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An additive-cubic functional equation in a Banach space

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Abstract

In this article, we consider the following functional equation:

$$2h(x+y, z+w) + 2h(x-y, z-w) + 12h(x, z) = h(x+y, 2z+w) + h(x-y, 2z-w). \tag{1}$$

Using the direct and fixed point methods, we obtain the Hyers-Ulam stability of the proposed functional equation.

Keywords: Hyers-Ulam stability, additive-cubic functional equation, direct method, fixed point method.

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1. Introduction and preliminaries

In 1940, Ulam [33] mentioned a question concerning the stability of (group) homomorphisms which motivated the study of the stability problems of functional equations. Hyers [13] then obtained a partial answer to the question for additive mappings in Banach spaces. The stability of functional equations has been also known as the Hyers-Ulam stability. Later it was extended by Aoki [2] for additive mappings and, by Rassias [30], for linear mappings by concerning an unbounded Cauchy difference. Replacing the unbounded Cauchy difference by a general control function, Găvruta [9] also extended the Rassias theorem. Hyers himself contributed a number of notable articles such as [14–16]. Recently, Park gave the definition of additive ρ-functional inequalities and proved the Hyers-Ulam stability of those inequalities in Banach spaces in [24, 25, 27]. The stability problems of various functional equations and functional inequalities have been studied extensively (see [1, 6, 10, 11, 19–21, 23, 34]).

In this article, we let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ denote the sets of positive integers, real numbers, positive real numbers, and complex numbers, respectively. Also, we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. We begin with a useful result in the theory of fixed point.

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Theorem 1.1 ([3, 7]). *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $a \in \mathcal{X}$. For a strict Lipschitz contraction $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ with the Lipschitz constant $\alpha < 1$, either*

(1) $d(\mathcal{J}^n a, \mathcal{J}^{n+1} a) = \infty$ *for all* $n \in \mathbb{N}_0$ *or there exists* $n_0 \in \mathbb{N}$ *for which* $d(\mathcal{J}^n a, \mathcal{J}^{n+1} a) < \infty$ *for all* $n \geqslant n_0$;
(2) $\mathcal{J}^n a \to b^*$, *where* b^* *is a unique fixed point of* $\mathcal{J}$ *in* $\mathcal{X}_{n_0} := \{b \in \mathcal{X} : d(\mathcal{J}^{n_0} a, b) < \infty\}$;
(3) $d(b, b^*) \leqslant \frac{1}{1-\alpha} d(b, \mathcal{J} b)$ *for all* $b \in \mathcal{X}_{n_0}$.

Applications for the stability of functional equations for proving fixed point theorems and applications in nonlinear analysis were introduced by Isac and Rassias [17] in 1996. A large number of research articles concerning the stability problems of some functional equations and various definitions of stability by using the fixed pointed method have been widely studied in [4, 5, 8, 26, 28, 29, 31, 32] and others.

Jun and Kim [18] introduced the following cubic functional equation:

$$h(2x + y) + h(2x - y) = 2h(x - y) + 2h(x - y) + 12h(x). \tag{1.1}$$

They established the general solution and the Hyers-Ulam-Rassias stability problem of (1.1) for mapping from a real vector space to a Banach space. The Hyers-Ulam stability of the additive-quadratic functional equation, which is additive in the first variable and quadratic in the second variable:

$$h(x + y, z + w) + h(x - y, z - w) = 2h(x, z) + 2h(x, w),$$

was found in [12].

In this paper, first, we consider the functional equation (1) which is additive-cubic. Second, we prove the Hyers-Ulam stability of the functional equation (1) by using the direct method. Finally, we prove the Hyers-Ulam stability of the functional equation (1) using the fixed point method.

2. Hyers-Ulam stability of the additive-cubic functional equation: direct method

Throughout this article, let $\mathcal{X}$ and $\mathcal{Y}$ be a (complex) normed space and a (complex) Banach space, respectively. For a given mapping $h : \mathcal{X}^2 \to \mathcal{Y}$, we define, for all $x, y, z, w \in \mathcal{X}$,

$$Dh(x, y, z, w) := 2h(x + y, z + w) + 2h(x - y, z - w) + 12h(x, z) - h(x + y, 2z + w) - h(x - y, 2z - w).$$

We also denote the class of mappings $\{g : \mathcal{X}^2 \to \mathcal{Y} : g(x, 0) = g(0, y) = 0 \text{ for all } x, y \in \mathcal{X}\}$ by $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$.

Next, we introduce the concept of additive-cubic mapping.

