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Analyzing existence, uniqueness, and stability of neutral fractional Volterra-Fredholm integro-differential equations

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Abstract

This paper explores the investigation of a Volterra-Fredholm integro-differential equation that incorporates Caputo fractional derivatives and adheres to specific order conditions. The study rigorously establishes both the existence and uniqueness of analytical solutions by applying the Banach principle. Additionally, it presents a unique outcome regarding the existence of at least one solution, supported by exacting conditions derived from the Krasnoselskii fixed point theorem. Furthermore, the paper encompasses neutral Volterra-Fredholm integro-differential equations, thus extending the applicability of the findings. Additionally, the paper explores the concept of Ulam stability for the obtained solutions, providing valuable insights into their long-term behavior. To emphasis the practical significance and reliability of the results, an illustrative example is included, effectively demonstrating the applicability of the theoretical discoveries.

Keywords: Volterra-Fredholm integro-differential equation, Caputo fractional derivatives, Banach contraction principle, Krasnoselskii fixed point theorem, Arzela-Ascoli theorem, Ulam stability.

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1. Introduction

Fractional calculus has emerged as a powerful tool for modeling and analyzing complex phenomena in various scientific and engineering disciplines. This mathematical framework extends the conventional calculus by allowing for fractional-order derivatives and integrals, enabling the representation of systems with memory and non-local interactions. In recent years, the study of Volterra-Fredholm integro-differential equations (IDEs) with Caputo fractional derivatives has gained prominence due to its applicability in diverse fields. This research explores the theoretical and practical aspects of such equations, with a particular focus on their existence, uniqueness, and stability. The theoretical foundations of

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fractional calculus and IDEs have been substantially advanced through a series of pivotal works. Ahmad and Sivasundaram [2] and Wu and Liu [45] have contributed to the existence and uniqueness results of solutions for fractional IDEs, while Kilbas et al. [21] and Zhou et al. [47] have provided essential theories regarding fractional differential equations. Further developments in this area are presented by Hamoud and Ghadle [18] and Ndiaye and Mansal [31], who have explored the uniqueness of solutions for fractional Volterra-Fredholm IDEs and extended the scope to include Caputo fractional derivatives. Moreover, Dahmani [11] and Feckan et al. [13] have presented new existence and uniqueness results for high-dimensional fractional differential systems. Additionally, the works of Wang et al. [44], Ahmad et al. [1], and Smart [39] have contributed to the understanding of fractional differential equations with diverse characteristics. Recent research by Hamarashid et al. [15, 16] has introduced novel numerical algorithms for approximating solutions of nonlinear boundary integro-differential equations and presented numerical results for the existence of Volterra-Fredholm integral equations of nonlinear boundary integro-differential type. Moreover, Srivastava and Saxena [40] have investigated fractional integro-differential equations with multivariable confluent hypergeometric functions as their kernels. The concept of Ulam stability in the context of fractional calculus has also gained significance. Ahmad et al. [3] have explored the Hyers-Ulam stability of a coupled system of fractional differential equations of Hilfer-Hadamard type. Additionally, Raghavendran et al. [32] have proposed the Aboodh transform for solving fractional integro-differential equations, offering a novel approach to numerical solutions. The dynamical behavior of random fractional integro-differential equations has been studied by Begum et al. [7], Dong et al. [12], and Wang et al. [42], enhancing our understanding of these complex systems. Columbu et al. [10] studied properties of unbounded solutions in a class of chemotaxis models. Their work focuses on understanding the behavior and properties of solutions within this specific class of models, shedding light on potential instability in chemotaxis systems. Li et al. [22] explored the combined effects ensuring boundedness in an attraction-repulsion chemotaxis model involving production and consumption. This investigation likely touches upon crucial stability aspects that govern the behavior of these systems under different conditions. Li et al. [23] investigated properties of solutions to porous medium problems with various sources and boundary conditions. This exploration likely contributes insights into stability aspects in systems described by porous medium problems, possibly shedding light on factors influencing system behavior. Li and Viglialoro [27] delved into boundedness considerations for a nonlocal reaction chemotaxis model, even in attraction-dominated scenarios. This study might offer valuable perspectives on stability aspects within such models, especially in regimes where attraction dynamics dominate. [4, 5, 28, 29, 33, 35–37, 41] provided remarks on oscillation of second-order neutral differential equations. While not directly related to PDEs, their insights into oscillatory behavior could inform discussions on system dynamics and stability in certain differential equation models. Bohner and Li [8] studied the oscillation of second-order p-Laplace dynamic equations with nonpositive neutral coefficients. Their findings on oscillatory behavior could potentially contribute to understanding the stability properties of certain dynamic equations. Li and Rogovchenko's works [24–26] provided oscillation criteria for various types of second and third-order neutral differential equations. Although not directly related to PDEs, these criteria may offer valuable insights into stability conditions governing differential equation models. Moaaz et al. [30] explored oscillation criteria for even-order neutral differential equations with distributed deviating arguments. While focused on differential equations, their findings might offer parallels or insights applicable to stability analysis in certain PDE systems. In this context, our research aims to contribute to the ongoing advancements in the field by further investigating Volterra-Fredholm integro-differential equations with Caputo fractional derivatives. We focus on the existence, uniqueness, and stability of solutions, aiming to enhance both the theoretical foundations and practical tools for modeling and analyzing complex systems. This article combines theoretical insights and practical applications, building upon the significant contributions of past research to the field of fractional calculus and integro-differential equations.

In this paper, we present a novel approach to the study of Volterra-Fredholm IDEs that incorporates Caputo fractional derivatives, setting our work apart from existing literature in several ways. This paper offers a unique perspective by focusing on the interplay between Caputo fractional derivatives and

IDEs, thereby extending the applicability of our research to real-world phenomena with fractional-order behaviors. Additionally, we leverage both the Banach fixed-point theorem and the Krasnoselskii fixed-point theorem to rigorously establish the existence and uniqueness of solutions, ensuring the robustness of our findings. Furthermore, we delve into the relatively unexplored realm of Ulam stability in fractional calculus, shedding light on the long-term behavior of fractional-order systems. Our research also extends to Volterra-Fredholm neutral IDEs, making our findings relevant to a broader range of systems and phenomena. To emphasize the practical significance of our work, we provide a compelling illustrative example, showcasing the applicability of our theoretical discoveries in real-world scenarios. Collectively, these unique elements make this paper a valuable and novel contribution to the field of fractional calculus and integro-differential equations.

2. Preliminaries

In this section, we concentrate on the prevalent definitions used in fractional calculus, including the Riemann-Liouville (RL) fractional derivative and the Caputo fractional (CF) derivative, as previously discussed in academic literature [14, 16, 18, 21, 31, 39, 47]. Let us consider the Banach space $C(E,\mathbb{R})$ equipped with the infinity norm defined as $\|\eta\|_\infty = \sup\{|\eta(\xi)| : \xi \in E = [\zeta_0, b]\}$, where η belongs to $C(E,\mathbb{R})$.

Definition 2.1 ([21])**.** The fractional integral of a function φ with the RL definition of order $\delta > 0$ is given by

$$J^\delta \varphi(\xi) = \frac{1}{\Gamma(\delta)} \int_0^\xi (\xi-\zeta)^{\delta-1}\varphi(\zeta)d\zeta, \quad \text{for } \xi > 0, \delta \in \mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the set of positive real numbers, and $J^0\varphi(\xi) = \varphi(\xi)$.

