J. Math. Computer Sci., 33 (2024), 368–380

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Global error analysis of discontinuous Galerkin methods for systems of boundary value problems

Helmi Temimi

Department of Mathematics and Natural Sciences, Gulf University for Science and Technology, Hawally 32093, Kuwait.

Abstract

This paper introduces a novel approach for solving systems of boundary value problems (BVPs) by employing the recently developed Discontinuous Galerkin (DG) method, which removes the necessity for auxiliary variables. This marks the initial installment in a sequence of publications dedicated to exploring DG methods for solving partial differential equations (PDEs). In fact, through a systematic application of the DG method to each spatial variable within the PDE, employing the method of lines, we convert the initial problem into a system of ordinary differential equations (ODEs). In the current study, we developed a global error analysis of the DG method applied to systems of ODEs. Our analysis shows that using p-degree piecewise polynomials and h-mesh step size, the DG solutions achieve optimal $O(h^{p+1})$ convergence rates in the $\mathcal{L}^2$-norm.

Keywords: Discontinuous Galerkin, superconvergence, systems, boundary value problems, optimal rate.

2020 MSC: 65L101, 65L60.

©2024 All rights reserved.

1. Introduction

The Discontinuous Galerkin (DG) technique made its debut in 1973 when Reed and Hill introduced it in their groundbreaking work [19]. Their primary objective was to address the hyperbolic equation of neutron transport. In their pioneering efforts, they devised a locally conservative and parallelizable method within the framework of discontinuous Galerkin, capable of effectively handling intricate geometries while dispensing with the need for continuity across element boundaries. These highly desirable characteristics have propelled the DG method to widespread adoption for tackling a diverse array of differential equations including, but not limited to, Troesch's problem [23], α-synuclein spreading in Parkinson's disease [14], fully coupled hydro-mechanical modeling of two-phase flow in deformable fractured porous media [18] and recovery of the conductivity in electrical impedance tomography [17]. A notable refinement of the DG approach emerged with the introduction of the Runge-Kutta DG scheme by Cockburn et al.. This enhancement was elaborated upon in a series of works [10–13], with the aim of addressing nonlinear hyperbolic conservation laws. Over time, the DG methodology has been extended to encompass scenarios involving higher-order derivatives. For a comprehensive and up-to-date exploration of the evolution and applications of the DG method, Shu has presented a thorough overview in his surveys [20, 21].

Email address: temimi.h@gust.edu.kw (Helmi Temimi)

doi: 10.22436/jmcs.033.04.04

Received: 2023-09-08 Revised: 2023-11-16 Accepted: 2023-12-15

The DG method has evolved in various ways over the last three decades. In a notable work [13], Cockburn and Shu introduced an important DG scheme called the Local Discontinuous Galerkin (LDG) method. This method is designed to solve general convection-diffusion problems and was inspired by efficient numerical experiments conducted by Bassi and Rebay [5] for simulating compressible Navier-Stokes equations. By introducing auxiliary variables that reduce the original problem into a first-order system, the LDG methods ensure the stability and optimal convergence of the scheme when using suitable numerical fluxes. Since its introduction, the LDG method has been successfully applied to various linear and nonlinear problems, including second-order elliptic, parabolic, and hyperbolic problems. Another successful variation of the DG method for solving higher-order differential equations is the so-called ultra-weak discontinuous Galerkin method, the subject of the current study, originally proposed by Despres for linear elliptic partial differential equations [15]. The idea of this DG method is to shift all spatial derivatives to the test function in the weak formulation through m integration by parts, where m denotes the order of the differential equation. This ultra-weak DG scheme offers several advantages over existing methods in the literature. Notably, unlike the LDG method, this DG method can be smoothly applied without the need for introducing auxiliary variables or expanding the original equation into a larger system, resulting in reduced memory and computational costs. Furthermore, compared to other DG methods, our approach guarantees optimal convergence and superconvergence without using internal penalty terms. Another advantage is that the current scheme achieves superconvergence results [25] that can be used to construct a posteriori error estimates by solving a local problem on each element. Moreover, the careful choice of the numerical fluxes ensures the robust stability of the DG scheme.

In 2007, Adjerid and Temimi [1] developed a groundbreaking Discontinuous Galerkin (DG) scheme for higher-order initial value problems, eliminating the need for auxiliary variables. They achieved a remarkable convergence rate of $p+1$ using p-degree piecewise polynomials. In a related work, Chen and Shu [8] introduced ultra-weak discontinuous Galerkin (UWDG) techniques for second to fifth-order ODE boundary value problems. Their approach combined DG spatial discretization with TVD high-order Runge-Kutta temporal discretization, showing both stability and optimal $(p+1)$-th convergence rate, contrary to initial sub-optimal estimates.

In [2], Adjerid and Temimi built upon their prior research and applied it within the framework of the wave equation. They achieved this by employing the method of lines. This innovative approach involved the implementation of the standard finite element method in the spatial dimension and the discontinuous Galerkin method in the temporal dimension. Their study demonstrated the effectiveness of this novel technique by establishing optimal error estimates in both spatial and temporal domains. Additionally, they conducted a comparative analysis against existing methodologies. Moreover, their investigation unveiled superconvergence within each space-time element, particularly at the intersection points of the Lobatto polynomials in space and the Jacobi polynomials in time. Subsequently, Temimi in [24] addressed one-dimensional second-order boundary value problems using the DG scheme, revealing that the leading term of the discretization error in each element is generated by specific combination of Jacobi polynomials. He proved that for a DG solution of degree p, there's a superconvergence rate of $\mathcal{O}(h^{p+2})$ at the polynomial roots. Expanding on this, Baccouch and Temimi in [3] extended the DG error analysis to second-order boundary value problems, showing that with p-degree piecewise polynomials, the UWDG solution and its derivative exhibit $\mathcal{O}(h^{2p})$ superconvergence at upwind and downwind endpoints. In their recent work detailed in [4], Baccouch and Temimi introduced a novel DG scheme for solving the wave equation via the method of lines, demonstrating that the DG solution achieves optimal convergence rates in the $\mathcal{L}^2$-norm.

In the current work, we developed a global error analysis of the DG method for systems of second-order BVPs in terms of convergence criteria. We have established that the DG solutions show optimal $O(h^{p+1})$ rate of convergence in the $\mathcal{L}^2$-norm when using p-degree piecewise polynomials and h-mesh step size. In forthcoming studies, our intention is to combine the findings from our current analysis with our prior research integrating the method of lines. This integration will pave the way for the creation of a fully DG scheme for solving higher-dimensional partial differential equations. We anticipate that this innovative scheme will surpass existing methods in terms of both accuracy and computational efficiency.

This manuscript is summarized in this way. We develop a DG scheme applied to systems of boundary value problems in Section 2. In Section 3, we present the DG global error analysis. Then, we carry out several computational simulations to exhibit the full agreement with theoretical findings in Section 4. Finally, some concluding remarks are stated in Section 5.

