J. Math. Computer Sci., 33 (2024), 275–289

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

A novel decision-making technique based on T-rough bipolar fuzzy sets

Nosheen Malik[a], Muhammad Shabir[a], Tareq M. Al-shami[b,c], Rizwan Gul[a], Murad Arar[d,*]

[a]Department of Mathematics, Quaid-i-Azam University, Islamabad, 45320, Pakistan.

[b]Department of Mathematics, Sana'a University, P.O. Box 1247, Sana'a, Yemen.

[c]Department of Engineering Mathematics & Physics, Faculty of Engineering & Technology, Future University, New Cairo, Egypt.

[d]Department of Mathematics, College of Sciences and Humanities in Aflaj, Prince Sattam bin Abdulaziz University, Riyadh, Saudi Arabia.

Abstract

In this paper, we hybridize the bipolar fuzzy set (BFS) theory with T-rough sets (T-RSs) and initiate the novel idea of T-rough BFSs (T-RBFSs). The concept presented in this article has never been discussed earlier. Moreover, we investigate the axiomatic systems of T-RBFSs in detail. Meanwhile, we address a decision-making (DM) problem having data endowed with fuzziness and bipolarity in the framework of the T-RBFSs. We also propose an algorithm for this application. This algorithm facilitates tackling the case when there is a team of decision-makers instead of a single decision-maker and when the objects of one set need to be approximated by grading the objects of some other set. Moreover, a practical application of T-RBFSs in DM problems is given, accompanied by a practical example, which provides the optimal and the worst decision between some objects. Finally, a comparative analysis of the recommended study with several prevailing approaches is given to endorse the advantages of the suggested research.

Keywords: T-rough sets, bipolar fuzzy sets, T-rough bipolar fuzzy sets, decision-Making.

2020 MSC: 03E72, 62C86.

1. Introduction

We frequently deal with ambiguous and uncertain data while simulating real-world issues in several domains. The complexity and dimensions of this uncertainty are growing. Researchers and scientists present several tools to address these uncertainties, like fuzzy set (FS) theory [7, 10, 11, 50] and rough set (RS) theory [3, 37, 38]. Data analysis in various sciences also requires multiple problems associated with DM. Many intersecting applications of several hybrid fuzzy structures in DM problems are discussed in [8, 18, 30, 51]. But, in all these applications, there is only one decision-maker looking at only one side of the information.

Pawlak first created the RS theory [37, 38] as a practical mathematical strategy to navigate uncertain data. It has gotten broad attention in the research domains in both real-life dilemmas and the theory itself

*Corresponding author
Email address: muradshhada@gmail.com (Murad Arar)

doi: 10.22436/jmcs.033.03.06

Received: 2023-11-27 *Revised:* 2023-12-11 *Accepted:* 2023-12-13

[4, 6, 9, 23]. It has found practical uses in several disciplines, including knowledge discovery, data analysis, machine learning, conflict analysis, approximate classification, etc. Meanwhile, in the current databases, the attribute values might be both symbolic and real-valued, and the RS theory has trouble coping with such matters. To eliminate this issue, Dubois and Prade [15] conceived the notion of rough FS (RFS). Various models generated by combined RS with FS theories have been explored [24, 34, 35, 40, 44, 59].

Two main questions may arise here. First, "What is the need for bipolarity?" and second, "Why do we study T-roughness in the BFSs?". As a matter of fact, bipolarity is a vital feature in many problems of data analysis and mathematical modeling; it means observing the positive as well as the negative aspects of the data. The positive side depicts possible or feasible information, while the negative side depicts impossible or adverse information. The FSs successfully approximate the presence of an uncertain property in some objects. Still, they cannot measure the presence of its counter property in the same objects. For instance, let an FS ω approximating the sweetness of food, give the membership value $\omega(a) = 0.3$ to a particular food item "a". But the sourness in "a" might not be equal to 0.7. In ceratin situations, it might be less than 0.7 or even zero. So, the complement of an FS may not work in all situations. To fix this issue, Zhang [53] introduced the BFSs, which are characterized by a positive MF (PMF) ranging in $[0, 1]$, which demonstrates the degree of fulfillment of the elements for some property, and a negative MF (NMF) ranging in $[-1, 0]$, which quantifies the degree of fulfillment of those elements to the counter property. The BFSs have the potential to tackle the bipolarity and fuzziness of the data.

Studying such bipolarity is because human decisions are customarily dependent on bipolar thinking. For instance, the hardness and softness of rocks, sweetness and sourness of food, sympathy and hostility are the counter characteristics of the information in the DM process. Lee [27] pioneered several fundamental operations on BFSs. Also, Lee [28] conduct a comparative analysis of the BFSs with several other extensions of FSs. Dubois and Prade [16] offered three types of bipolarity.

BFS theory has not only been employed in bipolar logical reasoning but also used in other application domains like computational psychiatry [56], medical science [29, 58], bipolar quantum logic-based computing [54, 57], physics and philosophy [55]. Additionally, several initiatives have been made to integrate RS theory with BFS theory [22, 47, 48].

The second question is, "Why do we study T-roughness in the BFSs?". We know that the membership values of an FS or a BFS are just the upshots of estimation and are not exact. By describing the lower and upper rough approximations, we become closer to these membership values' certain and uncertain parts. Knowing about the definite part of these values will help us to refine the results and decisions. Pawlak used the equivalence relation (ER) to collect objects for the rough approximations. But, in many real-world problems, having an ER between the objects to be studied becomes very difficult. Davvaz [13] initiated the concept of T-roughness using a more generalized relationship between the objects. The T-rough approximations magically work to estimate the values of one set with the help of BFS in some other set. Thus, the decisions based on the T-RSs are more effective and consistent. Numerous studies on fuzzy and bipolar fuzzy frameworks have been undertaken. For instance, different kinds of fuzzy ideals in semigroups were studied in [41–43]. Akram and Dudek [1] offered the idea of regular bipolar fuzzy graphs. RSs are hybridized with FSs in [12, 15, 17]. Malik and Shabir [33] established the idea of roughness in fuzzy bipolar soft sets DM applications. Yao [49] discussed the decision analysis using probabilistic RSs. In [19], authors have interpreted the roughness of a set based on (α, β)-indiscernibility. Kim et al. [26] devised various characteristics in bipolar fuzzy topological spaces. Mahmood and Rehman [32] pioneered the concept of bipolar complex FSs and their applications in generalized similarity measures. Yang et al. [47] projected the transformation method of bipolar fuzzy RSs. Wei et al. [46] pioneered bipolar fuzzy Hamacher aggregation operators with applications in DM. Stanujkic et al. [36] expanded the MULTIMOORA approach in a bipolar fuzzy environment. Han et al. [21] offered a comprehensive TOPSIS strategy for improving the accuracy of bipolar disorder. The notion of complex BFS was given by Gulistan et al. [20]. Mahmood et al. [31] fostered a multiple criteria DM using BFSs. Riaz and Tehrim [39] analyzed the VIKOR strategy for BFSs via connection numbers of SPA theory-based metric spaces. Discussing the three-way decision approach via some environments of FSs was the goal of some articles

[14, 45, 52]. Recently, Al-shami [5] has investigated the relationships between objects and a bipolar soft set, which opened the door for novel theoretical applications via bipolar topological structures.

1.1. Motivation and research gap

BFS theory has become a popular framework among researchers to deal with the ambiguity and bipolarity in information. Based on the aforementioned research survey, the knowledge gap, uniqueness, and motives of this script are outlined as follows.

- ⧫ Theories of RSs and FSs have grown in popularity in recent eras. They offer an innovative theoretical paradigm for solving challenging issues in an uncertain scenario. However, in the process of DM under the FS and RS environment, only one decision-maker looks at only one side of the information. We address a broader situation in which a team of decision-makers (experts) wishes to decide on a specific object, considering the positivity and negativity observed in the objects.
- ⧫ In the existing approaches to the roughness of BFSs, bipolar fuzzy relations have been used to construct approximations of a BFS. This strategy restricts their effectiveness as the construction of bipolar fuzzy relations is not an easy task. As a result, instead of using bipolar fuzzy relations, in the present article, we employ a set-valued map (SVM) T and introduce a novel model termed T-RBFS. The SVM T provides the opportunity to classify the alternatives according to their characteristics into bunches (image set of T) in the context of the given scenario.
- ⧫ Another limitation of all the existing algorithms is that they decide between the objects based on the values assigned to the same objects. Sometimes, it is also necessary to decide between the objects of one set, using the values assigned to the objects of some other set. This case has not been addressed till now in the bipolar fuzzy environment. Thus, we will establish the notions of the T-RBFS model by hybridizing BFSs and T-RSs in this study. The T-RBFS model can solve the uncertain and incomplete problems in the RS theory, and handle the bipolarity and fuzziness. Moreover, we propose an efficient computational algorithm to address the DM problem in the framework of T-RBFSs.