Definition 2.2 ([47])**.** The RL derivative of order δ, where δ is confined to the interval $(0,1)$ and the lower limit is set to zero, is defined for a function $\varphi : [0,1) \to \mathbb{R}$ as follows:

$${}^L D^\delta \varphi(\xi) = \frac{1}{\Gamma(1-\delta)} \frac{d}{d\xi} \int_0^\xi \frac{\varphi(\zeta)}{(\xi-\zeta)^\delta} d\zeta, \quad \text{for } \xi > 0.$$

Definition 2.3 ([47])**.** The CF derivative of order δ, where δ falls within the range of 0 to 1, is applicable to a function $\varphi : [0,1) \to \mathbb{R}$. It can be represented as:

$$D^\delta \varphi(\xi) = \frac{1}{\Gamma(1-\delta)} \int_0^\xi \frac{\varphi^{(0)}(\zeta)}{(\xi-\zeta)^\delta} d\zeta, \quad \text{for } \xi > 0.$$

Definition 2.4 ([21])**.** The CF derivative of the function $\varphi(\xi)$ is defined as follows. For δ values between $n-1$ and n (exclusive), it is given by:

$${}^c D^\delta \varphi(\xi) = \frac{1}{\Gamma(n-\delta)} \int_0^\xi (\xi-\zeta)^{n-\delta-1} \frac{d^n\varphi(\zeta)}{d\zeta^n} d\zeta.$$

For δ equal to n, it is simply the n^{th} derivative of $\varphi(\xi)$:

$${}^c D^\delta \varphi(\xi) = \frac{d^n\varphi(\xi)}{dx^n}.$$

The parameter δ in this definition can be a real or even complex number, representing the order of the derivative.

Definition 2.5 ([21])**.** The RL fractional derivative of order $\delta > 0$ is typically expressed as:

$$D^\delta \varphi(\xi) = D^i J^{i-\delta} \varphi(\xi), \quad \text{where } i-1 < \delta \leqslant i.$$

Definition 2.6 ([16, 47]). In the context of a metric space (χ, d), a function $\varphi : \chi \to \chi$ is defined as a contraction mapping if $\exists$ a non-negative real number $0 \leqslant k < 1$ such that for all ξ and ν in χ, the following inequality holds:

$$d(\varphi(\xi), \varphi(\nu)) \leqslant k\, d(\xi, \nu).$$

Theorem 2.7 (Banach's fixed point theorem, see [16, 47]). *Let Ψ be a nonempty closed subset of a Banach space χ. Then, for any contraction mapping T from Ψ to itself, $\exists$ a unique fixed point.*

Theorem 2.8 (Arzela-Ascoli theorem, see [47]). *A sequence of functions that is both bounded and equicontinuous within the closed and bounded interval $[a, b]$ possesses a subsequence that converges uniformly.*

Theorem 2.9 (Krasnoselskii fixed point theorem, see [47]). *In a Banach space χ, let ζ be a nonempty closed and convex subset. Within ζ, there exist two functions $\mathcal{H}$ and $\mathcal{K}$ with the following properties:*

1. *$\mathcal{H}$ is a contraction mapping;*
2. *$\mathcal{K}$ is compact and continuous;*
3. *for all ξ and ν in ζ such that $\mathcal{H}\xi + \mathcal{K}\nu$ remains within ζ.*

Under these conditions, $\exists$ a ν in ζ for which $\mathcal{H}\nu + \mathcal{K}\nu = \nu$.

3. Volterra-Fredholm integro-differential equation

In this section, we will investigate both the existence and uniqueness of solutions and their Ulam stability results for Volterra-Fredholm IDE, offering valuable insights for theoretical foundations. Through solved examples, we'll illustrate the significance of our findings in understanding the dependable behavior of fractional-order systems.

3.1. Existence and uniqueness results

In this subsection, we explore into the CF Volterra-Fredholm IDE, given by:

$$^{c}D^{\delta}\eta(\zeta) = \varphi(\zeta)\eta(\zeta) + \vartheta(\zeta, \eta(\zeta)) + \int_{\zeta_0}^{\zeta} Z_1(\zeta, \aleph, \eta(\aleph))d\aleph + \int_{\zeta_0}^{b} Z_2(\zeta, \aleph, \eta(\aleph))d\aleph. \tag{3.1}$$

This equation is accompanied by the initial condition:

$$\eta(\zeta_0) = \eta_0. \tag{3.2}$$

In the above expressions, $^{c}D^{\delta}$ denotes CF derivative with $0 < \delta \leqslant 1$, and $\eta : E \to \mathbb{R}$, where $E = [\zeta_0, b]$, represents the continuous function under consideration. Additionally, $\vartheta : E \times \mathbb{R} \to \mathbb{R}$ and $Z_n : E \times E \times \mathbb{R} \to \mathbb{R}$, where $n = 1, 2$, are continuous functions. Before commencing our main results and their proofs, we present the following lemma along with some essential hypotheses.

(A1) Consider continuous functions Z_1 and $Z_2 : E \times E \times \mathbb{R} \to \mathbb{R}$ defined on the set $D = \{(\zeta, \aleph) : 0 \leqslant \zeta_0 \leqslant \aleph \leqslant \zeta \leqslant b\}$. They satisfy the following conditions:

$$|Z_1(\xi, \aleph, \eta_1(\aleph)) - Z_1(\xi, \aleph, \eta_2(\aleph))| \leqslant \lambda^*_{z_1}\|\eta_1(\aleph) - \eta_2(\aleph)\|,$$
$$|Z_2(\xi, \aleph, \eta_1(\aleph)) - Z_2(\xi, \aleph, \eta_2(\aleph))| \leqslant \lambda^*_{z_2}\|\eta_1(\aleph) - \eta_2(\aleph)\|.$$

(A2) The function $\vartheta : E \times \mathbb{R} \to \mathbb{R}$ is continuous, and it satisfies the condition

$$|\vartheta(\zeta, \eta_1) - \vartheta(\zeta, \eta_2)| \leqslant \lambda^*_{\vartheta}\|\eta_1 - \eta_2\|.$$

(A3) The function $\varphi : E \to \mathbb{R}$ is continuous, and the constants $\lambda^*_{z_1}$, $\lambda^*_{z_2}$, and λ^*_{ϑ} are all positive.

Lemma 3.1. *If $\eta_0(\zeta) \in C(E, \mathbb{R})$, then $\eta(\zeta) \in C(E, \mathbb{R}^+)$ constitutes a solution to problem (3.1)-(3.2) if and only if it complies with the following conditions:*

$$\begin{aligned}\eta(\zeta) = \eta_0 &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph \\ &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph,\end{aligned} \tag{3.3}$$

for $\zeta \in E$.

Proof. This can be readily demonstrated by utilizing the integral operator (2.1) on both sides of equation (3.1), resulting in the integral equation (3.3). □

To establish the foundation of our investigation, we now present the following theorem that addresses the uniqueness of solutions in the context of Volterra-Fredholm IDE, as described in equations (3.1)-(3.2).

Theorem 3.2. *Assuming that conditions (A1)-(A3) are fulfilled, and given two positive real numbers β and γ with $0 < \beta < 1$, if they satisfy the following equations:*

$$\left[\frac{\|\varphi\|_\infty + \lambda_\vartheta^*}{\Gamma(\delta+1)} + \frac{(\lambda_{z_1}^* + \lambda_{z_2}^*)b}{(\delta+1)\Gamma(\delta)}\right]b^\delta = \beta, \quad |\eta_0| + \left[\frac{\vartheta_0}{\Gamma(\delta+1)} + \frac{(z_1^*+z_2^*)b}{(\delta+1)\Gamma(\delta)}\right]b^\delta = (1-\beta)\gamma,$$

then, the IVP (3.1)-(3.2) possesses a unique continuous solution over the interval $[\zeta_0, b]$, where $\vartheta_0 = \max\{|\vartheta(\aleph, 0)| : \aleph \in E\}$, $z_1^* = \max\{|Z_1(\xi,\aleph,0)| : (\xi,\aleph) \in D\}$, and $z_2^* = \max\{|Z_2(\xi,\aleph,0)| : (\xi,\aleph) \in D\}$.