2. Model problem

In this section, we develop the DG formulation of the system of BVPs given by

$$\ddot{\mathbf{y}}+\mathbf{C}\dot{\mathbf{y}}+\mathbf{K}\mathbf{y}=\mathbf{f}(x), \quad x\in(a,b), \tag{2.1a}$$

subject to mixed boundary conditions

$$\mathbf{y}(a)=\mathbf{y}_l, \qquad \dot{\mathbf{y}}(b)=\mathbf{y}_{rx}.$$

Dirichlet boundary conditions scenario is also explored.

$$\mathbf{y}(a)=\mathbf{y}_l, \qquad \mathbf{y}(b)=\mathbf{y}_r.$$

where

$$\mathbf{y}(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\\ \vdots\\ y_n(x)\end{pmatrix}, \quad \mathbf{y}_l=\begin{pmatrix} y_{1,l}\\ y_{2,l}\\ \vdots\\ y_{n,l}\end{pmatrix}, \quad \mathbf{y}_{rx}=\begin{pmatrix} y_{1,rx}\\ y_{2,rx}\\ \vdots\\ y_{n,rx}\end{pmatrix}, \quad \mathbf{y}_r=\begin{pmatrix} y_{1,r}\\ y_{2,r}\\ \vdots\\ y_{n,r}\end{pmatrix},$$

and where C and K are two matrices defined respectively by $C=(C_{ij})_{1\leqslant i,j\leqslant n}$ and $K=(K_{ij})_{1\leqslant i,j\leqslant n}$. In our analysis, we assume that the system of BVPs (2.1) has one and only one vector solution $\mathbf{y}$. The conditions on the vector function $\mathbf{f}$ for the existence and uniqueness of the solution vector to the general system of BVPs are given in [16].

To implement the discontinuous Galerkin method, we initially create a partition, $h=\frac{(b-a)}{N}$, for $k=0,1,2,\dots,N$ we let $x_k=a+k\cdot h$ and $I_k=(x_k,x_{k+1})$, we also construct a finite dimensional space $\mathbf{S}^{N,p}$

$$\mathbf{S}^{N,p}=\{\mathbf{Y}:[a,b]\to\mathbb{R}^n, \quad \mathbf{Y}|_{I_k}\in[\mathcal{P}_p]^n\},$$

where $\mathcal{P}_p$ designates the p-degree polynomial space. Next, we multiply (2.1a) by a vector test function $\mathbf{v}(x)\in[H^2([a,b])]^n$, we integrate the resulting system of equations over I_k and we integrate it by parts twice to derive the weak discontinuous Galerkin (DG) formulation for (2.1). Then, for $k=0,1,\dots,N$ we have

$$\int_{I_k}\left(\ddot{\mathbf{v}}^t\mathbf{y}+\dot{\mathbf{v}}^t\mathbf{C}\mathbf{y}+\mathbf{v}^t\mathbf{K}\mathbf{y}\right)dx+\mathbf{v}^t\dot{\mathbf{y}}|_{x_k}^{x_{k+1}}-\dot{\mathbf{v}}^t\mathbf{y}|_{x_k}^{x_{k+1}}+\mathbf{v}^t\mathbf{K}\mathbf{y}|_{x_k}^{x_{k+1}}=\int_{I_k}\mathbf{v}^t\mathbf{f}dx. \tag{2.2}$$

In (2.2), replacing $\mathbf{y}$ by $\mathbf{Y}_k(x)=\mathbf{Y}|_{[x_k,x_{k+1}]}\in\mathbf{S}^{N,p}$ and $\mathbf{v}$ by $\mathbf{V}\in\mathbf{S}^{N,p}$ leads to

$$\begin{aligned}\int_{I_k}\left(\ddot{\mathbf{V}}^t\mathbf{Y}_k+\dot{\mathbf{V}}^t\mathbf{C}\mathbf{Y}_k+\mathbf{V}^t\mathbf{K}\mathbf{Y}_k\right)dx&+\mathbf{V}^t(x_{k+1}^-)\widehat{\dot{\mathbf{Y}}}_k(x_{k+1})-\mathbf{V}^t(x_k^+)\widehat{\dot{\mathbf{Y}}}_k(x_k)-\dot{\mathbf{V}}(x_{k+1}^-)^t\widehat{\mathbf{Y}}_k(x_{k+1})\\&+\dot{\mathbf{V}}^t(x_k^+)\widehat{\mathbf{Y}}_k(x_k)+\mathbf{V}^t(x_{k+1}^-)\mathbf{K}\widehat{\mathbf{Y}}_k(x_{k+1})-\mathbf{V}^t(x_k^+)\mathbf{K}\widehat{\mathbf{Y}}_k(x_k)=\int_{I_k}\mathbf{V}^t\mathbf{f}dx, \quad \forall\, V\in\mathbf{S}^{N,p},\end{aligned}$$

where $\mathbf{Y}_k=[Y_{1,k},Y_{2,k},\dots,Y_{n,k}]^t$ and where $\widehat{\mathbf{Y}}_k(x_k)$, $\widehat{\mathbf{Y}}_k(x_{k+1})$, $\widehat{\dot{\mathbf{Y}}}_k(x_k)$, and $\widehat{\dot{\mathbf{Y}}}_k(x_{k+1})$ denote the numerical fluxes defined by Cheng et al. [8] as follows

$$\begin{cases}\widehat{\mathbf{Y}}_0(a)=\mathbf{y}_l, & \widehat{\mathbf{Y}}_0(x_1)=\mathbf{Y}_0(x_1^-),\\ \widehat{\mathbf{Y}}_k(x_k)=\mathbf{Y}_{k-1}(x_k^-), & \widehat{\mathbf{Y}}_k(x_{k+1})=\mathbf{Y}_k(x_{k+1}^-), \quad k=1,\dots,N,\end{cases}$$

$$\begin{cases} \widehat{\dot{\mathbf{Y}}}_k(x_k) = \dot{\mathbf{Y}}_k(x_k^+), & \widehat{\dot{\mathbf{Y}}}_k(x_{k+1}) = \dot{\mathbf{Y}}_{k+1}(x_{k+1}^+), \quad k = 0, \dots, N-1, \\ \widehat{\dot{\mathbf{Y}}}_N(x_N) = \dot{\mathbf{Y}}_N(x_N^+), & \widehat{\dot{\mathbf{Y}}}_N(b) = \mathbf{y}_{rx}. \end{cases}$$

The numerical fluxes, for the case of Dirichlet boundary conditions are given by

$$\begin{cases} \widehat{\mathbf{Y}}_0(a) = \mathbf{y}_l, & \widehat{\mathbf{Y}}_0(x_1) = \mathbf{Y}_0(x_1^-), \\ \widehat{\mathbf{Y}}_k(x_k) = \mathbf{Y}_{k-1}(x_k^-), & \widehat{\mathbf{Y}}_k(x_{k+1}) = \mathbf{Y}_k(x_{k+1}^-), \quad k = 1, \dots, N-1, \\ \widehat{\mathbf{Y}}_N(x_N) = \mathbf{Y}_{N-1}(x_N^-), & \widehat{\mathbf{Y}}_N(b) = \mathbf{y}_r, \end{cases}$$

and

$$\begin{cases} \widehat{\dot{\mathbf{Y}}}_0(a) = \dot{\mathbf{Y}}_0(a^+), & \widehat{\dot{\mathbf{Y}}}_0(x_1) = \dot{\mathbf{Y}}_1(x_1^+), \\ \widehat{\dot{\mathbf{Y}}}_k(x_k) = \dot{\mathbf{Y}}_k(x_k^+), & \widehat{\dot{\mathbf{Y}}}_k(x_{k+1}) = \dot{\mathbf{Y}}_{k+1}(x_{k+1}^+), \quad k = 1, \dots, N-1, \\ \widehat{\dot{\mathbf{Y}}}_N(x_N) = \dot{\mathbf{Y}}_N(x_N^+), & \widehat{\dot{\mathbf{Y}}}_N(b) = \dot{\mathbf{Y}}_N(b^-) + \frac{p}{h}(\mathbf{Y}_N(b^-) - \mathbf{y}_r). \end{cases}$$

