to\mathcal{P}(\mathcal{U})$ such that $\theta(\tilde{\rho}_1)=\{\bar{0},\bar{1},\bar{2}\}=\theta(\tilde{\rho}_2)$. Clearly, $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ is soft field with following SMbs, $\tilde{q}_o=\{(\tilde{\rho}_1,\bar{0}),(\tilde{\rho}_2,\bar{0})\}$, $\tilde{q}_1=\{(\tilde{\rho}_1,\bar{0}),(\tilde{\rho}_2,\bar{1})\}$, $\tilde{q}_2=\{(\tilde{\rho}_1,\bar{0}),(\tilde{\rho}_2,\bar{2})\}$, $\tilde{q}_3=\{(\tilde{\rho}_1,\bar{1}),(\tilde{\rho}_2,\bar{0})\}$, $\tilde{q}_4=\{(\tilde{\rho}_1,\bar{1}),(\tilde{\rho}_2,\bar{1})\}$, $\tilde{q}_5=\{(\tilde{\rho}_1,\bar{1}),(\tilde{\rho}_2,\bar{2})\}$, $\tilde{q}_6=\{(\tilde{\rho}_1,\bar{2}),(\tilde{\rho}_2,\bar{0})\}$, $\tilde{q}_7=\{(\tilde{\rho}_1,\bar{2}),(\tilde{\rho}_2,\bar{1})\}$, $\tilde{q}_8=\{(\tilde{\rho}_1,\bar{2}),(\tilde{\rho}_2,\bar{2})\}$.

1. $((\theta,\mathcal{X}),\tilde{+})$ is an abelian group with Cayley's table given in Table 14.

Table 14: Cayley's table for soft filed $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ over soft addition $\tilde{+}$.

$\tilde{+}$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_0$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_1$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_0$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_3$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_6$
$\tilde{q}_2$	$\tilde{q}_2$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_5$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_8$	$\tilde{q}_6$	$\tilde{q}_7$
$\tilde{q}_3$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_2$
$\tilde{q}_4$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_3$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_6$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_0$
$\tilde{q}_5$	$\tilde{q}_5$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_8$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_2$	$\tilde{q}_0$	$\tilde{q}_1$
$\tilde{q}_6$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_3$	$\tilde{q}_4$	$\tilde{q}_5$
$\tilde{q}_7$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_6$	$\tilde{q}_1$	$\tilde{q}_2$	$\tilde{q}_0$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_3$
$\tilde{q}_8$	$\tilde{q}_8$	$\tilde{q}_6$	$\tilde{q}_7$	$\tilde{q}_2$	$\tilde{q}_0$	$\tilde{q}_1$	$\tilde{q}_5$	$\tilde{q}_3$	$\tilde{q}_4$

2. $((\phi,\mathcal{X}),\tilde{\cdot})$ is an abelian group where $\phi:\mathcal{X}\to\mathcal{P}(\mathcal{U})$ such that $\phi(\tilde{\rho}_1)=\{\bar{1},\bar{2}\}$, $\phi(\tilde{\rho}_2)=\{\bar{1},\bar{2}\}$. Table 15 is representing the corresponding Cayley's table.

Table 15: Cayley's table for soft filed $((\theta,\mathcal{X}),\tilde{+},\tilde{\cdot})$ over soft multiplication $\tilde{\cdot}$.

$\tilde{\cdot}$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_4$	$\tilde{q}_4$	$\tilde{q}_5$	$\tilde{q}_7$	$\tilde{q}_8$
$\tilde{q}_5$	$\tilde{q}_5$	$\tilde{q}_4$	$\tilde{q}_8$	$\tilde{q}_7$
$\tilde{q}_7$	$\tilde{q}_7$	$\tilde{q}_8$	$\tilde{q}_4$	$\tilde{q}_5$
$\tilde{q}_8$	$\tilde{q}_8$	$\tilde{q}_7$	$\tilde{q}_5$	$\tilde{q}_4$

3. Distributive properties hold.

6. Conclusion

The key objective of this work is to study various properties of soft algebraic structures and define them in terms of traditional algebraic structures. This study uses SMbs and SEts to define several soft algebraic structures. To guarantee understanding, an SSbGp is developed, and every possible SSbGp of the examined SGp is carefully examined using examples. Two essential elements of SGps, the soft inverse element and the soft identity element, are generalized in this study. Moreover, a generalized formula for determining how many possible SSbGps there could be for a specific SGp is given. It is observed that when an SGp is constructed, its number of SSbGps is larger than the number of subgroups of the corresponding classical group. The soft semigroup is also defined. Examples are provided for the definitions of SRg and soft field. Additionally, each of these definitions is confirmed by demonstrating a few of the associated preset theorems. Additionally, an SRg theorem analogous to the classical ring definition is offered with the aid of the suggested definition. The basis for the construction of soft algebraic structures, such as soft vector spaces, soft subfields, and soft subrings, is given in this study. It also offers a basis for extending the theory of soft sets to algebra, topology, and functional analysis, among other branches of mathematics. Since soft sets are more adaptable than classical sets and can be used to solve a wide range of real-world issues, this paper could be expanded to include more real-world problem modeling.

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