Proof. Define the operator $T : C(E, \mathbb{R}) \to C(E, \mathbb{R})$ as follows:

$$\begin{aligned}(T\eta)(\zeta) = \eta_0 &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph \\ &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi + \int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi\right)d\aleph.\end{aligned}$$

Furthermore, let's define Φ_γ as the set of functions $\eta \in C(E, \mathbb{R})$ such that $\|\eta\|_\infty \leqslant \gamma$ for some $\gamma > 0$. Our objective is to demonstrate the existence of a fixed point for the operator T within the subset $\Phi_\gamma \subset C(E, \mathbb{R})$. The fixed point is essentially the one and only solution to the IVP described in equations (3.1)-(3.2). To prove this, we'll divide the process into two separate steps.

Step 1. Our aim is to demonstrate that the operator T preserves functions within the set Φ_γ. Based on the previously stated hypotheses, for any function η belonging to the set Φ_γ and for all ζ in the interval E, we can establish the following:

$$\begin{aligned}|(T\eta)(\zeta)| &\leqslant |\eta_0| + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\varphi(\aleph)||\eta(\aleph)|d\aleph \quad + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\vartheta(\aleph,\ \eta(\aleph))|d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta}|Z_1(\xi,\ \aleph,\ \eta(\aleph))|d\xi + \int_{t_0}^{b}|Z_2(\xi,\ \aleph,\ \eta(\aleph))|d\xi\right)d\aleph \\ &\leqslant |\eta_0| + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\|\varphi\|_\infty\|\eta\|_\infty d\aleph + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\big(|\vartheta(\aleph,\ \eta(\aleph)) - \vartheta(\aleph,\ 0)| \\ &\quad + |\vartheta(\aleph,\ 0)|\big)d\aleph + \frac{1}{\Gamma(\delta)}\int_{t_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta}\big(|Z_1(\xi,\ \aleph,\ \eta(\aleph)) - Z_1(\xi,\ \aleph,\ 0)| + |Z_1(\xi,\ \aleph,\ O)|\big)d\xi\right. \\ &\quad \left. + \int_{t_0}^{b}\big(|Z_2(\xi,\ \aleph,\ \eta(\aleph)) - Z_2(\xi,\ \aleph,\ 0)| + |Z_2(\xi,\ \aleph,\ 0)|\big)d\xi\right)d\aleph\end{aligned}$$

$$\begin{aligned}
&\leqslant |\eta_0| + \frac{\|\varphi\|_\infty b^\delta \gamma}{\Gamma(\delta+1)} + \frac{b^\delta}{\Gamma(\delta+1)}(\lambda^*_\vartheta \gamma + \vartheta_0) + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}(\lambda^*_{z_1}\gamma + z_1^*) + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}(\lambda^*_{z_2}\gamma + z_2^*)\\
&\leqslant |\eta_0| + \frac{\|\varphi\|_\infty b^\delta \gamma}{\Gamma(\delta+1)} + \frac{b^\delta}{\Gamma(\delta+1)}\lambda^*_\vartheta \gamma + \frac{b^\delta}{\Gamma(\delta+1)}\vartheta_0 + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}\lambda^*_{z_1}\gamma + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)} z_1^*\\
&\quad + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}\lambda^*_{z_2}\gamma + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)} z_2^*\\
&\leqslant |\eta_0| + b^\delta\left(\frac{\vartheta_0}{\Gamma(\delta+1)} + \frac{(z_1^*+z_2^*)b}{(\delta+1)\Gamma(\delta)}\right) + b^\delta\gamma\left(\frac{\|\varphi\|_\infty + \lambda^*_\vartheta}{\Gamma(\delta+1)} + \frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right)\\
&= (1-\beta)\gamma + \beta\gamma = \gamma.
\end{aligned}$$

Hence, we can conclude that $\|T\eta\| \leqslant \gamma$, which implies that $T\eta \in \Phi_\gamma$, thereby establishing that $T\Phi_\gamma$ is a subset of Φ_γ.

Step 2. Our objective is to demonstrate that T is a contraction mapping. Consider η_1 and η_2 both belonging to Φ_γ,

$$\begin{aligned}
|(T\eta_1)(\zeta) - (T\eta_2)(\zeta)| &\leqslant \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\varphi(\aleph)||\eta_1(\aleph)-\eta_2(\aleph)|d\aleph\\
&\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\vartheta(\aleph,\eta_1(\aleph)) - \vartheta(\aleph,\eta_2(\aleph))|d\aleph\\
&\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta}|Z_1(\xi,\aleph,\eta_1(\aleph)) - Z_1(\xi,\aleph,\eta_2(\aleph))|d\xi\right.\\
&\quad \left. + \int_{\aleph}^{b}|Z_2(\xi,\aleph,\eta_1(\aleph)) - Z_2(\xi,\aleph,\eta_2(\aleph))|d\xi\right)d\aleph\\
&\leqslant \frac{\|\varphi\|_\infty b^\delta}{\Gamma(\delta+1)}\|\eta_1-\eta_2\| + \frac{\lambda^*_\vartheta b^\delta}{\Gamma(\delta+1)}\|\eta_1-\eta_2\| + \frac{\lambda^*_{z_1} b^{\delta+1} + \lambda^*_{z_2} b^{\delta+1}}{(\delta+1)\Gamma(\delta)}\|\eta_1-\eta_2\|\\
&= \left(\frac{\|\varphi\|_\infty + \lambda^*_\vartheta}{\Gamma(\delta+1)} + \frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right) b^\delta\|\eta_1-\eta_2\| = \epsilon\|\eta_1-\eta_2\|,
\end{aligned}$$

where $\epsilon = \left[\frac{\|\varphi\|_\infty + \lambda^*_\vartheta}{\Gamma(\delta+1)} + \frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right] b^\delta < 1$, we get $\|T\eta_1 - T\eta_2\| \leqslant \epsilon\|\eta_1-\eta_2\|$. This establishes that T satisfies as a contraction mapping. Consequently, in accordance with Theorem 2.7, ∃ a fixed point denoted as $\eta \in C(E, \mathbb{R})$ such that $T\eta = \eta$. This fixed point represents the only solution to the IVP outlined in equations (3.1) through (3.2). This concludes the proof of the theorem. □

To set the groundwork for our investigation, we now present the following theorem that addresses the existence of solutions in the context of Volterra-Fredholm IDE, as described in equations (3.1)-(3.2).

Theorem 3.3. *Assume that*

(A4) $\|\vartheta(\zeta,\eta)\| \leqslant \mu(\zeta), \forall(\zeta,\eta) \in E\times\chi$, $\|Z_1(\zeta,\aleph,\eta)\| \leqslant \sigma(\zeta), \forall(\zeta,\aleph,\eta) \in E\times E\times\chi$, *and* $\|Z_2(\zeta,\aleph,\eta)\| \leqslant \rho(\zeta), \forall(\zeta,\aleph,\eta) \in E\times E\times\chi$, *where* μ, σ, *and* $\rho \in C(E, \mathbb{R}^+)$.

This implies that there is at least one solution to the problem described in (3.1)-(3.2) *over the interval* $[\zeta_0, b]$.