Therefore, the discrete formulation involves finding $\mathbf{Y}_k(x) = \mathbf{Y}|_{[x_k, x_{k+1}]} \in \mathbf{S}^{N,p}$ such that, $\forall\ \mathbf{V} \in \mathbf{S}^{N,p}$,

$$\begin{aligned} \int_{I_0} \left(\ddot{\mathbf{V}}^t \mathbf{Y}_0 + \dot{\mathbf{V}}^t \mathbf{C} \mathbf{Y}_0 + \mathbf{V}^t \mathbf{K} \mathbf{Y}_0\right) dx + \mathbf{V}^t(x_1^-)\dot{\mathbf{Y}}_1(x_1^+) - \mathbf{V}^t(a^+)\dot{\mathbf{Y}}_0(a^+) \\ - \dot{\mathbf{V}}(x_1^-)^t \mathbf{Y}_0(x_1^-) + \dot{\mathbf{V}}^t(a^+)\mathbf{y}_l + \mathbf{V}^t(x_1^-)\mathbf{K}\mathbf{Y}_0(x_1^-) - \mathbf{V}^t(a^+)K\mathbf{y}_l = \int_{I_0} \mathbf{V}^t \mathbf{f} dx, \end{aligned} \tag{2.3a}$$

For $k = 1, \dots, N-1$,

$$\begin{aligned} \int_{I_k} \left(\ddot{\mathbf{V}}^t \mathbf{Y}_k + \dot{\mathbf{V}}^t \mathbf{C} \mathbf{Y}_k + \mathbf{V}^t \mathbf{K} \mathbf{Y}_k\right) dx + \mathbf{V}^t(x_{k+1}^-)\dot{\mathbf{Y}}_{k+1}(x_{k+1}^+) - \mathbf{V}^t(x_k^+)\dot{\mathbf{Y}}_k(x_k^+) - \dot{\mathbf{V}}(x_{k+1}^-)^t \mathbf{Y}_k(x_{k+1}^-) \\ + \dot{\mathbf{V}}^t(x_k^+)\mathbf{Y}_{k-1}(x_k^-) + \mathbf{V}^t(x_{k+1}^-)\mathbf{K}\mathbf{Y}_k(x_{k+1}^-) - \mathbf{V}^t(x_k^+)\mathbf{K}\mathbf{Y}_{k-1}(x_k^-) = \int_{I_k} \mathbf{V}^t \mathbf{f} dx, \end{aligned}$$

and

$$\begin{aligned} \int_{I_N} \left(\ddot{\mathbf{V}}^t \mathbf{Y}_N + \dot{\mathbf{V}}^t \mathbf{C} \mathbf{Y}_N + \mathbf{V}^t \mathbf{K} \mathbf{Y}_N\right) dx + \mathbf{V}^t(b^-)\mathbf{y}_{rx} - \mathbf{V}^t(x_N^+)\dot{\mathbf{Y}}_N(x_N^+) - \dot{\mathbf{V}}(b^-)^t \mathbf{Y}_N(b^-) \\ + \dot{\mathbf{V}}^t(x_N^+)\mathbf{Y}_{N-1}(x_N^-) + \mathbf{V}^t(b^-)\mathbf{K}\mathbf{Y}_N(b^-) - \mathbf{V}^t(x_N^+)\mathbf{K}\mathbf{Y}_{N-1}(x_N^-) = \int_{I_N} \mathbf{V}^t \mathbf{f} dx. \end{aligned} \tag{2.3b}$$

When subjected Dirichlet boundary conditions, (2.3b) is written as

$$\begin{aligned} \int_{I_N} \left(\ddot{\mathbf{V}}^t \mathbf{Y}_N + \dot{\mathbf{V}}^t \mathbf{C} \mathbf{Y}_N + \mathbf{V}^t \mathbf{K} \mathbf{Y}_N\right) dx + \mathbf{V}^t(b^-)\left(\dot{\mathbf{Y}}_N(b^-) + \frac{p}{h}(\mathbf{Y}_N(b^-) - \mathbf{y}_r)\right) - \mathbf{V}^t(x_N^+)\dot{\mathbf{Y}}_N(x_N^+) \\ - \dot{\mathbf{V}}(b^-)^t \mathbf{y}_r + \dot{\mathbf{V}}^t(x_N^+)\mathbf{Y}_{N-1}(x_N^-) + \mathbf{V}^t(b^-)\mathbf{K}\mathbf{y}_r - \mathbf{V}^t(x_N^+)\mathbf{K}\mathbf{Y}_{N-1}(x_N^-) = \int_{I_N} \mathbf{V}^t \mathbf{f} dx. \end{aligned}$$

3. Global error analysis

In this section, we provide the error estimates in the $\mathcal{L}^2$-norm of the proposed scheme for the model problem (2.1).

3.1. Notation and definitions

The $\mathcal{L}^2$ inner product of two integrable functions, u and v, on an interval $I_k = (x_k, x_{k+1})$ is defined as

$$(u, v)_{I_k} = \int_{I_k} u(x)v(x)dx.$$

The $\mathcal{L}^2$-norm of an integrable function u on I_k is defined by

$$\|u\|_{0,I_k} = (u, u)_{I_k}^{1/2}.$$

We also recall the $H^s(I_k)$-norm and $[H^s(I_k)]^n$-norm for, respectively, real-valued function u in $H^s(I_k)$ and vector-valued function $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in $[H^s(I_k)]^n$ defined by $\forall\ 0 \leqslant \alpha \leqslant s$ and $1 \leqslant s \leqslant \infty$,

$$\|u\|_{s,I_k} = \left(\sum_{\alpha=0}^{s} \left\| \frac{d^\alpha u}{dx^\alpha} \right\|_{0,I_k}^2\right)^{1/2}, \quad \|\mathbf{v}\|_{s,I_k} = \left(\sum_{i=1}^{n} \|v_i\|_{s,I_k}^2\right)^{1/2}.$$

Moreover, we define the following norms on $\Omega = [a, b]$,

$$\|u\|_{s,\Omega} = \left(\sum_{k=0}^{N} \|u\|_{s,I_k}^2\right)^{1/2}, \qquad \|\mathbf{v}\|_{s,\Omega} = \left(\sum_{k=0}^{N} \|\mathbf{v}\|_{s,I_k}^2\right)^{1/2}.$$

To simplify the notation, we let

$$\|u\| = \|u\|_{0,\Omega}, \quad \|u\|_s = \|u\|_{s,\Omega},$$

and

$$\|\mathbf{v}\| = \|\mathbf{v}\|_{0,\Omega}, \quad \|\mathbf{v}\|_s = \|\mathbf{v}\|_{s,\Omega}.$$