1.2. Aim and contribution

The main contributions in this article are listed as follows.

- By combining T-RSs with BFS theory, a novel concept of T-RBFSs is proposed.
- Several significant structural properties of the T-RBFSs model are thoroughly investigated.
- A comprehensive DM approach under the framework of T-RBFSs has been developed. The DM methodology and the algorithm of the proposed technique are given.
- Practical case study also verifies the validity of the designed approach.

1.3. Framework of the paper

The rest of this script is organized as follows. Section 2 recalls some concepts necessary for understanding our research work. Section 3 studies the novel idea of T-RBFSs and their essential structural properties. Section 4 offers a novel DM framework based on the T-RBFS theory and verifies the DM strategy's main steps with a legitimate example. The comparative analysis is conducted in Section 5. Section 6 concludes our results and suggestions for more research directions.

2. Background Knowledge

In the current part, we will revise several key ideas related with RSs, T-RSs, FSs, and BFSs.

Definition 2.1 ([37, 38]). Assume that $\emptyset \neq \widetilde{\mathfrak{U}}$ is a finite universe, and $\sigma \subseteq \widetilde{\mathfrak{U}} \times \widetilde{\mathfrak{U}}$ is an ER. For $W \subseteq \widetilde{\mathfrak{U}}$, lower and upper rough approximations w.r.t. σ are described as:

$$\underline{W}_{\sigma} = \{\varkappa \in \widetilde{\mathfrak{U}} : [\varkappa]_{\sigma} \subseteq W\}, \quad \overline{W}^{\sigma} = \{\varkappa \in \widetilde{\mathfrak{U}} : [\varkappa]_{\sigma} \cap W \neq \emptyset\}.$$

The pair $\left(\underline{W}_{\sigma}, \overline{W}^{\sigma}\right)$ is called a RS of W in $\widetilde{\mathfrak{U}}$.

Pawlak's RS theory was initially offered using an ER. But, in real-world problems, the ER between the objects is often difficult to obtain because of the vagueness and inadequacy of human knowledge. For this reason, Davvaz [13] considered a generalized situation and presented the concept of T-RSs.

Definition 2.2 ([13]). Let $\widetilde{\mathfrak{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes and $Q \subseteq \widetilde{\mathcal{V}}$. Let $T : \widetilde{\mathfrak{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM, where $2^{\widetilde{\mathcal{V}}}$ represents the set of all non-empty subsets of $\widetilde{\mathcal{V}}$. The lower and upper inverses of Q under T are respectively characterized as:

$$\underline{(Q)}_{T} = \{\varkappa \in \widetilde{\mathfrak{U}} : T(\varkappa) \subseteq Q\}, \quad \overline{(Q)}^{T} = \{\varkappa \in \widetilde{\mathfrak{U}} : T(\varkappa) \cap Q \neq \emptyset\}.$$

If $\underline{(Q)}_{T} = \overline{(Q)}^{T}$, then, Q is T-definable, otherwise Q is said to be a T-RS.

These lower and upper inverses of the set Q are called the T-rough approximations of Q under T and the triplet $(\widetilde{\mathfrak{U}}, \widetilde{\mathcal{V}}, T)$ is the approximation space.

Example 2.3. Assume that $\widetilde{\mathfrak{U}} = \{\mathfrak{x}, \mathfrak{y}, \mathfrak{z}\}$ and $\widetilde{\mathcal{V}} = \{\mathfrak{a}, \mathfrak{b}\}$. Consider a SVM $T : \widetilde{\mathfrak{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ defined by $T(\mathfrak{x}) = \{\mathfrak{a}\}$, $T(\mathfrak{y}) = \{\mathfrak{b}\}$, $T(\mathfrak{z}) = \{\mathfrak{a}, \mathfrak{b}\}$. Then the lower and upper inverses of non-empty subsets of $\widetilde{\mathcal{V}}$ under T are given as

$$\underline{(\{\mathfrak{a}\})}_{T} = \{\mathfrak{x}\}, \quad \overline{(\{\mathfrak{a}\})}^{T} = \{\mathfrak{x}, \mathfrak{z}\}, \quad \underline{(\{\mathfrak{b}\})}_{T} = \{\mathfrak{y}\}, \quad \overline{(\{\mathfrak{b}\})}^{T} = \{\mathfrak{y}, \mathfrak{z}\}, \quad \underline{(\{\mathfrak{a}, \mathfrak{b}\})}_{T} = \{\mathfrak{x}, \mathfrak{y}, \mathfrak{z}\}, \quad \overline{(\{\mathfrak{a}, \mathfrak{b}\})}^{T} = \{\mathfrak{x}, \mathfrak{y}, \mathfrak{z}\}.$$

Definition 2.4 ([50]). An FS ξ over $\widetilde{\mathfrak{U}}$ is described through a function $\xi : \widetilde{\mathfrak{U}} \longrightarrow [0,1]$, where $\xi(\varkappa)$ signifies the degree to which $\varkappa$ belongs to ξ.

Definition 2.5 ([50]). Let ξ_1 and ξ_2 be two FSs over $\widetilde{\mathfrak{U}}$. Then

1. $\xi_1 = \xi_2 \Longleftrightarrow \xi_1(\varkappa) = \xi_2(\varkappa), \ \forall \ \varkappa \in \widetilde{\mathfrak{U}}$;
2. $\xi_1 \subseteq \xi_2 \Longleftrightarrow \xi_1(\varkappa) \leqslant \xi_2(\varkappa), \ \forall \ \varkappa \in \widetilde{\mathfrak{U}}$;
3. $\xi_1 \cup \xi_2 = \xi_1(\varkappa) \vee \xi_2(\varkappa)$;
4. $\xi_1 \cap \xi_2 = \xi_1(\varkappa) \wedge \xi_2(\varkappa)$;
5. $\xi_1^c = 1 - \xi_1(\varkappa)$.

Definition 2.6 ([53]). A BFS μ over $\widetilde{\mathfrak{U}}$ is a mathematical object having the form:

$$\mu = \left\{\left(\varkappa, \mu^{+}(\varkappa), \mu^{-}(\varkappa)\right) : \varkappa \in \widetilde{\mathfrak{U}}\right\},$$

where $\mu^{+} : \widetilde{\mathfrak{U}} \longrightarrow [0,1]$ and $\mu^{-} : \widetilde{\mathfrak{U}} \longrightarrow [-1,0]$ signifies the PMF and the NMF, respectively.

The collection of all BFSs in $\widetilde{\mathfrak{U}}$ is symbolized by $B_F(\widetilde{\mathfrak{U}})$.

Definition 2.7 ([27]). Let $\mu, \nu \in B_F(\widetilde{\mathfrak{U}})$. Then μ is contained in ν, that is $\mu \subseteq \nu$, if $\mu^{+}(\varkappa) \leqslant \nu^{+}(\varkappa)$ and $\mu^{-}(\varkappa) \geqslant \nu^{-}(\varkappa)$ for all $\varkappa \in \widetilde{\mathfrak{U}}$. Clearly, $\mu = \nu$ if $\mu \subseteq \nu$ and $\nu \subseteq \mu$.

Definition 2.8 ([27]). The whole BFS over $\widetilde{\mathfrak{U}}$ is expressed as $I_{\widetilde{\mathfrak{U}}} = \left(I^{+}, I^{-}\right)$, where $I^{+}(\varkappa) = 1$ and $I^{-}(\varkappa) = 0$, $\forall \ \varkappa \in \widetilde{\mathfrak{U}}$. The null BFS in $\widetilde{\mathfrak{U}}$ is symbolized as $O_{\widetilde{\mathfrak{U}}} = \left(O^{+}, O^{-}\right)$, where $O^{+}(\varkappa) = 0$ and $O^{-}(\varkappa) = -1$, $\forall \ \varkappa \in \widetilde{\mathfrak{U}}$. Hence, $I_{\widetilde{\mathfrak{U}}}(\varkappa) = (1,0)$ and $O_{\widetilde{\mathfrak{U}}}(\varkappa) = (0,-1)$, $\forall \ \varkappa \in \widetilde{\mathfrak{U}}$.

Definition 2.9 ([27]). Let $\mu, \nu \in B_F(\widetilde{\mathfrak{U}})$. Then

$$\mu \cup \nu = \left\{\left(\varkappa, \mu^{+}(\varkappa) \vee \nu^{+}(\varkappa), \mu^{-}(\varkappa) \wedge \nu^{-}(\varkappa)\right) : \varkappa \in \widetilde{\mathfrak{U}}\right\},$$
$$\mu \cap \nu = \left\{\left(\varkappa, \mu^{+}(\varkappa) \wedge \nu^{+}(\varkappa), \mu^{-}(\varkappa) \vee \nu^{-}(\varkappa)\right) : \varkappa \in \widetilde{\mathfrak{U}}\right\},$$
$$\mu^{c} = \left\{\left(\varkappa, 1 - \mu^{+}(\varkappa), -1 - \mu^{-}(\varkappa)\right) : \varkappa \in \widetilde{\mathfrak{U}}\right\}.$$

3. T-rough bipolar fuzzy sets (T-RBFSs)

In this segment, we propose the idea of T-RBFSs by characterizing the lower and upper inverses of the BFSs. Also, we investigate several significant structural characteristics of T-RBFSs in-depth with various concrete examples.