Proof. Choose a fixed $\left[\|\eta_0\| + \frac{b^\delta\|\varphi\|_\infty\|\eta\|_\infty}{\Gamma(\delta+1)} + \frac{b^\delta\|\mu\|_C}{\Gamma(\delta+1)} + \frac{b^{\delta+1}\|\sigma\|_C}{(\delta+1)\Gamma(\delta)} + \frac{b^{\delta+1}\|\rho\|_C}{(\delta+1)\Gamma(\delta)}\right] \leqslant r$ and define the set $\Phi_\gamma = \{\eta \in C; \|\eta\|_\infty \leqslant \gamma\}$. Within this context, we introduce the operators Θ and Ψ on $\Phi\gamma$ as follows:

$$(\Theta\eta)(\zeta) = \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph$$

$$+\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph\eta(\aleph))d\xi d\aleph,$$
$$(\Psi\eta)(\zeta)=\eta_0+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph.$$

When considering η_1 and η_2 from the set Φ_γ, we observe that:

$$\begin{aligned}\|\Theta\eta_1+\Psi\eta_2\| &\leqslant \Big\|\eta_0+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph\\ &\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\Big(\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi+\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi\Big)d\aleph\Big\|\\ &\leqslant \|\eta_0\|+\frac{b^\delta\|\varphi\|_\infty\|\eta\|_\infty}{\Gamma(\delta+1)}+\frac{b^\delta\|\mu\|_C}{\Gamma(\delta+1)}+\frac{b^{\delta+1}\|\sigma\|_C}{(\delta+1)\Gamma(\delta)}+\frac{b^{\delta+1}\|\rho\|_C}{(\delta+1)\Gamma(\delta)}\leqslant r.\end{aligned}$$

Hence, $\Theta\eta_1+\Psi\eta_2\in\Phi_\gamma$. Notably, the assumption (A4) ensures that Ψ acts as a contraction mapping. The continuity of functions ϑ, Z_1, and Z_2 specified in (3.1)-(3.2) implies the continuity of the operator Φ. Moreover, it's important to highlight that Θ stays uniformly bounded on Φ_γ as follows:

$$\|\Theta\eta\|\leqslant\frac{b^\delta\|\mu\|_C}{\Gamma(\delta+1)}+\frac{b^{\delta+1}\|\sigma\|_C}{(\delta+1)\Gamma(\delta)}+\frac{b^{\delta+1}\|\rho\|_C}{(\delta+1)\Gamma(\delta)}.$$

Now, we demonstrate the compactness of the operator Θ. Given that ϑ, Z_1, and Z_2 are bounded on the compact sets $\Omega_1=E\times\chi$ and $\Omega_2=E\times E\times\chi$, we can define $\sup_{(\zeta,\eta)\in\Omega_1}\|\vartheta(\zeta,\eta)\|=C_1$ and $\sup_{(\zeta,\aleph,\eta)\in\Omega_2}\|Z_i(\zeta,\aleph,\eta)\|=C_2$, where $i=1,2$. For $\zeta_1,\zeta_2\in[\zeta_0,b]$, and $\eta\in\Phi_\gamma$, we observe that:

$$\begin{aligned}&\|(\Theta\eta)(\zeta_1)-(\Theta\eta)(\zeta_2)\|\\ &=\Big\|\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_1} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph\\ &\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph\\ &\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_2} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph\Big\|\\ &\leqslant\frac{1}{\Gamma(\delta)}\Big\|\int_{\zeta_2}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph+\int_{\zeta_2}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_1} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph\\ &\quad+\int_{\zeta_2}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph-\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph\\ &\quad-\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_2} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph-\int_{\zeta_0}^{\zeta_2}(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph\\ &\quad+\int_{\zeta_0}^{\zeta_2}(\zeta_1-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph+\int_{\zeta_0}^{\zeta_2}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_1} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph\\ &\quad+\int_{\zeta_0}^{\zeta_2}(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph\Big\|\\ &\leqslant\frac{1}{\Gamma(\delta)}\Big\|\int_{\zeta_2}^{\zeta_1}(\zeta_1-\aleph)^{\delta-1}\big[\vartheta(\aleph,\eta(\aleph))+\int_{\aleph}^{\zeta_1} Z_1(\xi,\aleph,\eta(\aleph))d\xi+\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi\big]d\aleph\\ &\quad-\int_{\zeta_0}^{\zeta_2}\big[(\zeta_2-\aleph)^{\delta-1}-(\zeta_1-\aleph)^{\delta-1}\big]\vartheta(\aleph,\eta(\aleph))d\aleph\end{aligned}$$

$$
\begin{aligned}
&-\int_{\zeta_0}^{\zeta_2}\left[(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_2} Z_1(\xi,\aleph,\eta(\aleph))d\xi-(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{\zeta_1} Z_1(\xi,\aleph,\eta(\aleph))d\xi\right]d\aleph\\
&-\int_{\zeta_0}^{\zeta_2}\left[(\zeta_2-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi-(\zeta_1-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi\right]d\aleph\Big\|\\
&\leqslant \frac{C_1}{\Gamma(\delta+1)}|2(\zeta_1-\zeta_2)^{\delta}|+\frac{C_2\delta}{\Gamma(\delta+2)}|2(\zeta_1-\zeta_2)^{\delta+1}|+\frac{C_3\delta}{\Gamma(\delta+2)}|2(\zeta_1-\zeta_2)^{\delta+1}|\\
&\quad+\frac{C_1}{\Gamma(\delta+1)}|\zeta_2^{\delta}|+\frac{C_2\delta}{\Gamma(\delta+2)}|\zeta_2^{\delta+1}|+\frac{C_3\delta}{\Gamma(\delta+2)}|\zeta_2^{\delta+1}|-\frac{C_1}{\Gamma(\delta+1)}|\zeta_1^{\delta}|-\frac{C_2\delta}{\Gamma(\delta+2)}|\zeta_1^{\delta+1}|-\frac{C_3\delta}{\Gamma(\delta+2)}|\zeta_1^{\delta+1}|\\
&\leqslant \frac{C_1}{\Gamma(\delta+1)}|2(\zeta_1-\zeta_2)^{\delta}+\zeta_2^{\delta}-\zeta_1^{\delta}|+\frac{C_2\delta}{\Gamma(\delta+2)}|2(\zeta_1-\zeta_2)^{\delta}+\zeta_2^{\delta}-\zeta_1^{\delta}|+\frac{C_3\delta}{\Gamma(\delta+2)}|2(\zeta_1-\zeta_2)^{\delta}+\zeta_2^{\delta}-\zeta_1^{\delta}|.
\end{aligned}
$$

This quantity is independent of the choice of η. Therefore, Θ exhibits relative compactness on Φ_γ. Consequently, in accordance with the Arzela-Ascoli theorem, Θ is a compact operator on Φ_γ. All the conditions outlined in Theorem 2.9 are satisfied. Therefore, the conclusion of Theorem 2.9 applies, indicating that the problem (3.1)-(3.2) has at least one solution. This concludes the proof of the theorem. □

3.2. Ulam stability results

In this subsection, we will investigate the Ulam stability of the problem (3.1)-(3.2). Let's examine the following inequality:

$$\left|{}^{c}D^{\delta}\eta(\zeta)-\varphi(\zeta)\eta(\zeta)-\vartheta(\zeta,\eta(\zeta))-\int_{\zeta_0}^{\zeta} Z_1(\zeta,\aleph,\eta(\aleph))d\aleph-\int_{\zeta_0}^{b} Z_2(\zeta,\aleph,\eta(\aleph))d\aleph\right|\leqslant\varepsilon. \tag{3.4}$$

Definition 3.4. The stability of equation (3.1)-(3.2) in the sense of Ulam-Hyers is established when $\exists$ a positive constant $C_\varphi>0$ such that for every $\varepsilon>0$ and for each solution $\nu\in C(E,\mathbb{R})$ to inequality (3.4), $\exists$ a solution $\xi\in C(E,\mathbb{R})$ to equation (3.1)-(3.2) such that $|\nu(\zeta)-\xi(\zeta)|\leqslant\varepsilon C_\varphi$ holds for all $\zeta\in E$.

To establish the foundation of our investigation, we now present the following theorem that addresses the Ulam stability results in the context of the Volterra-Fredholm IDE (3.1)-(3.2).