3.2. Projections

We first introduce some one-dimensional projections. Let $I_k = (x_k, x_{k+1})$ be any interval and let $\mathcal{P}_p(I_k)$ be the l-degree polynomials space with $l \leqslant p$ on I_k. The L^2-projection onto $\mathcal{P}_p(I_k)$ is denoted by P_h, i.e., for a function $u \in L^2(I_k)$ the projection $P_h u$ is the only polynomial in $\mathcal{P}_p(I_k)$ that satisfies $\forall\ v \in \mathcal{P}_p(I_k)$,

$$\int_{I_k} (P_h u - u) v dx = 0.$$

Furthermore, we consider two one-dimensional Gauss-Radau projections P_h^- and P_h^+ defined in [3] as for any function $u \in L^2(I_k)$, $P_h^- u$ and $P_h^- u$ are the unique polynomials in $\mathcal{P}_p(I_k)$ that satisfy $\forall\ v \in \mathcal{P}_{p-1}(I_k)$,

$$\int_{I_k} (u - P_h^- u) v dx = 0, \quad (u - P_h^- u)(x_{k+1}^-) = 0, \quad (u - P_h^- u)'(x_{k+1}^-) = 0,$$

and

$$\int_{I_k} (u - P_h^+ u) v dx = 0, \quad (u - P_h^+ u)(x_k^+) = 0, \quad (u - P_h^+ u)'(x_k^+) = 0. \tag{3.2a}$$

Next, $\mathbf{P}_h^\pm \mathbf{v}$ Gauss-Radau projections are introduced for vector-valued function $\mathbf{v} = [v_1, v_2, \dots, v_n]^t$. The projection $\mathbf{P}_h^-$ for a vector-valued function $\mathbf{v}$ is defined as

$$\mathbf{P}_h^- \mathbf{v} = [P_h^- v_1, P_h^- v_2, \dots, P_h^- v_N]^t.$$

Moreover, we define a projection $\mathbf{P}_h^+$ for vector-valued function $\mathbf{v}$ as follows

$$\mathbf{P}_h^+ \mathbf{v} = [P_h^+ v_1, P_h^+ v_2, \dots, P_h^+ v_N]^t.$$

The existence and uniqueness of the projections $\mathbf{P}_h^- \mathbf{v}$ and $\mathbf{P}_h^+ \mathbf{v}$ are readily apparent. Additionally, the following interpolation error estimate holds. The proof can be found in Theorem 3.1.6 in [9].

Lemma 3.1. *For any* $\mathbf{v} \in \left[H^{p+1}(\Omega)\right]^n$, *there exists a constant* $C > 0$ *independent of* h *such that*

$$\left\|(\mathbf{v} - \mathbf{P}_h^{\pm}\mathbf{v})^{(s)}\right\| \leqslant Ch^{q-s} \|\mathbf{v}\|_q, \quad 2 \leqslant q \leqslant p+1. \tag{3.3}$$

Lastly, we restate the inverse property of the finite element space $\mathbf{S}^{N,p}$ needed later in the error analysis

$$\left\|\mathbf{v}^{(s)}\right\| \leqslant Ch^{-s} \|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{S}^{N,p}, \quad s = 0, 1, 2, \tag{3.4}$$

where C is a positive constant independent of both $\mathbf{v}$ and h.

3.3. *A priori error estimate*

In this paper, we use $\mathbf{e}$ to denote the error between the exact and numerical solutions, i.e.,

$$\mathbf{e} = \mathbf{y} - \mathbf{Y}_k, \quad x \in I_k, \quad k = 0, 1, \ldots, N.$$

We denote the projection error by

$$\boldsymbol{\epsilon} = \mathbf{y} - \mathbf{P}_h^{+}\mathbf{y}, \quad x \in I_k, \quad k = 0, 1, \ldots, N.$$

Then, we divide the error in two parts

$$\mathbf{e} = \boldsymbol{\epsilon} + \bar{\mathbf{e}}, \quad x \in I_k, \quad k = 0, 1, \ldots, N, \tag{3.5}$$

where $\bar{\mathbf{e}} = \mathbf{P}_h^{+}\mathbf{y} - \mathbf{Y}_k \in \mathbf{S}^{N,p}$.

We are now ready to present the error estimate for the numerical scheme.

Theorem 3.2. *Suppose that* $\mathbf{y} \in \left[H^{p+1}([a, b])\right]^n$ *and* $\mathbf{Y}_k$ *are, respectively, the solutions of* (2.1) *and* (2.3). *Then, for sufficiently small* h, *we have the following error estimates*

$$\|\mathbf{e}\| \leqslant Ch^{p+1}, \tag{3.6}$$

where C *is a positive constant depending on* u *and independent of* h.

Proof. Subtracting the DG discrete formulation (2.3) from the weak DG formulation (2.2), where we replaced $\mathbf{v}$ by $\mathbf{V} \in \mathbf{S}^{N,p}$, we get the following error equation on each element I_k: $\forall\ \mathbf{V} \in \mathbf{S}^{N,p}$ and $k = 0, 1, 2, \ldots, N$,

$$\begin{aligned}\int_{I_k} \left(\ddot{\mathbf{V}}^t - \dot{\mathbf{V}}^t\mathbf{C} + \mathbf{V}^t\mathbf{K}\right) \mathbf{e}\ dx + \mathbf{V}^t(x_{k+1}^-)\dot{\mathbf{e}}(x_{k+1}^-) - \mathbf{V}^t(x_k^+)\dot{\mathbf{e}}(x_k^-)&\\ - \dot{\mathbf{V}}^t(x_{k+1}^-)\mathbf{e}(x_{k+1}^-) + \dot{\mathbf{V}}^t(x_k^+)\mathbf{e}(x_k^-) + \mathbf{V}^t(x_{k+1}^-)\mathbf{C}\mathbf{e}(x_{k+1}^-) - \mathbf{V}^t(x_k^+)\mathbf{C}\mathbf{e}(x_k^-) &= \mathbf{0}.\end{aligned} \tag{3.7}$$

We now introduce the following bilinear form in order to simplify the proof:

$$\begin{aligned}A_k(\mathbf{e}, \mathbf{v}) = \int_{I_k} \left(\ddot{\mathbf{v}}^t + \dot{\mathbf{v}}^t\mathbf{C} + \mathbf{v}^t\mathbf{K}\right) \mathbf{e}\ dx + \mathbf{v}^t(x_{k+1}^-)\dot{\mathbf{e}}(x_{k+1}^-) - \mathbf{v}^t(x_k^+)\dot{\mathbf{e}}(x_k^-)&\\ - \dot{\mathbf{v}}^t(x_{k+1}^-)\mathbf{e}(x_{k+1}^-) + \dot{\mathbf{v}}^t(x_k^+)\mathbf{e}(x_k^-) + \mathbf{v}^t(x_{k+1}^-)\mathbf{C}\mathbf{e}(x_{k+1}^-) - \mathbf{v}^t(x_k^+)\mathbf{C}\mathbf{e}(x_k^-)&.\end{aligned} \tag{3.8}$$

We notice that (3.7) gives

$$A_k(\mathbf{e}, \mathbf{V}) = \mathbf{0}, \quad \forall\ \mathbf{V} \in \mathbf{S}^{N,p}. \tag{3.9}$$

Then, integrating by parts twice gives

$$A_k(\mathbf{e}, \mathbf{v}) = \int_{I_k} \mathbf{v}^t\left(\ddot{\mathbf{e}} + \mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e}\right)\ dx + \mathbf{v}^t(x_k^+)[\![\dot{\mathbf{e}}]\!](x_k) - \dot{\mathbf{v}}^t(x_k^+)[\![\mathbf{e}]\!](x_k) + \mathbf{v}^t(x_k^+)\mathbf{C}[\![\mathbf{e}]\!](x_k). \tag{3.10}$$