Definition 2.1. A mapping $h : \mathcal{X}^2 \to \mathcal{Y}$ is called *additive-cubic* if h is additive in the first variable and cubic in the second variable, that is, h satisfies the following system of equations

$$h(x, z) + h(y, z) = h(x + y, z)$$

and

$$2h(x, y + z) + 2h(x, y - z) + 12h(x, z) = h(x, 2y + z) + h(x, 2y - z)$$

for all $x, y, z \in \mathcal{X}$. We denote the class of additive-cubic mapping by $\mathcal{AC}(\mathcal{X}, \mathcal{Y})$.

Lemma 2.2. *If* $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ *satisfies* (1), *then* $h \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$.

Proof. The fact that h is cubic in the second variable can be obtained by taking $y = 0$. Next, if $y = w = 0$, then $2h(x, z) + 2h(x, z) + 12h(x, z) = h(x, 2z) + h(x, 2z)$. So,

$$8h(x, z) = h(x, 2z) \tag{2.1}$$

for all $x, z \in \mathcal{X}$. If $w = 0$, then $2h(x + y, z) + 2h(x - y, z) + 12h(x, z) = h(x + y, 2z) + h(x - y, 2z)$. Using (2.1), we obtain

$$2h(x, z) = h(x + y, z) + h(x - y, z)$$

for all $x, y, z \in \mathcal{X}$, which implies that h is additive in the first variable. This completes the proof. □

Now, we present our main results.

Theorem 2.3. *Let* $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ *be a mapping such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x, y \in \mathcal{X}$. *If* $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ *and*

$$\|Dh(x,y,z,w)\| \leqslant \varphi(x,y)\varphi(z,w) \tag{2.3}$$

for all $x, y, z, w \in \mathcal{X}$, *then there exists a unique mapping* $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ *such that*

$$\|h(x,z) - H(x,z)\| \leqslant \frac{1}{16}\varphi(x,0)\Phi(z,0) \tag{2.4}$$

for all $x, z \in \mathcal{X}$.

Proof. Replacing $y = w = 0$ in (2.3), we obtain

$$\|8h(x,z) - h(x,2z)\| \leqslant \frac{1}{2}\varphi(x,0)\varphi(z,0) \tag{2.5}$$

and so

$$\left\|8h\left(x,\frac{z}{2}\right) - h(x,z)\right\| \leqslant \frac{1}{2}\varphi(x,0)\varphi\left(\frac{z}{2},0\right)$$

for all $x, z \in \mathcal{X}$. Then, for each $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\left\|8^l h\left(x,\frac{z}{2^l}\right) - 8^m h\left(x,\frac{z}{2^m}\right)\right\| \leqslant \sum_{j=l}^{m-1}\left\|8^j h\left(x,\frac{z}{2^j}\right) - 8^{j+1} h\left(x,\frac{z}{2^{j+1}}\right)\right\| \leqslant \frac{1}{16}\sum_{j=l+1}^{m} 8^j \varphi(x,0)\varphi\left(\frac{z}{2^j},0\right) \tag{2.6}$$

for all $x, z \in \mathcal{X}$. Thus $\{8^n h(x, 2^{-n}z)\}$ is a Cauchy sequence and so it is a convergent sequence in $\mathcal{Y}$ due to the completeness of $\mathcal{Y}$. Now, we define a mapping $H : \mathcal{X}^2 \to \mathcal{Y}$ by

$$H(x,z) := \lim_{n\to\infty} 8^n h\left(x,\frac{z}{2^n}\right)$$

for all $x, z \in \mathcal{X}$. Next, choose $l = 0$ and let $m \to \infty$ in (2.6). Then we have (2.4). It follows from (2.2) and (2.3) that

$$\|DH(x,y,z,w)\| = \lim_{n\to\infty} 8^n \left\|Dh\left(x,y,\frac{z}{2^n},\frac{w}{2^n}\right)\right\| \leqslant \varphi(x,y)\lim_{n\to\infty} 8^n \varphi\left(\frac{z}{2^n},\frac{w}{2^n}\right) = 0$$

for all $x, y, z, w \in \mathcal{X}$. Hence, by Lemma 2.2, $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$. To prove the uniqueness property of H, let G be another additive-cubic mapping satisfying (2.4). Then

$$\begin{aligned}\|H(x,z) - G(x,z)\| &= 8^q \left\|H\left(x,\frac{z}{2^q}\right) - G\left(x,\frac{z}{2^q}\right)\right\| \\ &\leqslant 8^q \left\|H\left(x,\frac{z}{2^q}\right) - h\left(x,\frac{z}{2^q}\right)\right\| + 8^q \left\|h\left(x,\frac{z}{2^q}\right) - G\left(x,\frac{z}{2^q}\right)\right\| \\ &\leqslant 8^{q-1}\varphi(x,0)\Phi\left(\frac{z}{2^q},0\right)\end{aligned}$$

for all $x, z \in \mathcal{X}$. Therefore, $\|H(x,z) - G(x,z)\| \to 0$ when $q \to \infty$ and this confirms the uniqueness of H. This completes the proof. □