Definition 3.1. Assume that $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are two non-empty universes, $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ is a SVM, and $\mu \in B_F(\widetilde{\mathcal{V}})$. The T-rough approximations of μ are the lower and upper inverses of μ under T, which are the BFSs $\underline{T}(\mu)$ and $\overline{T}(\mu)$ in $\widetilde{\mathcal{U}}$, respectively, are described as:

$$\underline{T}(\mu) = \Big\{\Big(x, \bigwedge_{y\in T(x)} \mu^+(y), \bigvee_{y\in T(x)} \mu^-(y)\Big) : x \in \widetilde{\mathcal{U}}\Big\}, \quad \overline{T}(\mu) = \Big\{\Big(x, \bigvee_{y\in T(x)} \mu^+(y), \bigwedge_{y\in T(x)} \mu^-(y)\Big) : x \in \widetilde{\mathcal{U}}\Big\}.$$

If $\underline{T}(\mu) = \overline{T}(\mu)$, then μ is stated to be T- bipolar fuzzy definable; else, μ is a T-RBFS in $\widetilde{\mathcal{V}}$.

Let $\underline{T}(\mu)(x)$ and $\overline{T}(\mu)(x)$ be denoted by $\big(\underline{T}(\mu^+)(x), \underline{T}(\mu^-)(x)\big)$ and $\big(\overline{T}(\mu^+)(x), \overline{T}(\mu^-)(x)\big)$, respectively and let $T(x)$ contain the objects linked with x. Then, the knowledge concerning the object $x \in \widetilde{\mathcal{U}}$ interpreted by these inverses are as follows.

1. $\underline{T}(\mu^+)(x)$ provides the degree of definite fulfillment of the property of μ in the objects of $T(x)$.
2. $\underline{T}(\mu^-)(x)$ gives the degree of definite fulfillment of the counter property of μ in the objects of $T(x)$.
3. $\overline{T}(\mu^+)(x)$ demonstrates the degree of possible fulfillment of the property of μ in the objects of $T(x)$.
4. $\overline{T}(\mu^-)(x)$ signifies the degree of possible fulfillment of the counter property of μ in the objects of $T(x)$.

Following are a few cardinal structural properties of the T-RBFSs.

Theorem 3.2. *Assume that $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes, $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM and $\mu, \nu \in B_F(\widetilde{\mathcal{V}})$. Then, the following assertions hold:*

1. $\underline{T}(I_{\widetilde{\mathcal{V}}}) = I_{\widetilde{\mathcal{U}}} = \overline{T}(I_{\widetilde{\mathcal{V}}})$;
2. $\underline{T}(O_{\widetilde{\mathcal{V}}}) = O_{\widetilde{\mathcal{U}}} = \overline{T}(O_{\widetilde{\mathcal{V}}})$;
3. $\underline{T}(\mu) \subseteq \overline{T}(\mu)$;
4. $\overline{T}(\mu^c) = \big(\underline{T}(\mu)\big)^c$;
5. $\underline{T}(\mu^c) = \big(\overline{T}(\mu)\big)^c$;
6. $\underline{T}(\mu \cap \nu) = \underline{T}(\mu) \cap \underline{T}(\nu)$;
7. $\underline{T}(\mu \cup \nu) \supseteq \underline{T}(\mu) \cup \underline{T}(\nu)$;
8. $\overline{T}(\mu \cup \nu) = \overline{T}(\mu) \cup \overline{T}(\nu)$;
9. $\overline{T}(\mu \cap \nu) \subseteq \overline{T}(\mu) \cap \overline{T}(\nu)$;
10. $\mu \subseteq \nu \Longrightarrow \underline{T}(\mu) \subseteq \underline{T}(\nu)$ *and* $\overline{T}(\mu) \subseteq \overline{T}(\nu)$.

Proof.

(1) We have $I^+(y) = 1$ and $I^-(y) = 0$ for all $y \in \widetilde{\mathcal{U}}$, which yields $(\underline{T}(I_{\widetilde{\mathcal{V}}}))^+(x) = \bigwedge_{y\in T(x)} I^+(y) = 1$ and $(\underline{T}(I_{\widetilde{\mathcal{V}}}))^-(x) = \bigvee_{y\in T(x)} I^-(y) = 0$ for all $x \in \widetilde{\mathcal{U}}$. Thus, $\underline{T}(I_{\widetilde{\mathcal{V}}})(x) = \big((\underline{T}(I_{\widetilde{\mathcal{V}}}))^+(x), (\underline{T}(I_{\widetilde{\mathcal{V}}}))^-(x)\big) = (1, 0) = I_{\widetilde{\mathcal{U}}}(x)$ for all $x \in \widetilde{\mathcal{U}}$. This gives $\underline{T}(I_{\widetilde{\mathcal{V}}}) = I_{\widetilde{\mathcal{U}}}$. Similarly, one can show that $\overline{T}(I_{\widetilde{\mathcal{V}}}) = I_{\widetilde{\mathcal{U}}}$. Hence, $\underline{T}(I_{\widetilde{\mathcal{V}}}) = I_{\widetilde{\mathcal{U}}} = \overline{T}(I_{\widetilde{\mathcal{V}}})$.

(2) This is similar to the proof of (1).

(3) Straightforward.

(4) We have $(\mu^c)^+(y) = 1 - \mu^+(y)$ and $(\mu^c)^-(y) = -1 - \mu^-(y)$ for all $y \in \widetilde{\mathfrak{U}}$, by Definition 2.9, which yields

$$\begin{aligned}\overline{T}(\mu^c)(x) &= \left(\overline{T}(\mu^c)^+(x), \overline{T}(\mu^c)^-(x)\right)\\ &= \Big(\bigvee_{y\in T(x)} (\mu^c)^+(y), \bigwedge_{y\in T(x)} (\mu^c)^-(y)\Big)\\ &= \Big(\bigvee_{y\in T(x)} (1-\mu^+(y)), \bigwedge_{y\in T(x)} (-1-\mu^-(y))\Big)\\ &= \Big(1-\bigwedge_{y\in T(x)} \mu^+(y), -1-\bigvee_{y\in T(x)} \mu^-(y)\Big) = \left(\underline{T}(\mu)\right)^c(x)\end{aligned}$$

for all $x \in \widetilde{\mathfrak{U}}$. Hence, $\overline{T}(\mu^c) = \left(\underline{T}(\mu)\right)^c$.

(5) Similar to the proof of (4).

(6) By Definition 2.9, we have $(\mu\cap\nu)^+(y) = \mu^+(y)\wedge\nu^+(y)$ for all $y \in \widetilde{\mathfrak{U}}$. According to Definition 3.1 it implies that

$$\begin{aligned}\underline{T}(\mu\cap\nu)^+(x) &= \bigwedge_{y\in T(x)} \left(\mu^+(y)\wedge\nu^+(y)\right)\\ &= \Big(\bigwedge_{y\in T(x)} \mu^+(y)\Big)\wedge\Big(\bigwedge_{y\in T(x)} \nu^+(y)\Big) = \left(\underline{T}(\mu)^+(x)\wedge(\underline{T}(\nu)^-(x)\right) = \left(\underline{T}(\mu)\cap\underline{T}(\nu)\right)^+(x)\end{aligned}$$

for all $x \in \widetilde{\mathfrak{U}}$. Similarly, we can show that $\underline{T}(\mu\cap\nu)^-(x) = \left(\underline{T}(\mu)\cap\underline{T}(\nu)\right)^-(x)$ for all $x \in \widetilde{\mathfrak{U}}$.

Therefore, in the light of Definition 2.7, we get $\underline{T}(\mu\cap\nu) = \underline{T}(\mu)\cap\underline{T}(\nu)$.