Next, we add and subtract $\mathbf{P}_h^+\mathbf{v}$ to $\mathbf{v}$ and we apply (3.9) using $\mathbf{V} = P_h^+\mathbf{v} \in \mathbf{S}^{N,p}$, we get

$$A_k(e,\mathbf{v}) = A_k(\mathbf{e},\mathbf{v}-\mathbf{P}_h^+\mathbf{v}) + A_k(\mathbf{e},\mathbf{P}_h^+\mathbf{v}) = A_k(\mathbf{e},\mathbf{v}-\mathbf{P}_h^+\mathbf{v}). \tag{3.11}$$

We combine (3.11) and (3.10) and use the projection's properties (3.2a) leading to

$$\begin{aligned} A_k(\mathbf{e},\mathbf{v}) = &\int_{I_k} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\ddot{\mathbf{e}} + \mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx + (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t(x_k^+)[\![\dot{\mathbf{e}}]\!](x_k) \\ &- (\dot{\mathbf{v}}-\mathbf{P}_h^{\dot{+}}\mathbf{v})^t(x_k^+)[\![\mathbf{e}]\!](x_k) + (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t(x_k^+)\mathbf{C}[\![\mathbf{e}]\!](x_k) = \int_{I_k} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\ddot{\mathbf{e}} + \mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx. \end{aligned}$$

Utilizing (3.5) and making use of the characteristics of the projection operator $\mathbf{P}_h^+\mathbf{v}$, we obtain

$$A_k(\mathbf{e},\mathbf{v}) = \int_{I_k} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\ddot{\varepsilon} + \mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx, \tag{3.12}$$

where we used the fact that $\ddot{\mathbf{e}} = \ddot{\varepsilon} + \ddot{\mathbf{\bar{e}}}$. Then, we integrate (3.12) by parts and use $(\mathbf{v}-\mathbf{P}_h^+\mathbf{v})'(x_k^+) = 0$ and $\dot{\mathbf{e}}(x_{k+1}^-) = 0$ to obtain

$$A_k(\mathbf{e},\mathbf{v}) = -\int_{I_k} (\dot{\mathbf{v}}-\mathbf{P}_h^{\dot{+}}\mathbf{v})^t\dot{\varepsilon}\ dx + \int_{I_k} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx, \tag{3.13}$$

making (3.8) and (3.13) equal leads to

$$\begin{aligned} &\int_{I_k} \left(\ddot{\mathbf{v}}^t + \dot{\mathbf{v}}^t\mathbf{C} + \mathbf{v}^t\mathbf{K}\right)\mathbf{e}\ dx + \mathbf{v}^t(x_{k+1}^-)\dot{\mathbf{e}}(x_{k+1}^-) - \mathbf{v}^t(x_k^+)\dot{\mathbf{e}}(x_k^-) \\ &\quad - \dot{\mathbf{v}}^t(x_{k+1}^-)\mathbf{e}(x_{k+1}^-) + \dot{\mathbf{v}}^t(x_k^+)\mathbf{e}(x_k^-) + \mathbf{v}^t(x_{k+1}^-)\mathbf{C}\mathbf{e}(x_{k+1}^-) - \mathbf{v}^t(x_k^+)\mathbf{C}\mathbf{e}(x_k^-) \\ &\quad = -\int_{I_k} (\dot{\mathbf{v}}-\mathbf{P}_h^{\dot{+}}\mathbf{v})^t\dot{\varepsilon}\ dx + \int_{I_k} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx. \end{aligned} \tag{3.14}$$

Adding up (3.14) for $k = 0, 1, \ldots, N$ gives

$$\begin{aligned} &\int_{\Omega} \left(\ddot{\mathbf{v}}^t + \dot{\mathbf{v}}^t\mathbf{C} + \mathbf{v}^t\mathbf{K}\right)\mathbf{e}\ dx + \mathbf{v}^t(x_{N+1}^-)\dot{\mathbf{e}}(x_{N+1}^-) - \mathbf{v}^t(x_0^+)\dot{\mathbf{e}}(x_0^-) - \sum_{k=1}^{N}[\![\mathbf{v}^t]\!](x_k)\dot{\mathbf{e}}(x_k^-) - \dot{\mathbf{v}}^t(x_{N+1}^-)\mathbf{e}(x_{N+1}^-) \\ &\quad + \dot{\mathbf{v}}^t(x_0^+)\mathbf{e}(x_0^-) + \sum_{k=1}^{N}[\![\dot{\mathbf{v}}^t]\!](x_k)\mathbf{e}(x_k^-) + \mathbf{v}^t(x_{N+1}^-)\mathbf{C}\mathbf{e}(x_{N+1}^-) - \mathbf{v}^t(x_0^+)\mathbf{C}\mathbf{e}(x_0^-) - \sum_{k=1}^{N}[\![\mathbf{v}^t]\!](x_k)\mathbf{C}\mathbf{e}(x_k^-) \\ &= -\int_{\Omega} (\dot{\mathbf{v}}-\mathbf{P}_h^+\mathbf{v})^t\dot{\varepsilon}\ dx + \int_{\Omega} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx. \end{aligned} \tag{3.15}$$

Using the boundary conditions, (3.15) reduces to

$$\begin{aligned} &\int_{\Omega} \left(\ddot{\mathbf{v}}^t + \dot{\mathbf{v}}^t\mathbf{C} + \mathbf{v}^t\mathbf{K}\right)\mathbf{e}\ dx - \sum_{k=1}^{N}[\![\mathbf{v}^t]\!](x_k)\dot{\mathbf{e}}(x_k^-) + \sum_{k=1}^{N}[\![\dot{\mathbf{v}}^t]\!](x_k)\mathbf{e}(x_k^-) \\ &\quad - \sum_{k=1}^{N}[\![\mathbf{v}^t]\!](x_k)\mathbf{C}\mathbf{e}(x_k^-) = -\int_{\Omega} (\dot{\mathbf{v}}-\mathbf{P}_h^{\dot{+}}\mathbf{v})^t\dot{\varepsilon}\ dx + \int_{\Omega} (\mathbf{v}-\mathbf{P}_h^+\mathbf{v})^t\,(\mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e})\ dx. \end{aligned} \tag{3.16}$$

Next, we use a duality argument to show the error estimate (3.6). We assume that $\mathbf{v}$ is a solution of the adjoint problem

$$\ddot{\mathbf{v}}^t + \dot{\mathbf{v}}^t\mathbf{C} + \mathbf{v}^t\mathbf{K} = \mathbf{e}^t, \quad x \in (a,b), \tag{3.17}$$

subject to the periodic boundary conditions. Then, (3.16) becomes

$$\|\mathbf{e}\|^2 = -\int_\Omega (\dot{\mathbf{v}} - \mathbf{P}_h^+\mathbf{v})^t \dot{\mathbf{e}}\ dx + \int_\Omega (\mathbf{v} - \mathbf{P}_h^+\mathbf{v})^t \left(\mathbf{C}\dot{\mathbf{e}} + \mathbf{K}\mathbf{e}\right)\ dx,$$

since $[\![\mathbf{v}^t]\!](x_k) = [\![\dot{\mathbf{v}}^t]\!](x_k) = 0$. Using the Cauchy-Schwarz inequality, we obtain

$$\|\mathbf{e}\|^2 \leqslant \left\|\dot{\mathbf{v}} - \mathbf{P}_h^+\mathbf{v}\right\| \|\dot{\mathbf{e}}\| + C_1 \left\|\mathbf{v} - \mathbf{P}_h^+\mathbf{v}\right\| \|\dot{\mathbf{e}}\| + C_2 \left\|\mathbf{v} - \mathbf{P}_h^+\mathbf{v}\right\| \|\mathbf{e}\|. \tag{3.18}$$