Theorem 2.4. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that*

$$\tilde{\Phi}(x,y) := \sum_{j=1}^{\infty} 27^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) < \infty \tag{2.7}$$

for all $x, y \in \mathcal{X}$. If $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies (2.3), then there exists a unique mapping $\tilde{H} \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ such that

$$\|h(x,z) - \tilde{H}(x,z)\| \leqslant \frac{1}{27}\varphi(x,0)\left[\tilde{\Phi}(z,z) + \tilde{\Phi}(z,0)\right] \tag{2.8}$$

for all $x, z \in \mathcal{X}$.

Proof. Replacing $y = 0$ and $z = w$ in (2.3), we have

$$\|2h(x,2z) + 11h(x,z) - h(x,3z)\| \leqslant \varphi(x,0)\varphi(z,z)$$

for all $x, z \in \mathcal{X}$. This combined with (2.5) yields that

$$\|27h(x,z) - h(x,3z)\| \leqslant \varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right] \tag{2.9}$$

and so

$$\left\|27h\left(x, \frac{z}{3}\right) - h(x,z)\right\| \leqslant \varphi(x,0)\left[\varphi\left(\frac{z}{3}, \frac{z}{3}\right) + \varphi\left(\frac{z}{3}, 0\right)\right]$$

for all $x, z \in \mathcal{X}$. Then, for each $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\begin{aligned}\left\|27^l h\left(x, \frac{z}{3^l}\right) - 27^m h\left(x, \frac{z}{3^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1} \left\|27^j h\left(x, \frac{z}{3^j}\right) - 27^{j+1} h\left(x, \frac{z}{3^{j+1}}\right)\right\| \\ &\leqslant \frac{1}{27} \sum_{j=l+1}^{m} 27^j \varphi(x,0)\left[\varphi\left(\frac{z}{3^j}, \frac{z}{3^j}\right) + \varphi\left(\frac{z}{3^j}, 0\right)\right]\end{aligned} \tag{2.10}$$

for all $x, z \in \mathcal{X}$. Thus $\{27^n h(x, 3^{-n}z)\}$ is a Cauchy sequence and so it is a convergent sequence in $\mathcal{Y}$. Now, we define a mapping $\tilde{H} : \mathcal{X}^2 \to \mathcal{Y}$ by

$$\tilde{H}(x,z) := \lim_{n\to\infty} 27^n h\left(x, \frac{z}{3^n}\right)$$

for all $x, z \in \mathcal{X}$. Next, choose $l = 0$ and let $m \to \infty$ in (2.10). Then we have (2.8). It follows from (2.3) and (2.7) that

$$\|D\tilde{H}(x,y,z,w)\| = \lim_{n\to\infty} 27^n \left\|Dh\left(x, y, \frac{z}{3^n}, \frac{w}{3^n}\right)\right\| \leqslant \varphi(x,y) \lim_{n\to\infty} 27^n \varphi\left(\frac{z}{3^n}, \frac{w}{3^n}\right) = 0$$

for all $x, y, z, w \in \mathcal{X}$. Hence, by Lemma 2.2, $\tilde{H} \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$. To prove the uniqueness property of $\tilde{H}$, let $\tilde{G}$ be another additive-cubic mapping satisfying (2.8). Then

$$\begin{aligned}\|\tilde{H}(x,z) - \tilde{G}(x,z)\| &= 27^q \left\|\tilde{H}\left(x, \frac{z}{3^q}\right) - \tilde{G}\left(x, \frac{z}{3^q}\right)\right\| \\ &\leqslant 27^q \left\|\tilde{H}\left(x, \frac{z}{3^q}\right) - h\left(x, \frac{z}{3^q}\right)\right\| + 27^q \left\|h\left(x, \frac{z}{3^q}\right) - \tilde{G}\left(x, \frac{z}{3^q}\right)\right\| \\ &\leqslant 2 \cdot 27^{q-1} \varphi(x,0)\left[\tilde{\Phi}\left(\frac{z}{3^q}, \frac{z}{3^q}\right) + \tilde{\Phi}\left(\frac{z}{3^q}, 0\right)\right]\end{aligned}$$

for all $x, z \in \mathcal{X}$. Therefore, $\|\tilde{H}(x,z) - \tilde{G}(x,z)\| \to 0$ when $q \to \infty$ and this confirms the uniqueness of $\tilde{H}$. This completes the proof. □

Proof. By letting $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in \mathcal{X}$, we immediately obtain the result. □

Theorem 2.5. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi\left(2^j x, 2^j y\right) < \infty \tag{2.11}$$

for all $x, y \in \mathcal{X}$. Suppose that $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies (2.3). Then there exists a unique mapping $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ such that

$$\|h(x,z) - H(x,z)\| \leqslant \frac{1}{16} \varphi(x,0)\Psi(z,0) \tag{2.12}$$

for all $x, z \in \mathcal{X}$.