(7) In the view of Definition 2.9, we have $(\mu\cup\nu)^+(y) = \mu^+(y)\vee\nu^+(y)$ for all $y \in \widetilde{\mathfrak{U}}$, which yields using Definition 3.1 that

$$\begin{aligned}\underline{T}(\mu\cup\nu)^+(x) &= \bigwedge_{y\in T(x)} \left(\mu^+(y)\vee\nu^+(y)\right)\\ &\geqslant \Big(\bigwedge_{y\in T(x)} \mu^+(y)\Big)\vee\Big(\bigwedge_{y\in T(x)} \nu^+(y)\Big) = \left(\underline{T}(\mu)^+(x)\vee(\underline{T}(\nu)^-(x)\right) = \left(\underline{T}(\mu)\cup\underline{T}(\nu)\right)^+(x)\end{aligned}$$

for all $x \in \widetilde{\mathfrak{U}}$. Similarly, we can show that $\underline{T}(\mu\cup\nu)^-(x) \leqslant \left(\underline{T}(\mu)\cup\underline{T}(\nu)\right)^-(x)$ for all $x \in \widetilde{\mathfrak{U}}$. Thus, in the light of Definition 2.7, we obtain $\underline{T}(\mu\cup\nu) \supseteq \underline{T}(\mu)\cup\underline{T}(\nu)$.

(8) Analogous to the proof of (6).

(9) Analogous to the proof of (7).

(10) Let $\mu\subseteq\nu$. According to Definitions 2.7 and 3.1, it follows that

$$\underline{T}(\mu) = \Big\{\Big(x, \bigwedge_{y\in T(x)} \mu^+(y), \bigvee_{y\in T(x)} \mu^-(y)\Big) : x\in\widetilde{\mathfrak{U}}\Big\} \subseteq \Big\{\Big(x, \bigwedge_{y\in T(x)} \nu^+(y), \bigvee_{y\in T(x)} \nu^-(y)\Big) : x\in\widetilde{\mathfrak{U}}\Big\} = \underline{T}(\nu).$$

Thus, $\underline{T}(\mu)\subseteq\underline{T}(\nu)$. Similarly, one can show $\overline{T}(\mu)\subseteq\overline{T}(\nu)$. □

We present a simple example for the illustration.

Example 3.3. Let $\widetilde{\mathfrak{U}} = \{\kappa_1, \kappa_2, \kappa_3\}$ and $\widetilde{\mathcal{V}} = \{\nu_1, \nu_2, \nu_3, \nu_4\}$. Construct a SVM $T : \widetilde{\mathfrak{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ as:

$$T(\kappa) = \begin{cases}\{\nu_1, \nu_2\}, & \text{if } \kappa=\kappa_1,\\ \{\nu_1, \nu_3, \nu_4\}, & \text{if } \kappa=\kappa_2,\\ \{\nu_2, \nu_4\}, & \text{if } \kappa=\kappa_3.\end{cases}$$

Let $\mu \in B_F(\widetilde{\mathcal{V}})$ be defined as:

$$\mu = \{(\nu_1, 0.6, -0.3), (\nu_2, 0.3, -0.5), (\nu_3, 0.5, -0.4), (\nu_4, 0.8, -0.1)\}.$$

The lower inverse of μ under T is calculated as follows:

$$\underline{T}(\mu) = \left\{\left(\kappa_i, \bigwedge_{\nu \in T(\kappa_i)} \mu^+(\nu), \bigvee_{\nu \in T(\kappa_i)} \mu^-(\nu)\right) : i = 1, 2, 3\right\} = \{(\kappa_1, 0.3, -0.3), (\kappa_2, 0.5, -0.1), (\kappa_3, 0.3, -0.1)\}.$$

Similarly, the upper inverse of μ under T is calculated as follows:

$$\overline{T}(\mu) = \left\{\left(\kappa_i, \bigvee_{\nu \in T(\kappa_i)} \mu^+(\nu), \bigwedge_{\nu \in T(\kappa_i)} \mu^-(\nu)\right) : i = 1, 2, 3\right\} = \{(\kappa_1, 0.6, -0.5), (\kappa_2, 0.8, -0.4), (\kappa_3, 0.8, -0.5)\}.$$

Now, we verify the inclusion in part (7) of Theorem 3.2. For this, take another BFS ν in $\widetilde{\mathcal{V}}$, defined as:

$$\nu = \{(\nu_1, 0.4, -0.4), (\nu_2, 0.7, -0.3), (\nu_3, 0.7, -0.1), (\nu_4, 0.6, -0.5)\}.$$

Now, we evaluate the following quantities:

$$\begin{aligned}
\underline{T}(\nu) &= \{(\kappa_1, 0.4, -0.3), (\kappa_2, 0.4, -0.1), (\kappa_3, 0.6, -0.3)\},\\
\mu \cup \nu &= \{(\nu_1, 0.6, -0.4), (\nu_2, 0.7, -0.5), (\nu_3, 0.7, -0.4), (\nu_4, 0.8, -0.5)\},\\
\underline{T}(\mu \cup \nu) &= \{(\kappa_1, 0.6, -0.4), (\kappa_2, 0.6, -0.4), (\kappa_3, 0.7, -0.5)\},\\
\underline{T}(\mu) \cup \underline{T}(\nu) &= \{(\kappa_1, 0.4, -0.3), (\kappa_2, 0.5, -0.1), (\kappa_3, 0.6, -0.3)\}.
\end{aligned}$$

From the above calculations and Definition 2.7, it can be observed that $\underline{T}(\mu \cup \nu) \supseteq \underline{T}(\mu) \cup \underline{T}(\nu)$. Similarly, the inequality (9) of the same theorem can also be observed.

It is important to note that in the approximation space $(\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}, T)$ if we substitute the SVM $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ by another SVM $T_1 : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$, such that $T_1(x)$ contains $T(x)$ for all $x \in \widetilde{\mathcal{U}}$. Then, the upper inverse of $\mu \in B_F(\widetilde{\mathcal{V}})$ under T_1 also contains the upper inverse of μ under T. However, this containment is reversed in lower inverses, as highlighted in the subsequent result.

Proposition 3.4. *Let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes and $T, T_1 : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be SVMs, such that $T(x) \subseteq T_1(x)$ for all $x \in \widetilde{\mathcal{U}}$. Then, $\overline{T}(\mu) \subseteq \overline{T_1}(\mu)$ and $\underline{T}(\mu) \supseteq \underline{T_1}(\mu)$ for all $\mu \in B_F(\widetilde{\mathcal{V}})$.*

Proof. Take any $\mu \in BF(\widetilde{\mathcal{V}})$. For all $x \in \widetilde{\mathcal{U}}$, we have $\bigvee_{y \in T(x)} \mu^+(y) \leqslant \bigvee_{y \in T_1(x)} \mu^+(y)$ and $\bigwedge_{y \in T(x)} \mu^-(y) \geqslant \bigwedge_{y \in T_1(x)} \mu^-(y)$, since $T(x) \subseteq T_1(x)$. So, Definition 2.7 gives $\overline{T}(\mu) \subseteq \overline{T_1}(\mu)$. Similarly, one can verify that $\underline{T_1}(\mu) \subseteq \underline{T_1}(\mu)$. □

Proposition 3.5. *Let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes and $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM, such that $T(x)$ is a singleton subset of $\widetilde{\mathcal{V}}$, for all $x \in \widetilde{\mathcal{U}}$. Then, $\underline{T}(\mu) = \overline{T}(\mu)$ for all $\mu \in B_F(\widetilde{\mathcal{V}})$.*

Proof. Straightforward. □

For every ordered pair (a, b), where $a \in [0, 1]$ and $b \in [-1, 0]$, we can construct a BFS $\mu_{ab} = (\mu^+_{ab}, \mu^-_{ab})$ in $\widetilde{\mathcal{V}}$, such that μ^+_{ab} and μ^-_{ab} are constant maps defined by $\mu^+_{ab}(x) = a$ and $\mu^-_{ab}(x) = b$ for all $x \in \widetilde{\mathcal{V}}$. Such a BFS μ_{ab} will be termed as a constant BFS in $\widetilde{\mathcal{V}}$.

Proposition 3.6. *Assume that $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes and $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM. If μ is a constant BFS in $\widetilde{\mathcal{V}}$, then $\underline{T}(\mu) = \overline{T}(\mu)$.*

Proof. Since μ is a constant BFS in $\widetilde{\mathcal{V}}$, we can take $\mu = \mu_{ab} \in B_F(\widetilde{\mathcal{V}})$ for some $a \in [0,1]$ and $b \in [-1,0]$. That is, $\mu^+_{ab}(y) = a$ and $\mu^-_{ab}(y) = b$ for all $y \in \widetilde{\mathcal{V}}$. Then, $\bigwedge_{y\in T(x)} \mu^+_{ab}(y) = a = \bigvee_{y\in T(x)} \mu^+_{ab}(y)$ and $\bigwedge_{y\in T(x)} \mu^-_{ab}(y) = b = \bigvee_{y\in T(x)} \mu^-_{ab}(y)$ for any $x \in \widetilde{\mathcal{U}}$. Thus,

$$\begin{aligned}\underline{T}(\mu_{ab}) &= \left\{\left(x, \bigwedge_{y\in T(x)} \mu^+_{ab}(y), \bigvee_{y\in T(x)} \mu^-_{ab}(y)\right) : x \in \widetilde{\mathcal{U}}\right\}\\ &= \left\{\left(x, \bigvee_{y\in T(x)} \mu^+_{ab}(y), \bigwedge_{y\in T(x)} \mu^-_{ab}(y)\right) : x \in \widetilde{\mathcal{U}}\right\} = \overline{T}(\mu_{ab}).\end{aligned}$$

Hence, $\underline{T}(\mu) = \overline{T}(\mu)$. □

The converse of Proposition 3.6 does not hold, in general, but it may hold by imposing a further condition on the map T. This can be seen in the following result.