Moreover, using the interpolation error estimate (3.3) gives

$$\left\|\dot{\mathbf{v}} - \mathbf{P}_h^+\mathbf{v}\right\| \|\dot{\mathbf{e}}\| \leqslant (C_3 h \|\mathbf{v}\|_2)(C_4 h^p) \leqslant C_3 C_4 h^{p+1} \|\mathbf{v}\|_2 \tag{3.19}$$

and using the inverse property (3.4),

$$\begin{aligned}
\left\|\mathbf{v} - \mathbf{P}_h^+\mathbf{v}\right\| \|\dot{\mathbf{e}}\| &\leqslant (C_5 h^2 \|\mathbf{v}\|_2)\left(\|\dot{\bar{\mathbf{e}}}\| + \|\dot{\epsilon}\|\right) \\
&\leqslant (C_5 h^2 \|\mathbf{v}\|_2)\left(C_6 h^{-1} \|\bar{\mathbf{e}}\| + C_4 h^p\right) \\
&\leqslant C_5\left(C_6 h \|\mathbf{e}\| + C_6 \|\epsilon h\| + C_4 h^{p+2}\right) \|\mathbf{v}\|_2 \\
&\leqslant C_5 h\left(C_6 h \|\mathbf{e}\| + C_6 C_7 h^{p+2} + C_4 h^{p+2}\right) \|\mathbf{v}\|_2 \\
&\leqslant \left(C_8 h \|\mathbf{e}\| + C_9 h^{p+2}\right) \|\mathbf{v}\|_2
\end{aligned} \tag{3.20}$$

and

$$\left\|\mathbf{v} - \mathbf{P}_h^+\mathbf{v}\right\| \|\mathbf{e}\| \leqslant (C_5 h^2 \|\mathbf{v}\|_2) \|\mathbf{e}\| \leqslant C_5 h^2 \|\mathbf{e}\| \|\mathbf{v}\|_2. \tag{3.21}$$

Combining (3.19), (3.20), and (3.21), (3.18) becomes

$$\|\mathbf{e}\|^2 \leqslant \left(C_3 C_4 h^{p+1} + C_1 C_8 h \|\mathbf{e}\| + C_1 C_9 h^{p+2} + C_2 C_5 h^2 \|\mathbf{e}\|\right) \|\mathbf{v}\|_2.$$

Thus, using the regularity estimate of the adjoint problem (3.17),

$$\|\mathbf{v}\|_2 \leqslant \tilde{C} \|\mathbf{e}\|, \quad \tilde{C} > 0.$$

to obtain

$$\|\mathbf{e}\|^2 \leqslant \tilde{C}\left(C_3 C_4 h^{p+1} + C_1 C_8 h \|\mathbf{e}\| + C_1 C_9 h^{p+2} + C_2 C_5 h^2 \|\mathbf{e}\|\right) \|\mathbf{e}\|.$$

Dividing by $\|\mathbf{e}\|$ leads to

$$\|\mathbf{e}\| \leqslant \tilde{C}\left(C_3 C_4 h^{p+1} + C_1 C_8 h \|\mathbf{e}\| + C_1 C_9 h^{p+2} + C_2 C_5 h^2 \|\mathbf{e}\|\right).$$

Thus, for small h, we get

$$\|\mathbf{e}\| \leqslant C h^{p+1},$$

which achieves the proof. □

4. Numerical examples

In order to validate our theory, we consider two systems of second-order differential equations subject to Neumann and Dirichlet boundary conditions. The numerical rate of convergence is defined by $\frac{\ln(\|e\|^{(N_1)}/\|e\|^{(N_2)})}{\ln(N_1/N_2)}$, where $\|e\|^{(N)}$ denotes the error $\|e\| = \|y_i - Y_i\|$, for $i = 1, 2, \ldots, n$ using N elements.

Example 4.1. Let us consider the following system of three second-order differential equations

$$\begin{cases} u_1'' - u_1' + u_2' + u_1 - u_2 - u_3 = f_1(x), \\ u_2'' + u_1' - 3u_3' + 3u_1 + u_2 + u_3 = f_2(x), \quad x \in [0,1], \\ u_3'' + 2u_1' + u_2' + 2u_3' + 2u_1 - 2u_2 = f_3(x), \end{cases} \tag{4.1a}$$

subject to Neumann boundary conditions

$$\begin{cases} u_1(0) = 0, \\ u_2(0) = 0, \\ u_3(0) = 0, \end{cases} \tag{4.1b}$$

and

$$\begin{cases} u_1'(1) = 1 - \tanh(1)^2, \\ u_2'(1) = 2 - 2\tanh(2)^2, \\ u_3'(1) = 3 - 3\tanh(3)^2. \end{cases} \tag{4.1c}$$

We choose f_1, f_2, and f_3 such that the exact solutions are given by

$$\begin{cases} u_1(x) = \tanh(x), \\ u_2(x) = \tanh(2x), \\ u_3(x) = \tanh(3x). \end{cases}$$

We solve problem (4.1) on uniform meshes with $N = 60, 65, \dots, 80$ steps and using different degrees $p = 2, 3, 4, 5$. The $\mathcal{L}^2$-norm of the errors $e_i = u_i - U_{i,DG}$ for $i = 1, 2, 3$ versus N shown in Tables 1, 2, and 3 exhibit an $O(h^{p+1})$ convergence. All the results shown in this example are for smooth solutions and are in full agreement with the theory.

Table 1: $\mathcal{L}^2$-norm of $u_1 - U_{1,DG}$ for problem (4.1) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_1\|\|$	Rate	$\|\|e_1\|\|$	Rate	$\|\|e_1\|\|$	Rate	$\|\|e_1\|\|$	Rate
10	1.3747e-05	$\cdots$	1.5485e-07	$\cdots$	1.9568e-09	$\cdots$	2.7397e-11	$\cdots$
15	4.0220e-06	3.03	3.0509e-08	4.00	2.5606e-10	5.01	2.3645e-12	6.04
20	1.6910e-06	3.01	9.6599e-09	4.00	6.0679e-11	5.00	4.1883e-13	6.01
25	8.6486e-07	3.00	3.9602e-09	4.00	1.9877e-11	5.00	1.0969e-13	6.00
30	5.0033e-07	3.00	1.9112e-09	4.00	7.9882e-12	5.00	3.6804e-14	5.99

Table 2: $\mathcal{L}^2$-norm of $u_2 - U_{2,DG}$ for problem (4.1) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_2\|\|$	Rate	$\|\|e_2\|\|$	Rate	$\|\|e_2\|\|$	Rate	$\|\|e_2\|\|$	Rate
10	9.4754e-05	$\cdots$	1.8592e-06	$\cdots$	4.7045e-08	$\cdots$	1.2800e-09	$\cdots$
15	2.6891e-05	3.10	3.6176e-07	4.03	6.1130e-09	5.03	1.1127e-10	6.02
20	1.1121e-05	3.06	1.1367e-07	4.02	1.4416e-09	5.02	1.9716e-11	6.01
25	5.6306e-06	3.05	4.6377e-08	4.01	4.7070e-10	5.01	5.1556e-12	6.01
30	3.2353e-06	3.03	2.2308e-08	4.01	1.8873e-10	5.01	1.7239e-12	6.00