Proof. It follows from (2.5) that

$$\left\| h(x,z) - \frac{1}{8} h(x,2z) \right\| \leqslant \frac{1}{16} \varphi(x,0)\varphi(z,0)$$

for all $x, z \in \mathcal{X}$. Then, for all $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\left\| \frac{1}{8^l} h(x,2^l z) - \frac{1}{8^m} h(x,2^m z) \right\| \leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{8^j} h(x,2^j z) - \frac{1}{8^{j+1}} h(x,2^{j+1} z) \right\| \leqslant \frac{1}{16} \sum_{j=l}^{m-1} \frac{1}{8^j} \varphi(x,0)\varphi(2^j z,0) \tag{2.13}$$

for all $x, z \in \mathcal{X}$. Then the completeness of $\mathcal{Y}$ implies that $\{8^{-n} h(x, 2^n z)\}$ is convergent for each $x, z \in \mathcal{X}$. Next, we define a mapping $H(x,z) : \mathcal{X}^2 \to \mathcal{Y}$ by

$$H(x,z) := \lim_{n\to\infty} \frac{1}{8^n} h(x, 2^n z)$$

for all $x, z \in \mathcal{X}$. Choose $l = 0$ and let $m \to \infty$ in (2.13). Then we have (2.12). Thus it follows from (2.3) and (2.11) that

$$\|DH(x,y,z,w)\| = \lim_{n\to\infty} \frac{1}{8^n} \|Dh(x,y,2^n z, 2^n w)\| \leqslant \varphi(x,y) \lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n z, 2^n w) = 0$$

for all $x, y, z, w \in \mathcal{X}$. By Lemma 2.2, we have $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$. Let G be another mapping in $\mathcal{AC}(\mathcal{X}, \mathcal{Y})$ satisfying (2.12). Then we have

$$\begin{aligned} \|H(x,z) - G(x,z)\| &= \frac{1}{8^q} \|H(x,2^q z) - G(x,2^q z)\| \\ &\leqslant \frac{1}{8^q} \|H(x,2^q z) - h(x,2^q z)\| + \frac{1}{8^q} \|h(x,2^q z) - G(x,2^q z)\| \\ &\leqslant \frac{1}{8^{q+1}} \varphi(x,0)\Psi(2^q z,0) \to 0 \text{ as } q \to \infty \end{aligned}$$

for all $x, z \in \mathcal{X}$ and so the uniqueness of H follows. This completes the proof. □

Theorem 2.6. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping satisfying*

$$\tilde{\Psi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{27^j} \varphi\left(3^j x, 3^j y\right) < \infty \tag{2.14}$$

for all $x, y \in \mathcal{X}$. Suppose that $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies (2.3). Then there exists a unique mapping $\tilde{H} \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ such that

$$\|h(x,z) - \tilde{H}(x,z)\| \leqslant \frac{1}{27} \varphi(x,0) \left[\tilde{\Psi}(z,z) + \tilde{\Psi}(z,0)\right] \tag{2.15}$$

for all $x, z \in \mathcal{X}$.

Proof. It follows from (2.9) that

$$\left\|h(x,z)-\frac{1}{27}h(x,3z)\right\|\leqslant\frac{1}{27}\varphi(x,0)\left[\varphi(z,z)+\varphi(z,0)\right]$$

for all $x,z\in\mathcal{X}$. Then, for all $m,l\in\mathbb{N}_0$ with $m>l$, we have

$$\begin{aligned}\left\|\frac{1}{27^l}h(x,3^lz)-\frac{1}{27^m}h(x,3^mz)\right\| &\leqslant\sum_{j=l}^{m-1}\left\|\frac{1}{27^j}h(x,3^jz)-\frac{1}{27^{j+1}}h(x,3^{j+1}z)\right\|\\ &\leqslant\sum_{j=l}^{m-1}\frac{1}{27^{j+1}}\varphi(x,0)\left[\varphi\left(3^jz,3^jz\right)+\varphi\left(3^jz,0\right)\right]\end{aligned}\tag{2.16}$$