Theorem 3.5. *Given that* (A1) *and* (A2) *hold true, problem* (3.1)-(3.2) *exhibits Ulam-Hyers stability when* $\epsilon<1$.

Proof. Consider $\varepsilon>0$, and let $\nu\in C(E,\mathbb{R})$ satisfies inequality (3.4). Also, let $\xi\in C(E,\mathbb{R})$ be the unique solution to the following problem. In this context, we recall that,

$$
\begin{aligned}
\eta(\zeta)=\eta_0&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi+\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi\right)d\aleph.
\end{aligned}
$$

By integrating inequality (3.4) and incorporating the initial condition of problem (3.2), we obtain:

$$
\begin{aligned}
&\Big|\eta(\zeta)-\eta_0-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta(\aleph))d\aleph\\
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta} Z_1(\xi,\aleph,\eta(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\xi,\aleph,\eta(\aleph))d\xi d\aleph\Big|\leqslant\varepsilon\frac{b^{\delta}}{\Gamma(\delta+1)}.
\end{aligned}
$$

Additionally, let's examine

$$
\begin{aligned}
&|\eta_1(\zeta)-\eta_2(\zeta)|\\
&\quad\leqslant\Big|\eta_1(\zeta)-\eta_0-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_2(\aleph)d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta_2(\aleph))d\aleph
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\xi,\aleph,\eta_2(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\xi,\aleph,\eta_2(\aleph))d\xi d\aleph\Bigg|\\
&\leqslant\Bigg|\eta_1(\zeta)-\eta_0-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph\\
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_2(\aleph)d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta_1(\aleph))d\aleph\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta_1(\aleph))d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta_2(\aleph))d\aleph\\
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph\\
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\xi,\aleph,\eta_2(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\xi,\aleph,\eta_2(\aleph))d\xi d\aleph\Bigg|\\
&\leqslant\Bigg|\eta_1(\zeta)-\eta_0-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta(\aleph,\eta_1(\aleph))d\aleph\\
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\xi,\aleph,\eta_1(\aleph))d\xi d\aleph\Bigg|\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\varphi(\aleph)|\,|\eta_1(\aleph)-\eta_2(\aleph)|d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\vartheta(\aleph,\eta_1(\aleph)-\vartheta(\aleph,\eta_2(\aleph)|d\aleph\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}|Z_1(\xi,\aleph,\eta_1(\aleph))-Z_1(\xi,\aleph,\eta_2(\aleph))|d\xi d\aleph\\
&+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}|Z_2(\xi,\aleph,\eta_1(\aleph))-Z_2(\xi,\aleph,\eta_2(\aleph))|d\xi d\aleph,
\end{aligned}
$$

$$
\begin{aligned}
\|\eta_1-\eta_2\|&\leqslant\frac{\varepsilon}{\Gamma(\delta+1)}+\left[\frac{\|\varphi\|_\infty+\lambda^*_\vartheta}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right]b^\delta\|\eta_1-\eta_2\|,\\
\|\eta_1-\eta_2\|&\leqslant\frac{\varepsilon}{\Gamma(\delta+1)}+\epsilon\|\eta_1-\eta_2\|,\\
\|\eta_1-\eta_2\|&\leqslant\frac{\varepsilon}{\Gamma(\delta+1)(1-\epsilon)}=\varepsilon M;\quad\text{where } M=\frac{1}{\Gamma(\delta+1)(1-\epsilon)}.
\end{aligned}
$$

Hence, we can conclude that the problem (3.1)-(3.2) exhibits Ulam-Hyers stability. This concludes the proof of the theorem. □

Example 3.6. We investigate the CF Volterra-Fredholm IDE (3.1)-(3.2) under the following parameters: $\delta=0.5$, $b=0.5$, $\lambda^*_\vartheta=0.2$, $\lambda^*_{Z_1}=0.3$, $\lambda^*_{Z_2}=0.3$, and $\|\varphi\|_\infty=0.1$. It now follows that,

$$
\begin{aligned}
\epsilon&=\left[\frac{\|\varphi\|_\infty+\lambda^*_\vartheta}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right]b^\delta\\
&=\left[\frac{0.1+0.2}{\Gamma(\frac{1}{2}+1)}+\frac{(0.3+0.3)(0.5)}{(\frac{1}{2}+1)\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}}\\
&=\left[\frac{0.3}{\Gamma(\frac{3}{2})}+\frac{(0.6)(0.5)}{(\frac{3}{2})\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}}=\left[\frac{0.3}{0.886}+\frac{0.30}{(1.5)(1.772)}\right](0.707)=0.318<1.
\end{aligned}
$$

If we set $\varepsilon = 0.5$, then the value of M can be computed as

$$M = \frac{1}{\Gamma(\delta+1)(1-\epsilon)} = \frac{1}{\Gamma(\frac{3}{2})(1-0.318)} = \frac{1}{0.604} = 1.655.$$

Now, when we multiply ε by M, we get $\varepsilon M = 0.5 \times 1.655 = 0.8275$. Since all the conditions of Theorem 3.2 are satisfied, $\exists$ a unique and stable solution to the provided equation.

4. Neutral Volterra-Fredholm integro-differential equation

In this section, we delve into the investigation of both the existence and uniqueness of solutions, as well as the Ulam stability results for neutral Volterra-Fredholm IDE. This exploration offers valuable insights for theoretical foundations, and we will demonstrate the significance of our findings through solved examples.

4.1. Existence and uniqueness results

In this subsection, we explore into the CF neutral Volterra-Fredholm IDE, given by:

$$^{c}D^{\delta}\Big[\eta(\zeta)-\vartheta_1(\zeta,\eta(\zeta))\Big] = \varphi(\zeta)\eta(\zeta)+\vartheta_2(\zeta,\eta(\zeta))+\int_{\zeta_0}^{\zeta} Z_1(\zeta,\aleph,\eta(\aleph))d\aleph+\int_{\zeta_0}^{b} Z_2(\zeta,\aleph,\eta(\aleph))d\aleph. \tag{4.1}$$

This equation is accompanied by the initial condition:

$$\eta(\zeta_0) = \eta_0. \tag{4.2}$$

In the above expressions, $^{c}D^{\delta}$ denotes CF derivative with $0 < \delta \leqslant 1$, and $\eta : E \to \mathbb{R}$, where $E = [\zeta_0, b]$, represents the continuous function under consideration. Additionally, $\vartheta_n : E \times \mathbb{R} \to \mathbb{R}$ and $Z_n : E \times E \times \mathbb{R} \to \mathbb{R}$, where $n = 1, 2$, are continuous functions. Before commencing our main results and their proofs, we present the following lemma along with some essential hypotheses.

(B1) Consider continuous functions Z_1 and $Z_2 : E \times E \times \mathbb{R} \to \mathbb{R}$ defined on the set $D = \{(\zeta,\aleph) : 0 \leqslant \zeta_0 \leqslant \aleph \leqslant \zeta \leqslant b\}$. They satisfy the following conditions:

$$|Z_1(\tau,\aleph,\eta_1(\aleph)) - Z_1(\tau,\aleph,\eta_2(\aleph))| \leqslant \lambda^*_{z_1}\|\eta_1(\aleph)-\eta_2(\aleph)\|,$$
$$|Z_2(\tau,\aleph,\eta_1(\aleph)) - Z_2(\tau,\aleph,\eta_2(\aleph))| \leqslant \lambda^*_{z_2}\|\eta_1(\aleph)-\eta_2(\aleph)\|.$$

(B2) The functions ϑ_1 and $\vartheta_2 : E \times \mathbb{R} \to \mathbb{R}$ are continuous, and they satisfy the conditions

$$|\vartheta_1(\zeta,\eta_1)-\vartheta_1(\zeta,\eta_2)| \leqslant \lambda^*_{\vartheta_1}\|\eta_1-\eta_2\|, \quad |\vartheta_2(\zeta,\eta_1)-\vartheta_2(\zeta,\eta_2)| \leqslant \lambda^*_{\vartheta_2}\|\eta_1-\eta_2\|.$$

(B3) The function $\varphi : E \to \mathbb{R}$ is continuous, and the constants $\lambda^*_{z_1}$, $\lambda^*_{z_2}$, $\lambda^*_{\vartheta_1}$, and $\lambda^*_{\vartheta_2}$ are all positive.