Table 3: $\mathcal{L}^2$-norm of $u_3 - U_{3,DG}$ for problem (4.1) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate
10	2.3577e-04	$\cdots$	7.2636e-06	$\cdots$	2.8101e-07	$\cdots$	1.1400e-08	$\cdots$
15	6.9761e-05	3.00	1.4484e-06	3.98	3.7108e-08	4.99	1.0110e-09	5.97
20	2.9410e-05	3.00	4.5991e-07	3.99	8.8123e-09	5.00	1.8057e-10	5.99
25	1.5051e-05	3.00	1.8871e-07	3.99	2.8881e-09	5.00	4.7411e-11	5.99
30	8.7075e-06	3.00	9.1102e-08	3.99	1.1606e-09	5.00	1.5892e-11	5.99

Example 4.2. Next, let us consider the following system of ten second-order differential equations

$$\mathbf{u}'' + K\mathbf{u}' + C\mathbf{u} = \mathbf{f}(x), \quad x \in [0,\pi], \tag{4.2a}$$

subject to Dirichlet boundary conditions

$$u_i(0) = 0, \qquad i = 1,2,\ldots,10, \tag{4.2b}$$

and

$$u_i(\pi) = 0, \qquad i = 1,2,\ldots,10, \tag{4.2c}$$

where

$$K = \begin{bmatrix} 0.20 & 0.00 & 0.00 & 0.10 & 0.25 & 0.10 & 0.25 & 0.00 & 0.05 & 0.05 \\ 0.10 & 0.10 & 0.20 & 0.15 & 0.15 & 0.15 & 0.15 & 0.05 & 0.20 & 0.25 \\ 0.25 & 0.15 & 0.00 & 0.05 & 0.25 & 0.20 & 0.25 & 0.10 & 0.00 & 0.05 \\ 0.15 & 0.25 & 0.05 & 0.20 & 0.05 & 0.20 & 0.20 & 0.00 & 0.15 & 0.25 \\ 0.25 & 0.05 & 0.15 & 0.10 & 0.25 & 0.10 & 0.05 & 0.05 & 0.25 & 0.20 \\ 0.10 & 0.15 & 0.15 & 0.00 & 0.05 & 0.00 & 0.15 & 0.05 & 0.10 & 0.05 \\ 0.10 & 0.20 & 0.10 & 0.05 & 0.25 & 0.05 & 0.15 & 0.20 & 0.15 & 0.25 \\ 0.15 & 0.05 & 0.15 & 0.05 & 0.10 & 0.20 & 0.25 & 0.05 & 0.25 & 0.05 \\ 0.25 & 0.00 & 0.25 & 0.00 & 0.05 & 0.25 & 0.15 & 0.20 & 0.05 & 0.20 \\ 0.00 & 0.10 & 0.20 & 0.05 & 0.20 & 0.15 & 0.25 & 0.15 & 0.20 & 0.20 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.25 & 0.15 & 0.20 & 0.15 & -0.10 & -0.05 & 0.05 & -0.15 & 0.00 & -0.05 \\ 0.00 & -0.15 & 0.25 & 0.25 & -0.10 & -0.05 & 0.10 & -0.10 & 0.10 & -0.25 \\ 0.00 & -0.10 & 0.05 & 0.00 & 0.20 & -0.05 & 0.15 & 0.10 & -0.15 & -0.10 \\ 0.00 & 0.10 & -0.05 & -0.20 & -0.05 & 0.00 & 0.20 & -0.15 & -0.20 & -0.20 \\ -0.10 & -0.15 & -0.05 & 0.00 & -0.25 & -0.10 & -0.20 & -0.25 & 0.05 & 0.25 \\ -0.10 & 0.20 & 0.10 & 0.15 & 0.20 & 0.00 & 0.10 & 0.25 & -0.15 & -0.05 \\ 0.25 & -0.20 & -0.15 & 0.10 & -0.05 & -0.25 & 0.15 & -0.15 & -0.10 & 0.00 \\ 0.00 & 0.25 & -0.25 & 0.20 & -0.20 & -0.20 & 0.10 & 0.25 & 0.25 & -0.15 \\ 0.10 & 0.10 & 0.25 & 0.15 & -0.25 & -0.10 & 0.05 & 0.05 & -0.10 & 0.20 \\ 0.10 & -0.20 & 0.25 & 0.05 & 0.10 & 0.15 & -0.25 & -0.15 & 0.25 & -0.20 \end{bmatrix}.$$

We choose f_i for $i = 1,2,\ldots,10$ such that the exact solutions are given by

$$u_i(x) = \sin(i \cdot x), \qquad i = 1,2,\ldots,10.$$

We solve problem (4.2) on uniform meshes with $N = 60, 65, \ldots, 80$ steps and using different degrees $p = 2,3,4,5$. Tables 4, 5, 6, and 7 exhibit the $\mathcal{L}^2$-norm of $(u_i - U_{i,DG})$ for $i = 3,5,8,10$ to conclude an $O(h^{p+1})$ optimal rate of convergence.

Table 4: $\mathcal{L}^2$-norm of $u_3 - U_{3,DG}$ for problem (4.2) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate	$\|\|e_3\|\|$	Rate
60	5.9533e-05	$\cdots$	4.1475e-07	$\cdots$	2.7607e-09	$\cdots$	1.6773e-11	$\cdots$
65	4.6792e-05	3.00	3.0100e-07	4.00	1.8473e-09	-5.01	1.0319e-11	6.06
70	3.7444e-05	3.00	2.2372e-07	4.00	1.2737e-09	-5.01	6.5859e-12	6.05
75	3.0430e-05	3.00	1.6972e-07	4.00	9.0118e-10	-5.01	4.3380e-12	6.05
80	2.5065e-05	3.00	1.3108e-07	4.00	6.5209e-10	-5.01	2.9367e-12	6.04

Table 5: $\mathcal{L}^2$-norm of $u_5 - U_{5,DG}$ for problem (4.2) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_5\|\|$	Rate	$\|\|e_5\|\|$	Rate	$\|\|e_5\|\|$	Rate	$\|\|e_5\|\|$	Rate
60	2.7463e-04	$\cdots$	3.1890e-06	$\cdots$	3.5185e-08	$\cdots$	3.4592e-10	$\cdots$
65	2.1595e-04	3.00	2.3155e-06	4.00	2.3574e-08	5.00	2.1398e-10	6.00
70	1.7287e-04	3.00	1.7217e-06	4.00	1.6271e-08	5.00	1.3716e-10	6.00
75	1.4053e-04	3.00	1.3066e-06	4.00	1.1522e-08	5.00	9.0664e-11	6.00
80	1.1578e-04	3.00	1.0094e-06	4.00	8.3432e-09	5.00	6.1553e-11	6.00

Table 6: $\mathcal{L}^2$-norm of $u_8 - U_{8,DG}$ for problem (4.2) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_8\|\|$	Rate	$\|\|e_8\|\|$	Rate	$\|\|e_8\|\|$	Rate	$\|\|e_8\|\|$	Rate
60	1.1230e-03	$\cdots$	2.0870e-05	$\cdots$	3.6834e-07	$\cdots$	5.7917e-09	$\cdots$
65	8.8337e-04	3.00	1.5157e-05	4.00	2.4685e-07	5.00	3.5837e-09	6.00
70	7.0733e-04	3.00	1.1272e-05	4.00	1.7042e-07	5.00	2.2977e-09	6.00
75	5.7511e-04	3.00	8.5552e-06	4.00	1.2070e-07	5.00	1.5191e-09	6.00
80	4.7390e-04	3.00	6.6099e-06	4.00	8.7408e-08	5.00	1.0315e-09	6.00

Table 7: $\mathcal{L}^2$-norm of $u_{10} - U_{10,DG}$ for problem (4.2) on uniform meshes.