for all $x,z\in\mathcal{X}$. This implies that $\{27^{-n}h(x,3^nz)\}$ is a convergent sequence for all $x,z\in\mathcal{X}$. Next, we define a mapping $\tilde{H}(x,z):\mathcal{X}^2\to\mathcal{Y}$ by

$$\tilde{H}(x,z):=\lim_{n\to\infty}\frac{1}{27^n}h(x,3^nz)$$

for all $x,z\in\mathcal{X}$. Choose $l=0$ and let $m\to\infty$ in (2.16). Then we have (2.15). Thus it follows from (2.3) and (2.14) that

$$\|D\tilde{H}(x,y,z,w)\|=\lim_{n\to\infty}\frac{1}{27^n}\|Dh\left(x,y,3^nz,3^nw\right)\|\leqslant\varphi(x,y)\lim_{n\to\infty}\frac{1}{27^n}\varphi\left(3^nz,3^nw\right)=0$$

for all $x,y,z,w\in\mathcal{X}$. By Lemma 2.2, we have $\tilde{H}\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$. Let $\tilde{G}$ be another mapping in $\mathcal{AC}(\mathcal{X},\mathcal{Y})$ satisfying (2.15). Then we have

$$\begin{aligned}\|\tilde{H}(x,z)-\tilde{G}(x,z)\| &=\frac{1}{27^q}\left\|\tilde{H}\left(x,3^qz\right)-\tilde{G}\left(x,3^qz\right)\right\|\\ &\leqslant\frac{1}{27^q}\left\|\tilde{H}\left(x,3^qz\right)-h\left(x,3^qz\right)\right\|+\frac{1}{27^q}\left\|h\left(x,3^qz\right)-\tilde{G}\left(x,3^qz\right)\right\|\\ &\leqslant\frac{2}{27^{q+1}}\varphi(x,0)\left[\tilde{\Psi}\left(3^qz,3^qz\right)+\tilde{\Psi}\left(3^qz,0\right)\right]\to 0\ \text{ as }\ q\to\infty\end{aligned}$$

for all $x,z\in\mathcal{X}$ and so the uniqueness of $\tilde{H}$ follows. This completes the proof. □

If $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$, then we obtain the following corollaries.

Corollary 2.7. *For all $r,\theta\in\mathbb{R}_0^+$ with $r\neq 3$, let $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ and*

$$\|Dh(x,y,z,w)\|\leqslant\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r)\tag{2.17}$$

for all $x,y,z,w\in\mathcal{X}$. Then there exists a unique mapping $H\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that

$$\|h(x,z)-H(x,z)\|\leqslant\begin{cases}\frac{\theta}{2(2^r-8)}\|x\|^r\|z\|^r, & \text{if } r>3,\\ \frac{\theta}{2(8-2^r)}\|x\|^r\|z\|^r, & \text{if } r<3,\end{cases}$$

for all $x,z\in\mathcal{X}$.

Corollary 2.8. *For all $r,\theta\in\mathbb{R}_0^+$ with $r\neq 3$, if $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ satisfies (2.17), then there exists a unique mapping $\tilde{H}\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that*

$$\|h(x,z)-\tilde{H}(x,z)\|\leqslant\begin{cases}\frac{3\theta}{3^r-27}\|x\|^r\|z\|^r, & \text{if } r>3,\\ \frac{3\theta}{27-3^r}\|x\|^r\|z\|^r, & \text{if } r<3,\end{cases}$$

for all $x,z\in\mathcal{X}$.

3. Hyers-Ulam stability of the additive-cubic functional equation: fixed point method

In this section, we use the fixed point method to prove the Hyers-Ulam stability of the additive-cubic functional equation (1).

Theorem 3.1. *Let* $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ *be a mapping such that there exists* $L \in \mathbb{R}_0^+$ *with* $L < 1$ *satisfying*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leqslant \frac{L}{8}\varphi(x, y) \tag{3.1}$$

for all $x, y \in \mathcal{X}$. *Then, for a mapping* $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ *satisfying* (2.3), *there exists a unique mapping* $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ *such that*

$$\|h(x, z) - H(x, z)\| \leqslant \frac{L}{16(1-L)}\varphi(x, 0)\varphi(z, 0) \tag{3.2}$$

for all $x, z \in \mathcal{X}$.