Lemma 4.1. *If $\eta_0(\zeta) \in C(E,\mathbb{R})$, then $\eta(\zeta) \in C(E,\mathbb{R}^+)$ constitutes a solution to problem (4.1)-(4.2) if and only if it complies with the following conditions:*

$$\begin{aligned}\eta(\zeta) &= \eta_0 - \vartheta_1(\zeta_0,\eta_0) + \vartheta_1(\zeta,\eta(\zeta)) \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta(\aleph))d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta(\aleph))d\tau d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta(\aleph))d\tau d\aleph,\end{aligned} \tag{4.3}$$

for $\zeta \in E$.

Proof. This can be readily demonstrated by utilizing the integral operator (2.1) on both sides of equation (4.1), resulting in the integral equation (4.3). □

To establish the foundation of our investigation, we now present the following theorem that addresses the uniqueness of solutions in the context of Volterra-Fredholm IDE, as described in equations (4.1)-(4.2).

Theorem 4.2. *Assuming that conditions* (B1)-(B3) *are fulfilled, and given two positive real numbers* β *and* γ *with* $0 < \beta < 1$, *if they satisfy the following equations:*

$$\lambda^*_{\vartheta_1} + \left[\frac{\|\varphi\|_\infty + \lambda^*_{\vartheta_2}}{\Gamma(\delta+1)} + \frac{(\lambda^*_{z_1} + \lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right] b^\delta = \beta,$$

$$|\eta_0| + |\vartheta_1(\zeta_0,\eta_0)| + \vartheta^*_1 + \left[\frac{\vartheta^*_2}{\Gamma(\delta+1)} + \frac{(z^*_1 + z^*_2)b}{(\delta+1)\Gamma(\delta)}\right] b^\delta = (1-\beta)\gamma,$$

then, the IVP (4.1)-(4.2) *possesses a unique continuous solution over the interval* $[\zeta_0, b]$, *where* $\vartheta^*_1 = \max\{|\vartheta_1(\aleph, 0)| : \aleph \in E\}$, $\vartheta^*_2 = \max\{|\vartheta_2(\aleph, 0)| : \aleph \in E\}$, $z^*_1 = \max\{|Z_1(\tau, \aleph, 0)| : (\tau, \aleph) \in D\}$, *and* $z^*_2 = \max\{|Z_2(\tau, \aleph, 0)| : (\tau, \aleph) \in D\}$.

Proof. Define the operator $T : C(E, \mathbb{R}) \to C(E, \mathbb{R})$ as follows:

$$\begin{aligned}(Th)(\zeta) = \eta_0 - \vartheta_1(\zeta_0, \eta_0) + \vartheta_1(\zeta, \eta(\zeta)) &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph \\ &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph, \eta(\aleph))d\aleph \\ &+ \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} Z_1(\tau, \aleph, \eta(\aleph))d\tau + \int_{\aleph}^{b} Z_2(\tau, \aleph, \eta(\aleph))d\tau\right)d\aleph.\end{aligned}$$

Furthermore, let's define Φ_γ as the set of functions $\eta \in C(E, \mathbb{R})$ such that $\|\eta\|_\infty \leqslant \gamma$ for some $\gamma > 0$. Our objective is to demonstrate the existence of a fixed point for the operator T within the subset $\Phi_\gamma \subset C(E, \mathbb{R})$. The fixed point is essentially the one and only solution to the IVP described in equations (4.1)-(4.2). To prove this, we'll divide the process into two separate steps.

Step 1. Our aim is to demonstrate that the operator T preserves functions within the set Φ_γ. Based on the previously stated hypotheses, for any function η belonging to the set Φ_γ and for all ζ in the interval E, we can establish the following:

$$\begin{aligned}|(Th)(\zeta)| \leqslant{} & |\eta_0| + |\vartheta_1(\zeta_0, \eta_0)| + |\vartheta_1(\zeta, \eta(\zeta))| + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}|\varphi(\aleph)||\eta(\aleph)|d\aleph \quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1} \\ & \times |\vartheta_2(\aleph,\ \eta(\aleph))|d\aleph + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} |Z_1(\tau,\ \aleph,\ \eta(\aleph))|d\tau + \int_{\zeta_0}^{b} |Z_2(\tau,\ \aleph,\ \eta(\aleph))|d\tau\right)d\aleph \\ \leqslant{} & |\eta_0| + |\vartheta_1(\zeta_0, \eta_0)| + |\vartheta_1(\zeta, \eta(\zeta)) - \vartheta_1(\zeta, 0)| + |\vartheta_1(\zeta, 0)| + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\|\varphi\|_\infty\|\eta\|_\infty d\aleph \\ & + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\big(|\vartheta_2(\aleph,\ \eta(\aleph)) - \vartheta_2(\aleph,\ 0)| + |\vartheta_2(\aleph,\ 0)|\big)d\aleph \\ & + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta} (\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} \big(|Z_1(\tau,\ \aleph,\ \eta(\aleph)) - Z_1(\tau,\ \aleph,\ 0)| + |Z_1(\tau,\ \aleph,\ O)|\big)d\tau\right. \\ & \left. + \int_{\zeta_0}^{b} \big(|Z_2(\tau,\ \aleph,\ \eta(\aleph)) - Z_2(\tau,\ \aleph,\ 0)| + |Z_2(\tau,\ \aleph,\ 0)|\big)d\tau\right)d\aleph \\ \leqslant{} & |\eta_0| + |\vartheta_1(\zeta_0, \eta_0)| + \lambda^*_{\vartheta_1} + \vartheta^*_1 + \frac{\|\varphi\|_\infty b^\delta\gamma}{\Gamma(\delta+1)} + \frac{b^\delta}{\Gamma(\delta+1)}(\lambda^*_{\vartheta_2}\gamma + \vartheta_2) + \frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}(\lambda^*_{z_1}\gamma + z^*_1)\end{aligned}$$

$$
\begin{aligned}
&+\frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}(\lambda^*_{z_2}\gamma+z^*_2)\\
&\leqslant |\eta_0|+|\vartheta_1(\zeta_0,\eta_0)|+\lambda^*_{\vartheta_1}+\vartheta^*_1+\frac{\|\varphi\|_\infty b^\delta\gamma}{\Gamma(\delta+1)}+\frac{b^\delta}{\Gamma(\delta+1)}\lambda^*_{\vartheta_2}\gamma+\frac{b^\delta}{\Gamma(\delta+1)}\vartheta_2+\frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}\lambda^*_{z_1}\gamma\\
&+\frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}z^*_1+\frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}\lambda^*_{z_2}\gamma+\frac{b^{(\delta+1)}}{(\delta+1)\Gamma(\delta)}z^*_2\\
&\leqslant |\eta_0|+|\vartheta_1(\zeta_0,\eta_0)|+\vartheta^*_1+b^\delta\left[\frac{\vartheta^*_2}{\Gamma(\delta+1)}+\frac{(z^*_1+z^*_2)b}{(\delta+1)\Gamma(\delta)}\right]+\gamma\left(\lambda^*_{\vartheta_1}+\left[\frac{\|\varphi\|_\infty+\lambda^*_{\vartheta}}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right]b^\delta\right)\\
&=(1-\beta)\gamma+\beta\gamma=\gamma.
\end{aligned}
$$

Hence, we can conclude that $\|T\eta\|\leqslant\gamma$, which implies that $T\eta\in\Phi_\gamma$, thereby establishing that $T\Phi_\gamma$ is a subset of Φ_γ.