Proposition 3.7. *Let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes, $\mu \in B_F(\widetilde{\mathcal{V}})$ and $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM, such that $T(x) = \widetilde{\mathcal{V}}$ for all $x \in \widetilde{\mathcal{U}}$. Then, $\underline{T}(\mu) = \overline{T}(\mu)$ if and only if μ is a constant BFS in $\widetilde{\mathcal{V}}$.*

Proof. For a constant BFS μ in $\widetilde{\mathcal{V}}$, we have $\underline{T}(\mu) = \overline{T}(\mu)$ by Proposition 3.6. To prove the converse statement, take a BFS μ in $\widetilde{\mathcal{V}}$, such that $\underline{T}(\mu) = \overline{T}(\mu)$. This gives

$$\left\{\left(x, \bigwedge_{y\in T(x)} \mu^+(y), \bigvee_{y\in T(x)} \mu^-(y)\right) : x \in \widetilde{\mathcal{U}}\right\} = \left\{\left(x, \bigvee_{y\in T(x)} \mu^+(y), \bigwedge_{y\in T(x)} \mu^-(y)\right) : x \in \widetilde{\mathcal{U}}\right\}.$$

Using $T(x) = \widetilde{\mathcal{V}}$ for all $x \in \widetilde{\mathcal{U}}$, the above equation gives

$$\bigwedge_{y\in\widetilde{\mathcal{V}}} \mu^+(y) = \bigvee_{y\in\widetilde{\mathcal{V}}} \mu^P(y) \quad \text{and} \quad \bigwedge_{y\in\widetilde{\mathcal{V}}} \mu^-(y) = \bigvee_{y\in\widetilde{\mathcal{V}}} \mu^-(y)$$

for all $y \in \widetilde{\mathcal{V}}$. This clearly indicates that μ is a constant BFS μ_{ab} in $\widetilde{\mathcal{V}}$, where $a = \bigvee_{y\in\widetilde{\mathcal{V}}} \mu^+(y)$ and $b = \bigvee_{y\in\widetilde{\mathcal{V}}} \mu^-(y)$. □

Using the lower and upper inverses, we define a binary relation $\approx$ on $B_F(\widetilde{\mathcal{V}})$ as follows.

Definition 3.8. Assume that $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are two non-empty universes, $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM and $\mu, \nu \in B_F(\widetilde{\mathcal{V}})$. Then, μ and ν are T-roughly similar if and only if $\underline{T}(\mu) = \underline{T}(\nu)$ and $\overline{T}(\mu) = \overline{T}(\nu)$.

We denote two T-roughly similar BFSs μ and ν by $\mu \approx \nu$. This binary relation on $B_F(\widetilde{\mathcal{V}})$ can be called the T-rough similarity relation on $BF(\widetilde{\mathcal{V}})$. The relation $\approx$ is surely an ER that induces a partition $B_F(\widetilde{\mathcal{V}})/\approx$ of $B_F(\widetilde{\mathcal{V}})$. An equivalence class of $\approx$ is a family of all BFSs in $\widetilde{\mathcal{V}}$ with the same lower and upper inverses. Some of the characteristics of the T-rough similarity relation $\approx$ are given in the following result.

Theorem 3.9. *Let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ be two non-empty universes and $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ be a SVM. Then, the following assertions hold for all $\mu, \nu \in B_F(\widetilde{\mathcal{V}})$:*

1. $\mu \subseteq \nu$ *and* $\nu \approx O_{\widetilde{\mathcal{V}}}$ *imply that* $\mu \approx O_{\widetilde{\mathcal{V}}}$;
2. $\mu \subseteq \nu$ *and* $\mu \approx I_{\widetilde{\mathcal{V}}}$ *imply that* $\nu \approx I_{\widetilde{\mathcal{V}}}$;

3. *if* $(\mu \cup \nu) \approx O_{\widetilde{\mathcal{V}}}$ *then* $\mu \approx O_{\widetilde{\mathcal{V}}}$ *and* $\nu \approx O_{\widetilde{\mathcal{V}}}$*;*
4. *if* $(\mu \cap \nu) \approx I_{\widetilde{\mathcal{V}}}$ *then* $\mu \approx I_{\widetilde{\mathcal{V}}}$ *and* $\nu \approx I_{\widetilde{\mathcal{V}}}$*.*

Proof.

(1) $\nu \approx O_{\widetilde{\mathcal{V}}}$ implies that $\overline{T}(\nu) = \overline{T}(O_{\widetilde{\mathcal{V}}})$ and $\underline{T}(\nu) = \underline{T}(O_{\widetilde{\mathcal{V}}})$. Also $\mu \subseteq \nu$ and Theorem 3.2 imply that $\overline{T}(\mu) \subseteq \overline{T}(\nu) = \overline{T}(O_{\widetilde{\mathcal{V}}}) \subseteq \overline{T}(\mu)$. Thus, $\overline{T}(\mu) = \overline{T}(O_{\widetilde{\mathcal{V}}})$. Similarly, $\underline{T}(\mu) = \underline{T}(O_{\widetilde{\mathcal{V}}})$. This shows that $\mu \approx O_{\widetilde{\mathcal{V}}}$.

(2) $\mu \approx I_{\widetilde{\mathcal{V}}}$ implies that $\overline{T}(\mu) = \overline{T}(I_{\widetilde{\mathcal{V}}})$ and $\underline{T}(\mu) = \underline{T}(I_{\widetilde{\mathcal{V}}})$. Also $\mu \subseteq \nu$ and Theorem 3.2 imply that $\overline{T}(\nu) \supseteq \overline{T}(\mu) = \overline{T}(I_{\widetilde{\mathcal{V}}}) \supseteq \overline{T}(\nu)$. Thus, $\overline{T}(\nu) = \overline{T}(I_{\widetilde{\mathcal{V}}})$. Similarly, $\underline{T}(\nu) = \underline{T}(I_{\widetilde{\mathcal{V}}})$. This indicates that $\nu \approx I_{\widetilde{\mathcal{V}}}$.

(3) This follows from (1).

(4) This follows from (2). □

4. Applications of T-RBFSs in DM problems

DM is choosing among several objects using the available knowledge and assets. It includes analyzing and evaluating data, considering numerous viewpoints, and making informed and effective choices compatible with an individual's or organization's goals and values. DM is a necessary skill for both personal and professional level. It is a vital problem-solving component and assists individuals and organizations navigate challenges and progress. Effective DM necessitates good judgment and the ability to communicate and collaborate with others to gather additional information and seek input and feedback. The researchers employ their expertise to develop algorithms to decide the optimal alternative from a collection $\widetilde{\mathcal{U}}$ of alternatives. In many situations, a team of decision-makers wishes to choose an optimal object. There may also be situations when the decision is to be taken among the objects of one set by grading the objects of some other set. By using the lower and upper inverses of the BFSs, we can estimate the definite and indefinite parts in the membership values of the objects. So, the decision made with the help of the T-rough approximations of BFSs are more refined and reliable than the decision made by just adding these membership values. The algorithm presented in this paper has three main advantages.

- It accommodates the opinions of more than one decision-maker.
- It gives approximations of the objects of one set by grading the objects of some other set.
- T-rough approximations of BFSs make the decision more refined and reliable.

Let $\widetilde{\mathcal{U}} = \{x_i : 1 \leqslant i \leqslant n_1\}$ be the collection of objects among which the decision is to be made and let $\widetilde{\mathcal{V}} = \{y_j : 1 \leqslant j \leqslant n_2\}$ be the collection of objects used to asses the objects of $\widetilde{\mathcal{U}}$. The objects of $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are linked by a SVM $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$. The collection of the BFSs in $\widetilde{\mathcal{V}}$ expresses the opinions of n_3 independent decision-makers about the objects of $\widetilde{\mathcal{V}}$ are denoted by $\eth = \{\mu_k : 1 \leqslant k \leqslant n_3\}$. The knowledge about the objects y_j of $\widetilde{\mathcal{V}}$, given by $\eth$, is denoted through a table with $\mu_k(y_j)$ as its $(j,k)^{th}$ entry. These μ_k are the BFSs in $\widetilde{\mathcal{V}}$, while the lower inverse $\underline{T}(\mu_k)$ and the upper inverse $\overline{T}(\mu_k)$ of μ_k are the BFSs in $\widetilde{\mathcal{U}}$. Recall that we write $\underline{T}(\mu_k)(x_i) = \left(\underline{T}(\mu_k^+)(x_i), \underline{T}(\mu_k^-)(x_i)\right)$ and $\overline{T}(\mu_k)(x_i) = \left(\overline{T}(\mu_k^+)(x_i), \overline{T}(\mu_k^-)(x_i)\right)$ for $x_i \in \widetilde{\mathcal{U}}$.