Proof. Consider the set $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with the generalized metric d defined by

$$d(f, g) = \inf\left\{\mu \in \mathbb{R}_0^+ : \|f(x, z) - g(x, z)\| \leqslant \mu\varphi(x, 0)\,\varphi(z, 0), \quad \forall x, z \in \mathcal{X}\right\},$$

where $\inf\emptyset = +\infty$ as usual. Then $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), d)$ is complete, see [22]. Define a mapping $\mathcal{J} : \mathcal{F}_0(\mathcal{X}, \mathcal{Y}) \to \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ by

$$\mathcal{J}f(x, z) := 8f\left(x, \frac{z}{2}\right)$$

for all $x, z \in \mathcal{X}$. For all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with $d(f, g) = \varepsilon$, we have

$$\|f(x, z) - g(x, z)\| \leqslant \varepsilon\varphi(x, 0)\,\varphi(z, 0)$$

for all $x, z \in \mathcal{X}$. Consequently, from (3.1), we have

$$\begin{aligned}\|\mathcal{J}f(x, z) - \mathcal{J}g(x, z)\| &= \left\|8f\left(x, \frac{z}{2}\right) - 8g\left(x, \frac{z}{2}\right)\right\| \\ &\leqslant 8\varepsilon\varphi(x, 0)\,\varphi\left(\frac{z}{2}, 0\right) \leqslant 8\varepsilon\frac{L}{8}\varphi(x, 0)\,\varphi(z, 0) = L\varepsilon\varphi(x, 0)\,\varphi(z, 0)\end{aligned}$$

for all $x, z \in \mathcal{X}$. Then we have $d(\mathcal{J}f, \mathcal{J}g) \leqslant L\varepsilon$, which means that

$$d(\mathcal{J}f, \mathcal{J}g) \leqslant Ld(f, g)$$

for all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$. It follows from (2.5) that

$$\left\|h(x, z) - 8h\left(x, \frac{z}{2}\right)\right\| \leqslant \frac{1}{2}\varphi(x, 0)\varphi\left(\frac{z}{2}, 0\right) \leqslant \frac{L}{16}\varphi(x, 0)\varphi(z, 0)$$

for all $x, z \in \mathcal{X}$ and so

$$d(h, \mathcal{J}h) \leqslant \frac{L}{16}.$$

From Theorem 1.1, there exists $H : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the following.

(1) H is a unique fixed point of $\mathcal{J}$, i.e.,

$$H(x, z) = 8H\left(x, \frac{z}{2}\right)$$

for all $x, z \in \mathcal{X}$. Thus there exists $\mu \in (0, \infty)$ satisfying

$$\|h(x, z) - H(x, z)\| \leqslant \mu\varphi(x, 0)\,\varphi(z, 0)$$

for all $x, z \in \mathcal{X}$.

(2) $d(\mathcal{J}^l h, H) \to 0$ as $l \to \infty$, which implies that

$$\lim_{l\to\infty} 8^l h\left(x, \frac{z}{2^l}\right) = H(x,z)$$

for all $x, z \in \mathcal{X}$.

(3) $d(h, H) \leqslant \frac{1}{1-L} d(h, \mathcal{J}h)$, which implies that

$$\|h(x,z) - H(x,z)\| \leqslant \frac{L}{16(1-L)} \varphi(x,0)\varphi(z,0)$$

for all $x, z \in \mathcal{X}$.

From (3.1) and for all $x, y \in \mathcal{X}$, we have $8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leqslant L^n \varphi(x,y)$ tends to zero as $n \to \infty$. As in the proof of Theorem 2.3, we can show that $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$. Therefore, we can conclude that there exists a unique mapping $H \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$, which satisfies (3.2). This completes the proof. □

Theorem 3.2. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\varphi\left(\frac{x}{3}, \frac{y}{3}\right) \leqslant \frac{L}{27} \varphi(x,y) \tag{3.3}$$

for all $x, y \in \mathcal{X}$. Then, for a mapping $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfying (2.3), there exists a unique mapping $\tilde{H} \in \mathcal{AC}(\mathcal{X}, \mathcal{Y})$ such that

$$\|h(x,z) - \tilde{H}(x,z)\| \leqslant \frac{L}{27(1-L)} \varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right] \tag{3.4}$$

for all $x, z \in \mathcal{X}$.