	$p=2$		$p=3$		$p=4$		$p=5$	
N	$\|\|e_{10}\|\|$	Rate	$\|\|e_{10}\|\|$	Rate	$\|\|e_{10}\|\|$	Rate	$\|\|e_{10}\|\|$	Rate
60	2.1876e-03	$\cdots$	5.0872e-05	$\cdots$	1.1227e-06	$\cdots$	2.2074e-08	$\cdots$
65	1.7214e-03	2.99	3.6954e-05	3.99	7.5254e-07	5.00	1.3660e-08	6.00
70	1.3787e-03	2.99	2.7486e-05	3.99	5.1960e-07	5.00	8.7591e-09	6.00
75	1.1212e-03	3.00	2.0865e-05	3.99	3.6804e-07	5.00	5.7913e-09	6.00
80	9.2409e-04	3.00	1.6122e-05	3.99	2.6656e-07	5.00	3.9327e-09	6.00

5. Conclusion

In this manuscript, we applied our newly developed DG method to a system of second-order BVPs. We showed that the DG solutions exhibit optimal $O(h^{p+1})$ convergence rates in the $\mathcal{L}^2$-norm, when we use p-degree piecewise polynomials and h-mesh step size. We validated the established theory through some numerical simulations. In our future research endeavors, we aim to expand the application of DG methods to address a variety of challenging real-world problems including, but not limited to, integro-differential equations [26], delayed-differential equations [29], and nonlinear ODEs [6, 7, 27, 28].

Acknowledgment

The author acknowledges the continuous support of the Kuwait Foundation for the Advancement of Sciences (Project Code: PR20-16SM-01). The author extends their gratitude to his consultant, Professor Mahboub Baccouch, for the invaluable guidance and advice provided throughout this project.

References

[1] S. Adjerid, H. Temimi, *A discontinuous Galerkin method for higher-order ordinary differential equations*, Comput. Methods Appl. Mech. Engrg., **197** (2007), 202–218. 1
[2] S. Adjerid, H. Temimi, *A discontinuous Galerkin method for the wave equation*, Comput. Methods Appl. Mech. Engrg., **200** (2011), 837–849. 1
[3] M. Baccouch, H. Temimi, *Analysis of optimal error estimates and superconvergence of the discontinuous Galerkin method for convection-diffusion problems in one space dimension*, Int. J. Numer. Anal. Model., **13** (2016), 403–434. 1, 3.2
[4] M. Baccouch, H. Temimi, *A high-order space-time ultra-weak discontinuous Galerkin method for the second-order wave equation in one space dimension*, J. Comput. Appl. Math., **389** (2021), 20 pages. 1
[5] F. Bassi, S. Rebay, *A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations*, J. Comput. Phys., **131** (1997), 267–279. 1
[6] M. Ben-Romdhane, H. Temimi, *An iterative numerical method for solving the Lane-Emden initial and boundary value problems*, Int. J. Comput. Methods, **15** (2018), 16 pages. 5
[7] M. Ben-Romdhane, H. Temimi, M. Baccouch, *An Iterative Finite Difference Method for Approximating the Two-Branched Solution of Bratu's Problem*, Appl. Numer. Math., **139** (2019), 62–76. 5
[8] Y. Cheng, C.-W. Shu, *A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives*, Math. Comp., **77** (2008), 699–730. 1, 2
[9] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, (1978). 3.2
[10] B. Cockburn, S. Y. Lin, C. W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: One dimensional systems*, J. Comput. Phys., **84** (1989), 90–113. 1
[11] B. Cockburn, S. Hou, C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. IV. The multidimensional case*, Math. Comp., **54** (1990), 545–581.
[12] B. Cockburn, C.-W. Shu, *The Runge-Kutta local projection* P^1*-discontinuous-Galerkin finite element method for scalar conservation laws*, RAIRO Modél. Math. Anal. Numér., **25** (1991), 337–361.
[13] B. Cockburn, C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., **35** (1998), 2440–2463. 1
[14] M. Corti , F. Bonizzoni, L. Dede', A. M. Quarteroni, P. F. Antonietti, *Discontinuous Galerkin methods for Fisher-Kolmogorov equation with application to α-synuclein spreading in Parkinson's disease*, Comput. Methods Appl. Mech. Engrg., **417** (2023), 17 pages. 1
[15] B. Després, *Sur une formulation variationnelle de type ultra-faible*, C. R. Acad. Sci. Paris Sér. I Math., **318** (1994), 939–944. 1
[16] H. B. Keller, *Numerical Methods for Two-Point Boundary-Value Problems*, Blaisdell Publishing Co. [Ginn and Co.], Waltham, Mass.-Toronto, Ont.-London, (1968). 2
[17] X. Li, W. Wang, *A high order discontinuous Galerkin method for the recovery of the conductivity in Electrical Impedance Tomography*, J. Comput. Appl. Math., **434** (2023), 20 pages. 1
[18] T. Ma, L. Jiang, W. Shen, W. Cao, C. Guo, H. M. Nick, *Fully coupled hydro-mechanical modeling of two-phase flow in deformable fractured porous media with discontinuous and continuous Galerkin method*, Comput. Geotech., **164** (2023). 1
[19] W. H. Reed, T. R. Hill, *Triangular mesh methods for the neutron transport equation*, Tech. Rep. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, (1973). 1
[20] C.-W. Shu, *Discontinuous Galerkin method for time-dependent problems: Survey and recent developments*, In: Recent developments in discontinuous Galerkin finite element methods for partial differential equations, IMA Vol. Math. Appl., **157** (2014), 25–62. 1
[21] C.-W. Shu, *High order WENO and DG methods for time-dependent convection-dominated PDEs: a brief survey of several recent developments*, J. Comput. Phys., **316** (2016), 598–613. 1
[22] G. Szego, *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, (1975).
[23] H. Temimi, *A discontinuous Galerkin finite element method for solving the Troesch's problem*, Appl. Math. Comput., **219** (2012), 521–529. 1
[24] H. Temimi, *Superconvergence of discontinuous Galerkin solutions for higher-order ordinary differential equations*, Appl. Numer. Math., **88** (2015), 46–65. 1
[25] H. Temimi, *Superconvergence Analysis of Discontinuous Galerkin Methods for Systems of Second-Order Boundary Value Problems*, Computation, **11** (2023), 1–16. 1
[26] H. Temimi, A. R. Ansari, A. M. Siddiqui, *An approximate solution for the static beam problem and nonlinear integro-differential equations*, Comput. Math. Appl., **62** (2011), 3132–3139. 5

[27] H. Temimi, M. Ben-Romdhane, *Numerical solution of Falkner-Skan equation by iterative transformation method*, Math. Model. Anal., **23** (2018), 139–151. 5

[28] H. Temimi, M. Ben-Romdhane, M. Baccouch, M. O. Musa, *A two-branched numerical solution of the two-dimensional Bratu's problem*, Appl. Numer. Math., **153** (2020), 202–216. 5

[29] H. Temimi, M. Ben-Romdhane, S. El-Borgi, Y.-J. Cha, *Time-delay effects on controlled seismically excited nonlinear structures*, Int. J. Struct. Stab. Dyn., **16** (2016), 22 pages. 5