Definition 4.1. The decision coefficient D has the values d_i w.r.t. each object $x_i \in \widetilde{\mathcal{U}}$, given by:

$$d_i = \sum_{k=1}^{n_3} \left(\underline{T}(\mu_k^+)(x_i) + \underline{T}(\mu_k^-)(x_i) + \overline{T}(\mu_k^+)(x_i) + \overline{T}(\mu_k^-)(x_i)\right). \tag{11}$$

4.1. *Proposed DM algorithm*

To choose the best object among the available ones, here we offer a DM algorithm in the framework of the T-RBFS model. The relevant steps are outlined as follows.

1. Input $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$.
2. Input the SVM $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$.
3. Input the BFSs μ_k.
4. Evaluate $\underline{T}(\mu_k)(x_i)$ and $\overline{T}(\mu_k)(x_i)$ for each $\mu_k \in \eth$ and for each $x_i \in \widetilde{\mathcal{U}}$.
5. Find the decision values d_i for each object $x_i \in \widetilde{\mathcal{U}}$, according to Definition 4.1.
6. Construct the decision table by rearranging the table obtained in Step 5 in the descending order of values of d_i. Choose m, so that, $d_m = \max_i d_i$. Then x_m is the optimal object to be decided.

A flowchart representation of the above algorithm is shown in Figure 1.

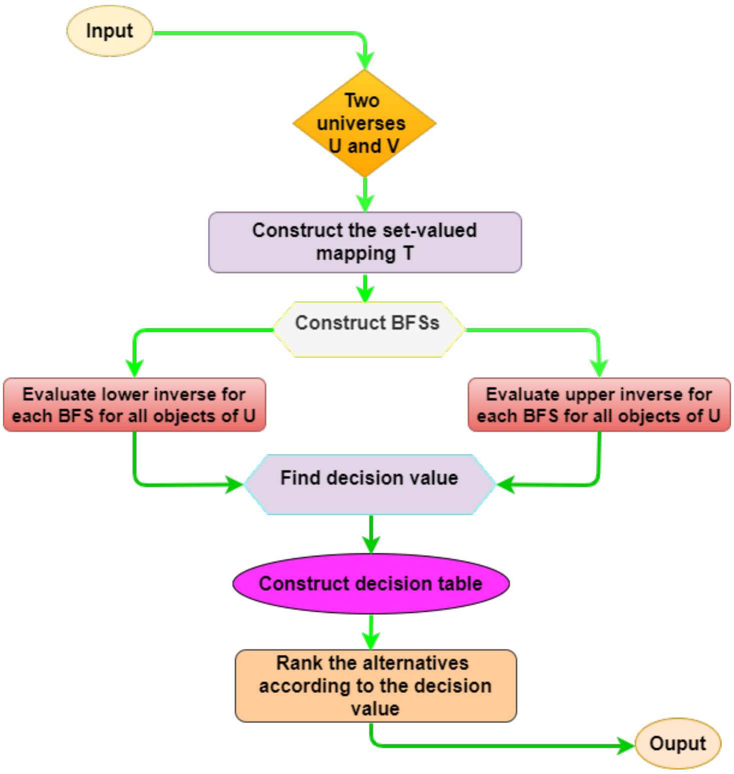

Figure 1: The method of the proposed DM algorithm.

4.2. *Case study*

To illustrate the feasibility, validity, and necessity of the designed idea and method, consider the following example.

Example 4.2. Assume that $\widetilde{\mathcal{U}} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a group of six teams from six different colleges participating in a science competition, such that each college has to prepare three different science models, which are being displayed in the competition without the title of the college (for the sake of unbiasedness). Let $\eth = \{\mu_1, \mu_2, \mu_3\}$ be a collection of BFSs in $\widetilde{\mathcal{V}}$, describing the opinions of three independent judges about the science models. Table 1 shows the models prepared and presented by team x_i.

1. Input $\widetilde{\mathcal{U}} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\widetilde{\mathcal{V}} = \{y_1, y_2, \ldots, y_{18}\}$.

2. The SVM $T : \widetilde{\mathcal{U}} \longrightarrow 2^{\widetilde{\mathcal{V}}}$ may be defined by using Table 1 as $T(x_i) = \{y_{3i-2}, y_{3i-1}, y_3\}$ for all $x_i \in \widetilde{\mathcal{U}}$.
3. The BFSs μ_k of $\eth$ showing the opinions of judges about the science models being displayed are given in Table 2.

Table 1: Models prepared by the team x_i.

Team	Models
x_1	y_1, y_2, y_3
x_2	y_4, y_5, y_6
x_3	y_7, y_8, y_9
x_4	y_{10}, y_{11}, y_{12}
x_5	y_{13}, y_{14}, y_{15}
x_6	y_{16}, y_{17}, y_{18}

Table 2: Tabular representation of $\eth$.

$\widetilde{\mathcal{V}}$	μ_1	μ_2	μ_3
y_1	$(0.7, -0.2)$	$(0.6, -0.2)$	$(0.6, -0.3)$
y_2	$(0.5, -0.3)$	$(0.6, -0.2)$	$(0.6, -0.3)$
y_3	$(0.9, -0.1)$	$(0.7, 0)$	$(0.8, -0.2)$
y_4	$(0.6, -0.3)$	$(0.6, -0.2)$	$(0.6, -0.3)$
y_5	$(0.9, 0)$	$(0.8, 0)$	$(0.8, -0.1)$
y_6	$(0.8, -0.1)$	$(0.6, -0.2)$	$(0.6, 0)$
y_7	$(0.7, -0.2)$	$(0.7, 0)$	$(0.6, -0.1)$
y_8	$(0.9, -0.1)$	$(0.8, 0)$	$(0.8, -0.1)$
y_9	$(0.8, -0.2)$	$(0.7, -0.2)$	$(0.7, 0)$
y_{10}	$(0.8, -0.1)$	$(0.7, -0.1)$	$(0.7, -0.2)$
y_{11}	$(0.8, -0.1)$	$(0.9, 0)$	$(0.9, 0)$
y_{12}	$(0.7, -0.1)$	$(0.8, -0.2)$	$(0.8, -0.1)$
y_{13}	$(0.8, -0.2)$	$(0.7, -0.1)$	$(0.7, -0.2)$
y_{14}	$(0.6, -0.2)$	$(0.5, -0.3)$	$(0.6, -0.2)$
y_{15}	$(0.7, -0.3)$	$(0.6, -0.1)$	$(0.7, -0.3)$
y_{16}	$(0.5, -0.3)$	$(0.6, -0.2)$	$(0.6, -0.1)$
y_{17}	$(0.6, -0.3)$	$(0.5, -0.2)$	$(0.6, -0.2)$
y_{18}	$(0.7, -0.1)$	$(0.7, -0.2)$	$(0.8, -0.1)$

4. $\underline{T}(\mu_k)(x_i)$ and $\overline{T}(\mu_k)(x_i)$ are calculated for each $\mu_k \in \eth$ and $x_i \in \widetilde{\mathcal{U}}$, in Table 3.

Table 3: Calculations of $\underline{T}(\mu_k)(x_i)$ and $\overline{T}(\mu_k)(x_i)$.

$\widetilde{\mathcal{U}}$	$\underline{T}(\mu_1)(x_i)$	$\underline{T}(\mu_2)(x_i)$	$\underline{T}(\mu_3)(x_i)$	$\overline{T}(\mu_1)(x_i)$	$\overline{T}(\mu_2)(x_i)$	$\overline{T}(\mu_3)(x_i)$	d_i
x_1	$(0.5, -0.1)$	$(0.6, 0)$	$(0.6, -0.2)$	$(0.9, -0.3)$	$(0.7, -0.2)$	$(0.8, -0.3)$	3.0
x_2	$(0.6, 0)$	$(0.6, 0)$	$(0.6, 0)$	$(0.9, -0.3)$	$(0.8, -0.2)$	$(0.8, -0.3)$	3.5
x_3	$(0.7, -0.1)$	$(0.7, 0)$	$(0.6, 0)$	$(0.9, -0.2)$	$(0.8, -0.2)$	$(0.8, -0.1)$	3.9
x_4	$(0.7, -0.1)$	$(0.7, 0)$	$(0.7, 0)$	$(0.8, -0.1)$	$(0.9, -0.2)$	$(0.9, -0.2)$	4.1
x_5	$(0.6, -0.2)$	$(0.5, -0.1)$	$(0.6, -0.2)$	$(0.8, -0.3)$	$(0.7, -0.3)$	$(0.7, -0.3)$	2.5
x_6	$(0.5, -0.1)$	$(0.5, -0.2)$	$(0.6, -0.1)$	$(0.7, -0.3)$	$(0.7, -0.2)$	$(0.8, -0.2)$	2.7

5. The decision values d_i for each $x_i \in \widetilde{\mathcal{U}}$ are calculated in the last column of Table 3.
6. Table 4 is the decision table.