Step 2. Our objective is to demonstrate that T is a contraction mapping. Consider η_1 and η_2 both belonging to Φ_γ.

$$
\begin{aligned}
&|(T\eta_1)(\zeta)-(T\eta_2)(\zeta)|\\
&\quad\leqslant |\vartheta_1(\zeta,\eta_1(\zeta))-\vartheta_1(\zeta,\eta_2(\zeta))|+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\varphi(\aleph)||\eta_1(\aleph)-\eta_2(\aleph)|d\aleph\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\vartheta_2(\aleph,\eta_1(\aleph))-\vartheta_2(\aleph,\eta_2(\aleph))|d\aleph\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta}|Z_1(\tau,\aleph,\eta_1(\aleph))-Z_1(\tau,\aleph,\eta_2(\aleph))|d\tau\right.\\
&\quad\left.+\int_{\aleph}^{b}|Z_2(\tau,\aleph,\eta_1(\aleph))-Z_2(\tau,\aleph,\eta_2(\aleph))|d\tau\right)d\aleph\\
&\quad\leqslant \lambda^*_{\vartheta_1}\|\eta_1-\eta_2\|+\frac{\|\varphi\|_\infty b^\delta}{\Gamma(\delta+1)}\|\eta_1-\eta_2\|+\frac{\lambda^*_{\vartheta_2}b^\delta}{\Gamma(\delta+1)}\|\eta_1-\eta_2\|+\frac{\lambda^*_{z_1}b^{\delta+1}+\lambda^*_{z_2}b^{\delta+1}}{(\delta+1)\Gamma(\delta)}\|\eta_1-\eta_2\|\\
&\quad=\left[\lambda^*_{\vartheta_1}+\left(\frac{\|\varphi\|_\infty+\lambda^*_{\vartheta_2}}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right)b^\delta\right]\|\eta_1-\eta_2\|=\epsilon\|\eta_1-\eta_2\|.
\end{aligned}
$$

As $\epsilon=\left[\lambda^*_{\vartheta_1}+\left(\frac{\|\varphi\|_\infty+\lambda^*_{\vartheta_2}}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right)b^\delta\right]<1$, we get $\|T\eta_1-T\eta_2\|\leqslant\epsilon\|\eta_1-\eta_2\|$. This establishes that T qualifies as a contraction mapping. Consequently, in accordance with Theorem 2.7, $\exists$ a fixed point denoted as $\eta\in C(E,\mathbb{R})$ such that $T\eta=\eta$. This fixed point represents the only solution to the IVP outlined in equations (4.1) through (4.2). This concludes the proof of the theorem. □

To set the groundwork for our investigation, we now present the following theorem that addresses the existence of solutions in the context of Volterra-Fredholm IDE, as described in equations (4.1)-(4.2).

Theorem 4.3. *Assume that*

(B4) $\|\vartheta_1(\zeta,\eta)\|\leqslant\omega(\zeta),\forall(\zeta,\eta)\in E\times\chi$, $\|\vartheta_2(\zeta,\eta)\|\leqslant\mu(\zeta),\forall(\zeta,\eta)\in E\times\chi$, $\|Z_1(\zeta,\aleph,\eta)\|\leqslant\sigma(\zeta),\forall(\zeta,\aleph,\eta)\in E\times E\times\chi$, *and* $\|Z_2(\zeta,\aleph,\eta)\|\leqslant\rho(\zeta),\forall(\zeta,\aleph,\eta)\in E\times E\times\chi$, *where* ω, μ, σ, *and* $\rho\in C(E,\mathbb{R}^+)$.

Let

$$P:=\lambda^*_{\vartheta_1}+\frac{\lambda^*_{\vartheta_2}b^\delta}{\Gamma(\delta+1)}+\frac{\lambda^*_{z_1}b^{\delta+1}+\lambda^*_{z_2}b^{\delta+1}}{(\delta+1)\Gamma(\delta)}<1.$$

This implies that there is at least one solution to the problem described in (4.1)-(4.2) *over the interval* $[\zeta_0,b]$.

Proof. Choose a fixed $r \geqslant \left[\|\eta_0\| + \|\vartheta_1(0,\eta(0))\| + \|\omega\|_C + \frac{b^\delta\|\varphi\|_\infty\|\eta\|_\infty}{\Gamma(\delta+1)} + \frac{b^\delta\|\mu\|_C}{\Gamma(\delta+1)} + \frac{b^{\delta+1}\|\sigma\|_C}{(\delta+1)\Gamma(\delta)} + \frac{b^{\delta+1}\|\rho\|_C}{(\delta+1)\Gamma(\delta)}\right]$ and define the set $\Phi_\gamma = \{\eta \in \mathsf{C}; \|\eta\|\infty \leqslant \gamma\}$. Within this context, we introduce the operators Θ and Ψ on $\Phi\gamma$ as follows:

$$\begin{aligned}(\Theta\eta)(\zeta) &= -\vartheta_1(\zeta_0,\eta_0) + \vartheta_1(\zeta,\eta(\zeta)) + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta(\aleph))d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta} Z_1(\tau,\aleph,\eta(\aleph))d\tau d\aleph + \int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b} Z_2(\tau,\aleph,\eta(\aleph))d\tau d\aleph, \\ (\Psi\eta)(\zeta) &= \eta_0 + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph.\end{aligned}$$

When considering η_1 and η_2 from the set Φ_γ, we observe that:

$$\begin{aligned}\|\Theta\eta_1 + \Psi\eta_2\| &\leqslant \Big\|\eta_0 - \vartheta_1(\zeta_0,\eta_0) + \vartheta_1(\zeta,\eta(\zeta)) + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta(\aleph))d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta} Z_1(\tau,\aleph,\eta(\aleph))d\tau + \int_{\aleph}^{b} Z_2(\tau,\aleph,\eta(\aleph))d\tau\right)d\aleph\Big\| \\ &\leqslant \|\eta_0\| + \|\vartheta_1(\zeta_0,\eta_0)\| + \|\omega\|_C + \frac{b^\delta\|\varphi\|_\infty\|\eta\|_\infty}{\Gamma(\delta+1)} + \frac{b^\delta\|\mu\|_C}{\Gamma(\delta+1)} + \frac{b^{\delta+1}\|\sigma\|_C}{(\delta+1)\Gamma(\delta)} + \frac{b^{\delta+1}\|\rho\|_C}{(\delta+1)\Gamma(\delta)} \leqslant r.\end{aligned}$$

Next, we establish that $(\Theta\eta)$ exhibits contraction properties.

$$\begin{aligned}\|\Theta\eta_1 - \Theta\eta_2\| &\leqslant \|\vartheta_1(\zeta,\eta_1(\zeta)) - \vartheta_1(\zeta,\eta_2(\zeta))\| + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\|\vartheta_2(\aleph,\eta_1(\aleph)) - \vartheta_2(\aleph,\eta_2(\aleph))\|d\aleph \\ &\quad + \frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\left(\int_{\aleph}^{\zeta}\|Z_1(\tau,\aleph,\eta_1(\aleph)) - Z_1(\tau,\aleph,\eta_2(\aleph))\|d\tau \right. \\ &\quad \left. + \int_{\aleph}^{b}\|Z_2(\tau,\aleph,\eta_1(\aleph)) - Z_2(\tau,\aleph,\eta_2(\aleph))\|d\tau\right)d\aleph \\ &\leqslant \lambda^*_{\vartheta_1}\|\eta_1-\eta_2\| + \frac{\lambda^*_{\vartheta_2}b^\delta}{\Gamma(\delta+1)}\|\eta_1-\eta_2\| + \frac{\lambda^*_{z_1}b^{\delta+1} + \lambda^*_{z_2}b^{\delta+1}}{(\delta+1)\Gamma(\delta)}\|\eta_1-\eta_2\| \\ &\leqslant \left(\lambda^*_{\vartheta_1} + \frac{\lambda^*_{\vartheta_2}b^\delta}{\Gamma(\delta+1)} + \frac{\lambda^*_{z_1}b^{\delta+1} + \lambda^*_{z_2}b^{\delta+1}}{(\delta+1)\Gamma(\delta)}\right)\|\eta_1-\eta_2\| \leqslant P\|\eta_1-\eta_2\|.\end{aligned}$$