Proof. Consider the generalized metric $\tilde{d}$ on $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ given by

$$\tilde{d}(f,g) = \inf\left\{\mu \in \mathbb{R}_0^+ : \|f(x,z) - g(x,z)\| \leqslant \mu\varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right], \ \forall x, z \in \mathcal{X}\right\},$$

where $\inf \emptyset = +\infty$. We can easily see that $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), \tilde{d})$ is complete, see [22]. Now, consider the linear mapping $\tilde{\mathcal{J}} : \mathcal{F}_0(\mathcal{X}, \mathcal{Y}) \to \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ defined by

$$\tilde{\mathcal{J}} f(x,z) := 27 f\left(x, \frac{z}{3}\right)$$

for all $x, z \in \mathcal{X}$. Let $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with $\tilde{d}(f,g) = \varepsilon$. Then we have

$$\|f(x,z) - g(x,z)\| \leqslant \varepsilon\varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right]$$

for all $a, c \in \mathcal{X}$. Also, from (3.3), we have

$$\begin{aligned}
\|\tilde{\mathcal{J}} f(x,z) - \tilde{\mathcal{J}} g(x,z)\| &= \left\|27 f\left(x, \frac{z}{3}\right) - 27 g\left(x, \frac{z}{3}\right)\right\| \\
&\leqslant 27\varepsilon\varphi(x,0)\left[\varphi\left(\frac{z}{3}, \frac{z}{3}\right) + \varphi\left(\frac{z}{3}, 0\right)\right] \\
&\leqslant 27\varepsilon\frac{L}{27}\varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right] = L\varepsilon\varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right]
\end{aligned}$$

for all $x, z \in \mathcal{X}$. Thus $\tilde{d}(\tilde{\mathcal{J}} f, \tilde{\mathcal{J}} g) \leqslant L\varepsilon$ and so

$$\tilde{d}(\tilde{\mathcal{J}} f, \tilde{\mathcal{J}} g) \leqslant L\tilde{d}(f,g)$$

for all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$. It follows from (2.9) that

$$\left\|h(x,z) - 27h\left(x, \frac{z}{3}\right)\right\| \leqslant \varphi(x,0)\left[\varphi\left(\frac{z}{3}, \frac{z}{3}\right) + \varphi\left(\frac{z}{3}, 0\right)\right] \leqslant \frac{L}{27}\varphi(x,0)\left[\varphi(z,z) + \varphi(z,0)\right]$$

for all $x, z \in \mathcal{X}$. Thus $\tilde{d}(h, \tilde{\mathcal{J}} h) \leqslant \frac{L}{27}$. It follows from Theorem 1.1 that there exists a mapping $\tilde{H} : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the following.

(1) $\tilde{H}$ is a unique fixed point of $\tilde{\mathcal{J}}$, i.e.,

$$\tilde{H}(x,z) = 27\tilde{H}\left(x,\frac{z}{3}\right)$$

for all $x,z \in \mathcal{X}$. Thus there exists $\mu \in (0,\infty)$ satisfying

$$\|h(x,z) - \tilde{H}(x,z)\| \leqslant \mu\varphi(x,0)\left[\varphi(z,z)+\varphi(z,0)\right]$$

for all $x,z \in \mathcal{X}$.

(2) $\tilde{d}(\tilde{\mathcal{J}}^l h, \tilde{H}) \to 0$ as $l \to \infty$, which implies that

$$\lim_{l\to\infty} 27^l h\left(x,\frac{z}{3^l}\right) = \tilde{H}(x,z)$$

for all $x,z \in \mathcal{X}$.

(3) $\tilde{d}(h,\tilde{H}) \leqslant \frac{1}{1-L}\tilde{d}(h,\tilde{\mathcal{J}}h)$, which implies that

$$\left\|h(x,z) - \tilde{H}(x,z)\right\| \leqslant \frac{L}{27(1-L)}\varphi(x,0)\left[\varphi(z,z)+\varphi(z,0)\right]$$

for all $x,z \in \mathcal{X}$.

From (3.3) and for all $x,y \in \mathcal{X}$, we have $27^n\varphi\left(\frac{x}{3^n},\frac{y}{3^n}\right) \leqslant L^n\varphi(x,y)$ tends to zero as $n \to \infty$. As in the proof of Theorem 2.4, we can show that $\tilde{H} \in \mathcal{AC}(\mathcal{X},\mathcal{Y})$. Therefore, we can conclude that there exists a unique mapping $\tilde{H} \in \mathcal{AC}(\mathcal{X},\mathcal{Y})$, which satisfies (3.4). This completes the proof. □

Theorem 3.3. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\varphi(x,y) \leqslant 8L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y \in \mathcal{X}$. Then, for a mapping $h \in \mathcal{F}_0(\mathcal{X},\mathcal{Y})$ satisfying (2.3), there exists a unique mapping $H \in \mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that

$$\|h(x,z) - H(x,z)\| \leqslant \frac{1}{16(1-L)}\varphi(x,0)\varphi(z,0) \tag{3.5}$$

for all $x,z \in \mathcal{X}$.