Table 4: Decision table.

x_i	d_i
x_4	4.1
x_3	3.9
x_2	3.5
x_1	3.0
x_6	2.7
x_5	2.5

From Table 4, we can see that $\max_{i} d_i = d_4 = 4.1$. Thus, x_4 is the winning team. Moreover, the ranking among the teams is given as:

$$x_4 \succeq x_3 \succeq x_2 \succeq x_1 \succeq x_6 \succeq x_5.$$

The graphic interpretation of the ranking order of the teams is depicted in Figure 2.

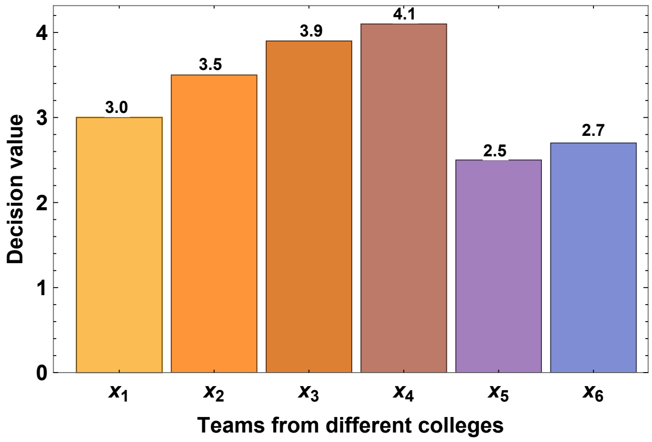

Figure 2: Ranking of teams.

5. Comparative study and discussion

According to the best of our knowledge, there does not exist any strategy or DM algorithm of T-RSs via BFSs environment in the literature. However, if we compare our devised strategy with the approaches discussed in [12, 13, 49, 50], we explore the following points.

1. These methods are unable to address the bipolarity in the DM process, which is a critical feature of human cognition and behavior. Therefore, the proposed DM approach in this article has wider practicability and stronger effectiveness.
2. The suggested approach is more efficient because it is capable of situations involving multiple experts.
3. The proposed DM technique is easy to understand and can be applied to real-life problems.
4. A characteristic comparative analysis of our proposed DM technique with different approaches is recapitulated in Table 5. The comparison is evaluated with features: membership function (MF), non-membership function (NMF), computational complexity, number of decision-makers, and ranking of alternatives.

Table 5: Characteristics comparison of various methods with our designed approach.

Methods	Characteristics				
	Handle MF	Handle NMF	Computational complexity	Decision-makers	Ranking of alternatives
Han et al. [22]	Yes	Yes	High	One	No
Alghamdi et al. [2]	Yes	Yes	Less	One	Yes
Jana et al. [25]	Yes	Yes	Less	One	Yes
Mahmood et al. [31]	Yes	Yes	Less	One	Yes
Yang et al. [48]	Yes	Yes	Less	One	No
Han et al. [21]	Yes	Yes	High	More than one	No
Proposed approach	Yes	Yes	Less	More than one	Yes

6. Conclusions

In the past few years, the RS theory has gained in popularity. It is a new theoretical mechanism to cope complicated issues in real-life dilemmas. There are numerous fields of life where the DM information does not involve the objects' positive aspects but also the negative aspects of the objects. Positive side reflects what is seen to be feasible, while negative side reveals what is thought to be impractical. In such cases of DM scenarios, the classical and fuzzy techniques are incapable.

In general, the main contributions made by this work are as follows.

- We have proposed a general strategy for roughness in BFSs and presented the idea of T-RBFSs.
- Some fundamental algebraic properties of the T-RBFSs and their inverses are studied in detail, and some effects of the set-valued mapping on these inverses are examined.
- Moreover, a similarity relation between the BFSs is defined based on their T-rough approximations.
- In order to demonstrate the application of the designed model with bipolar fuzzified data, we have built a unique strategy for addressing DM dilemmas using the theory of T-RBFSs. The DM procedure and an algorithm of the devised strategy have been elaborated. This algorithm has three main advantages. Firstly, it accommodates the views of more than one decision-maker. Secondly, it gives approximations of the objects of one set by grading the objects of some other set. Thirdly, the T-rough approximations make the decision more refined and reliable.
- Furthermore, a real-world illustration has been offered to indicate the effectiveness of the designed methodology.
- A comparison has been made using prevailing methodologies to demonstrate the supremacy of the recommended strategy.

We believe that our research sheds more light on the theoretical foundations of BFSs and may result in more sufficient mathematical methods to approximate reasoning in soft computing. At the same time, we acknowledge that there are still a number of directions in which this viewpoint has to be explored. In the future, we want to extend our work in the following directions.

- The practical usefulness of the designed method in addressing various models, like TOPSIS, VIKOR, ELECTRE, AHP, COPRAS, PROMETHEE, etc.
- Researchers might look at the properties of T-RBFSs concerning different algebraic structures.
- The attribute reduction of T-RBFSs should be analyzed, and comprehensive evaluations and comparisons with current studies should also be justified and explored.
- Another perspective is to analyze the topological characteristics of T-RBFSs to establish a robust platform for further study.

Acknowledgment

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

References

[1] M. Akram, W. A. Dudek, *Regular bipolar fuzzy graphs*, Neural Comput. Appl., **21** (2012), 197–205. 1

[2] M. A. Alghamdi, N. O. Alshehri, M. Akram, *Multi-criteria decision-making methods in bipolar fuzzy environment*, Int. J. Fuzzy Syst., **20** (2018), 2057–2064. 5

[3] T. M. Al-shami, *An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application*, Inform. Sci., **569** (2021), 110–124. 1

[4] T. M. Al-shami, *Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets*, Soft Comput., **25** (2021), 14449–14460. 1