Proof. Consider the complete metric space $(\mathcal{F}_0(\mathcal{X},\mathcal{Y}), d)$ given in the proof of Theorem 3.1. If we define a mapping $\mathcal{J} : \mathcal{F}_0(\mathcal{X},\mathcal{Y}) \to \mathcal{F}_0(\mathcal{X},\mathcal{Y})$ by

$$\mathcal{J}f(x,z) := \frac{1}{8}f(x,2z)$$

for all $x,z \in \mathcal{X}$, then it follows from (2.5) that

$$\left\|h(x,z) - \frac{1}{8}h(x,2z)\right\| \leqslant \frac{1}{16}\varphi(x,0)\varphi(z,0)$$

for all $x,z \in \mathcal{X}$. By using the same technique as in the proof of Theorems 2.5 and 3.1, there exists a unique mapping $H \in \mathcal{AC}(\mathcal{X},\mathcal{Y})$ satisfying (3.5). This completes the proof. □

Theorem 3.4. *Let $\varphi : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\varphi(x,y) \leqslant 27L\varphi\left(\frac{x}{3},\frac{y}{3}\right)$$

for all $x,y \in \mathcal{X}$. Then, for a mapping $h \in \mathcal{F}_0(\mathcal{X},\mathcal{Y})$ satisfying (2.3), there exists a unique mapping $\tilde{H} \in \mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that

$$\|h(x,z) - \tilde{H}(x,z)\| \leqslant \frac{1}{27(1-L)}\varphi(x,0)\left[\varphi(z,z)+\varphi(z,0)\right] \tag{3.6}$$

for all $x,z \in \mathcal{X}$.

Proof. Consider the complete metric space $(\mathcal{F}_0(\mathcal{X},\mathcal{Y}),\tilde{d})$ given in the proof of Theorem 3.2. Let the linear mapping $\tilde{\mathcal{J}}:\mathcal{F}_0(\mathcal{X},\mathcal{Y})\to\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ defined by

$$\tilde{\mathcal{J}}f(x,z):=\frac{1}{27}f(x,3z)$$

for all $x,z\in\mathcal{X}$. It follows from (2.9) that

$$\left\|h(x,z)-\frac{1}{27}h(x,3z)\right\|\leqslant\frac{1}{27}\varphi(x,0)\left[\varphi(z,z)+\varphi(z,0)\right]$$

for all $x,z\in\mathcal{X}$. By using the same technique as in the proof of Theorems 2.6 and 3.2, there exists a unique mapping $\tilde{H}\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ satisfying (3.6). This completes the proof. □

Taking $L=2^{3-r}$ and $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$ in Theorem 3.1, we have the following.

Corollary 3.5. *Let $r,\theta\in\mathbb{R}_0^+$ with $r>3$. If $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ satisfies (2.17), then there exists a unique mapping $H\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that*

$$\|h(x,z)-H(x,z)\|\leqslant\frac{\theta}{2(2^r-8)}\|x\|^r\|z\|^r$$

for all $x,z\in\mathcal{X}$.

Taking $L=3^{3-r}$ and $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$ in Theorem 3.2, we have the following.

Corollary 3.6. *Let $r,\theta\in\mathbb{R}_0^+$ with $r>3$. If $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ satisfies (2.17), then there exists a unique mapping $\tilde{H}\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that*

$$\|h(x,z)-\tilde{H}(x,z)\|\leqslant\frac{3\theta}{3^r-27}\|x\|^r\|z\|^r$$

for all $x,z\in\mathcal{X}$.

Taking $L=2^{r-3}$ and $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$ in Theorem 3.3, we have the following.

Corollary 3.7. *Let $r,\theta\in\mathbb{R}_0^+$ with $r<3$ and let $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ be a mapping satisfying (2.17). Then there exists a unique mapping $H\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that*

$$\|h(x,z)-H(x,z)\|\leqslant\frac{\theta}{2(8-2^r)}\|x\|^r\|z\|^r$$

for all $x,z\in\mathcal{X}$.

Taking $L=3^{r-3}$ and $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$ in Theorem 3.4, we have the following.

Corollary 3.8. *Let $r,\theta\in\mathbb{R}_0^+$ with $r<3$ and let $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ be a mapping satisfying (2.17). Then there exists a unique mapping $\tilde{H}\in\mathcal{AC}(\mathcal{X},\mathcal{Y})$ such that*

$$\|h(x,z)-\tilde{H}(x,z)\|\leqslant\frac{3\theta}{27-3^r}\|x\|^r\|z\|^r$$

for all $x,z\in\mathcal{X}$.

4. Conclusion

We have proved the Hyers-Ulam stability results of the additive-cubic functional equation (1) in Banach spaces by the direct and the fixed point methods.

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