[5] T. M. Al-shami, *Bipolar soft sets: relations between them and ordinary points and their applications*, Complexity, **2021** (2021), 14 pages. 1
[6] T. M. Al-shami, *Topological approach to generate new rough set models*, Complex Intell. Syst., **8** (2022), 4101–4113. 1
[7] T. M. Al-shami, *(2,1)-Fuzzy sets: properties, weighted aggregated operators and their applications to multi-criteria decision-making methods*, Complex Intell. Syst., **9** (2023), 1687–1705. 1
[8] T. M. Al-shami, J. C. R. Alcantud, A. Mhemdi, *New generalization of fuzzy soft sets: (a, b)-fuzzy soft sets*, AIMS Math., **8** (2023), 2995–3025. 1
[9] T. M. Al-shami, I. Alshammari, *Rough sets models inspired by supra-topology structures*, Artif. Intell. Rev., **56** (2023), 6855–6883. 1
[10] T. M. Al-shami, A. Mhemdi, *Approximation operators and accuracy measures of rough sets from an infra-topology view*, Soft Comput., **27** (2023), 1317–1330. 1
[11] T. M. Al-shami, A. Mhemdi, *Generalized frame for orthopair fuzzy sets: (m, n)-Fuzzy sets and their applications to multi-criteria decision-making methods*, Information, **14** (2023), 1–21. 1
[12] M. Banerjee, S. K. Pal, *Roughness of a fuzzy set*, Inform. Sci., **93** (1996), 235–246. 1, 5
[13] B. Davvaz, *A short note on algebraic T-rough sets*, Inform. Sci., **178** (2008), 3247–3252. 1, 2, 2.2, 5
[14] J. Deng, J. Zhan, E. Herrera-Viedma, F. Herrera, *Regret Theory-Based Three-Way Decision Method on Incomplete Multiscale Decision Information Systems With Interval Fuzzy Numbers*, IEEE Trans. Fuzzy Syst., **31** (2023), 982–996. 1
[15] D. Dubois, H. Prade, *Rough fuzzy sets and fuzzy rough sets*, Int. J. Gen. Syst., **17** (1990), 191–209. 1
[16] D. Dubois, H. Prade, *An introduction to bipolar representations of information and preferences*, Int. J. Intell. Syst., **23** (2008), 866–877. 1
[17] F. Feng, L. Changxing, B. Davvaz, M. I. Ali, *Soft sets combined with fuzzy sets and rough sets: a tentative approach*, Soft Comput., **14** (2010), 899–911. 1
[18] F. Feng, Y. B. Jun, X. Liu, L. Li, *An adjustable approach to fuzzy soft set based decision making*, J. Comput. Appl. Math., **234** (2010), 10–20. 1
[19] R. Gul, M. Shabir, *Roughness of a set by (α, β)-indiscernibility of Bipolar fuzzy relation*, Comput. Appl. Math., **39** (2020), 22 pages. 1
[20] M. Gulistan, N. Yaqoob, A. Elmoasry, J. Alebraheem, *Complex bipolar fuzzy sets: An application in a transport's company*, J. Intell. Fuzzy Syst., **40** (2021), 3981–3997. 1
[21] Y. Han, Z. Lu, Z. Du, Q. Luo, S. Chen, *A YinYang bipolar fuzzy cognitive TOPSIS method to bipolar disorder diagnosis*, Comput. Methods Programs Biomed., **158** (2018), 1–10. 1, 5
[22] Y. Han, P. Shi, S. Chen, *Bipolar-Valued Rough Fuzzy Set and Its Applications to the Decision Information System*, IEEE Trans. Fuzzy Syst., **23** (2015), 2358–2370. 1, 5
[23] R. A. Hosny, B. A. Asaad, A. A. Azzam, T. M. Al-shami, *Various topologies generated from E_j-neighbourhoods via ideals*, Complexity, **2021** (2021), 11 pages. 1
[24] H. Z. Ibrahim, T. M. Al-shami, M. Arar, M. Hosny, *k_m^n-Rung Picture Fuzzy Information in a Modern Approach to Multi-Attribute Group Decision-Making*, Complex Intell. Syst., (2023). 1
[25] C. Jana, M. Pal, J. Q. Wang, *Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process*, J. Ambient Intell. Human. Comput., **10** (2019), 3533–3549. 5
[26] J. Kim, S. K. Samanta, P. K. Lim, J. G. Lee, K. Hur, *Bipolar fuzzy topological spaces*, Ann. Fuzzy Math. Inform., **17** (2019), 205–229. 1
[27] K. M. Lee, *Bipolar-valued fuzzy sets and their basic operations*, In: Proceedings of the International Conference, Bangkok, Thailand, (2000), 307–317. 1, 2.7, 2.8, 2.9
[28] K. M. Lee, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets*, J. Korean Inst. Intell. Syst., **14** (2004), 125–129. 1
[29] M. Lu, J. R. Busemeyer, *Do traditional chinese theories of Yi Jing ('Yin-Yang'and Chinese medicine go beyond western concepts of mind and matter*, Mind Matter, **12** (2014), 37–59. 1
[30] X. Ma, Q. Liu, J. Zhan, *A survey of decision making methods based on certain hybrid soft set models*, Artif. Intell. Rev., **47** (2017), 507–530. 1
[31] T. Mahmood, S. Abdullah, M. Bilal, S. Rashid, *Multiple criteria decision making based on bipolar valued fuzzy set*, Ann. Fuzzy Math. Inform., **11** (2016), 1003–1009. 1, 5
[32] T. Mahmood, U. Ur Rehman, *A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures*, Int. J. Intell. Syst., **37** (2021), 535–567. 1
[33] N. Malik, M. Shabir, *Rough fuzzy bipolar soft sets and applications in decision-making problems*, Soft Comput., **23** (2019), 1603–1614. 1
[34] N. Malik, M. Shabir, T. M. Al-shami, R. Gul, M. Arar, M. Hosny, *Rough bipolar fuzzy ideals in semigroups*, Complex Intell. Syst., **9** (2023), 7197–7212. 1
[35] N. Malik, M. Shabir, T. M. Al-shami, R. Gul, A. Mhemdi, *Medical decision-making techniques based on bipolar soft information*, AIMS Math., **8** (2023), 18185–18205. 1
[36] D. Stanujkic, D. Karabasevic, E. K. Zavadskas, F. Smarandache, W. K. M. Brauers, *A bipolar fuzzy extension of the MULTIMOORA method*, Informatica, **30** (2019), 135–152. 1
[37] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci., **11** (1982), 341–356. 1, 2.1
[38] Z. Pawlak, A. Skowron, *Rudiments of rough sets*, Inform. Sci., **177** (2007), 3–27. 1, 2.1

[39] M. Riaz, S. T. Tehrim, *A robust extension of VIKOR method for bipolar fuzzy sets using connection numbers of SPA theory based metric spaces*, Artif. Intell. Rev., **54** (2021), 561–591. 1

[40] R. Sahu, S. R. Dash, S. Das, *Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory*, Decis. Mak.: Appl. Manag. Eng., **4** (2021), 104–126. 1

[41] M. Shabir, M. I. Ali, *Soft ideals and generalized fuzzy ideals in semigroups*, New Math. Nat. Comput., **5** (2009), 599–615. 1

[42] M. Shabir, Y. B. Jun, Y. Nawaz, *Characterizations of regular semgroups by (α, β)-fuzzy ideals*, Comput. Math. Appl., **59** (2010), 161–175.

[43] M. Shabir, Y. B. Jun, Y. Nawaz, *Semigroups characterized by $(\epsilon, \epsilon \vee qk)$-fuzzy ideals*, Comput. Math. Appl., **60** (2010), 1473–1493. 1

[44] H. Sharma, P. Sivaprakasam, M. Angamuthu, *Generalized Z-fuzzy soft β-covering based rough matrices and its application to MAGDM problem based on AHP method*, Decis. Mak.: Appl. Manag. Eng., **6** (2023), 134–152. 1

[45] J. Wang, X. Ma, Z. Xu, J. Zhan, *Regret Theory-Based Three-Way Decision Model in Hesitant Fuzzy Environments and Its Application to Medical Decision*, IEEE Trans. Fuzzy Syst., **30** (2022), 5361–5375. 1

[46] G. Wei, F. E. Alsaadi, T. Hayat, A. Alsaedi, *Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making*, Int. J. Fuzzy Syst., **20** (2018), 1–12. 1

[47] H.-L. Yang, S.-G. Li, Z.-L. Guo, C.-H. Ma, *Transformation of bipolar fuzzy rough set models*, Knowl. Based Syst., **27** (2012), 60–68. 1

[48] H.-L. Yang, S.-G. Li, S. Wang, J. Wang, *Bipolar fuzzy rough set model on two different universes and its application*, Knowl. Based Syst., **35** (2012), 94–101. 1, 5

[49] Y. Yao, *Three-way decisions with probabilistic rough sets*, Inform. Sci., **180** (2010), 341–353. 1, 5

[50] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **8** (1965), 338–353. 1, 2.4, 2.5, 5

[51] J. Zhan, M. I. Ali, N. Mehmood, *On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods*, Appl. Soft Comput., **56** (2017), 446–457. 1

[52] J. Zhan, W. Wang, J. C. R. Alcantud, J. Zhan, *A three-way decision approach with prospect-regret theory via fuzzy set pair dominance degrees for incomplete information systems*, Inf. Sci., **617** (2022), 310–330. 1

[53] W. R. Zhang, *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis*,In: NAFIPS/IFIS/NASA'94. Proceedings of the first international joint conference of the North American fuzzy information processing society biannual conference. The Industrial Fuzzy Control and Intellige., IEEE, (1994), 305–309. 1, 2.6

[54] W. R. Zhang, *Bipolar quantum logic gates and quantum cellular combinatoricsa logical extension to quantum entanglement*, J. Quantum Inf. Sci., **3** (2013), 93–105. 1

[55] W. R. Zhang, *G-CPT Symmetry of Quantum Emergence and Submergence–An Information Conservational Multiagent Cellular Automata Unification of CPT Symmetry and CP Violation for Equilibrium-Based Many-World Causal Analysis of Quantum Coherence and Decoherence*, J. Quantum Inf. Sci., **6** (2016), 62–97. 1

[56] W. R. Zhang, A. K. Pandurangi, K. E. Peace, Y.-Q. Zhang, Z. Zhao, *MentalSquares: a generic bipolar support vector machine for psychiatric disorder classification, diagnostic analysis and neurobiological data mining*, Int. J. Data Min. Bioinform., **5** (2011), 532–557. 1

[57] W. R. Zhang, K. E. Peace, *Causality is logically definable—toward an equilibrium-based computing paradigm of quantum agents and quantum intelligence (QAQI)(Survey and research)*, J. Quantum Inf. Sci., **4** (2014), 227–268. 1

[58] W.-R. Zhang, J. H. Zhang, Y. Shi, S.-S. Chen, *Bipolar linear algebra and yinyang-N-element cellular networks for equilibrium-based biosystem simulation and regulation*, J. Biol. Syst., **17** (2009), 547–576. 1

[59] B. Zhou, J. Chen, Q. Wu, D. Pamucar, W. Wang, L. Zhou, *Risk priority evaluation of power transformer parts based on hybrid FMEA framework under hesitant fuzzy environment*, Facta Univ. Ser.: Mech. Eng., **20** (2022), 399–420. 1