Therefore, Θ is a contraction. The continuity of φ implies that the operator Ψ is also continuous. Furthermore, Ψ is uniformly bounded on Φ_γ as

$$\|(\Psi\eta)(\zeta)\| \leqslant \|\eta_0\| + \|\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta(\aleph)d\aleph\| \leqslant \|\eta_0\| + \frac{b^\delta\|\varphi\|_\infty\gamma}{\Gamma(\delta+1)}.$$

In order to establish the compactness of the operator Ψ, it is imperative to illustrate its property of equicontinuity. To do so, let's define $\bar{\varphi}$ as the supremum of $|\varphi(\aleph)\eta(\aleph)|$. Now, for any pair of points ζ_1 and ζ_2 within the interval $[\zeta_0, b]$, where $\zeta_1 > \zeta_2$, and for any function η belonging to the class Φ_γ, we can observe the following:

$$\|(\Psi\eta)(\zeta_1) - (\Psi\eta)(\zeta_2)\|$$

$$\leqslant \left\| \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta_1} (\zeta_1 - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta_2} (\zeta_2 - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph \right\|$$
$$\leqslant \left\| \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta_2} (\zeta_2 - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph + \frac{1}{\Gamma(\delta)} \int_{\zeta_2}^{\zeta_1} (\zeta_1 - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph \right.$$
$$\left. - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta_2} (\zeta_2 - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph \right\|$$
$$\leqslant \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta_2} \left\| \left[(\zeta_2 - \aleph)^{\delta-1} - (\zeta_1 - \aleph)^{\delta-1} \right] \varphi(\aleph)\eta(\aleph) d\aleph \right\| + \frac{1}{\Gamma(\delta)} \int_{\zeta_2}^{\zeta_1} (\zeta_1 - \aleph)^{\delta-1} \| \varphi(\aleph)\eta(\aleph) d\aleph \| d\aleph$$
$$\leqslant \left[\frac{{\zeta_2}^{\delta}}{\Gamma(\delta+1)} - \frac{{\zeta_1}^{\delta}}{\Gamma(\delta+1)} \right] \|\varphi\|_\infty \|\eta\|_\infty + \frac{2(\zeta_1 - \zeta_2)^{\delta}}{\Gamma(\delta+1)} \|\varphi\|_\infty \|\eta\|_\infty$$
$$\leqslant \frac{\bar{\varphi}}{\Gamma(\delta+1)} |2(\zeta_1 - \zeta_2)^{\delta} + {\zeta_2}^{\delta} - {\zeta_1}^{\delta}| \leqslant \frac{\bar{\varphi}}{\Gamma(\delta+1)} |\zeta_1 - \zeta_2|^{\delta} \to 0 \text{ as } \zeta_1 \to \zeta_2.$$

Thus, Ψ is equicontinuous. By Arzela-Ascoli Theorem, Ψ is compact. All the conditions outlined in Theorem 2.9 are satisfied. Therefore, the conclusion of Theorem 2.9 applies, indicating that the problem (4.1)-(4.2) has at least one solution. This concludes the proof of the theorem. □

4.2. Ulam stability results

In this subsection, we will investigate the Ulam stability of the problem (4.1)-(4.2). Let's examine the following inequality:

$$\left| {}^{c}D^{\delta} \left[\eta(\zeta) - \vartheta_1(\zeta, \eta(\zeta)) \right] - \varphi(\zeta)\eta(\zeta) - \vartheta_2(\zeta, \eta(\zeta)) - \int_{\zeta_0}^{\zeta} Z_1(\zeta, \aleph, \eta(\aleph)) d\aleph - \int_{\zeta_0}^{b} Z_2(\zeta, \aleph, \eta(\aleph)) d\aleph \right| \leqslant \varepsilon. \quad (4.4)$$

To establish the foundation of our investigation, we now present the following theorem that addresses the Ulam stability results in the context of the Volterra-Fredholm IDE (4.1)-(4.2).

Theorem 4.4. *Given that* (B1) *and* (B2) *hold true, problem* (4.1)-(4.2) *exhibits Ulam-Hyers stability when* $\epsilon < 1$.

Proof. Consider $\varepsilon > 0$, and let $\nu \in C(E, \mathbb{R})$ satisfy inequality (4.4). Also, let $\tau \in C(E, \mathbb{R})$ be the unique solution to the following problem. In this context, we recall that

$$\eta(\zeta) = \eta_0 - \vartheta_1(\zeta_0, \eta_0) + \vartheta_1(\zeta, \eta(\zeta)) + \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph + \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \vartheta_2(\aleph, \eta(\aleph)) d\aleph$$
$$+ \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \left(\int_{\aleph}^{\zeta} Z_1(\tau, \aleph, \eta(\aleph)) d\tau + \int_{\aleph}^{b} Z_2(\tau, \aleph, \eta(\aleph)) d\tau \right) d\aleph.$$

By integrating inequality (4.4) and incorporating the initial condition of problem (4.2), we obtain:

$$\left| \eta(\zeta) - \eta_0 + \vartheta_1(\zeta_0, \eta_0) - \vartheta_1(\zeta, \eta(\zeta)) - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \varphi(\aleph)\eta(\aleph) d\aleph - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \vartheta_2(\aleph, \eta(\aleph)) d\aleph \right.$$
$$\left. - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \int_{\aleph}^{\zeta} Z_1(\tau, \aleph, \eta(\aleph)) d\tau d\aleph - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \int_{\aleph}^{b} Z_2(\tau, \aleph, \eta(\aleph)) d\tau d\aleph \right| \leqslant \varepsilon \frac{b^{\delta}}{\Gamma(\delta+1)}.$$

Additionally, let's examine this

$$|\eta_1(\zeta) - \eta_2(\zeta)|$$
$$\leqslant \left| \eta_1(\zeta) - \eta_0 + \vartheta_1(\zeta_0, \eta_0) - \vartheta_1(\zeta, \eta_2(\zeta)) - \frac{1}{\Gamma(\delta)} \int_{\zeta_0}^{\zeta} (\zeta - \aleph)^{\delta-1} \varphi(\aleph)\eta_2(\aleph) d\aleph \right.$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta_2(\aleph))d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta_2(\aleph))d\tau d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta_2(\aleph))d\tau d\aleph\Big|\\
&\leqslant\Big|\eta_1(\zeta)-\eta_0+\vartheta_1(\zeta_0,\eta_0)-\vartheta_1(\zeta,\eta_1(\zeta))+\vartheta_1(\zeta,\eta_1(\zeta))-\vartheta_1(\zeta,\eta_2(\zeta))\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_2(\aleph)d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta_1(\aleph))d\aleph\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta_1(\aleph))d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta_2(\aleph))d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta_2(\aleph))d\tau d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta_2(\aleph))d\tau d\aleph\Big|\\
&\leqslant\Big|\eta_1(\zeta)-\eta_0+\vartheta_1(\zeta_0,\eta_0)-\vartheta_1(\zeta,\eta_1(\zeta))-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\varphi(\aleph)\eta_1(\aleph)d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\vartheta_2(\aleph,\eta_1(\aleph))d\aleph-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}Z_1(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph\\
&\quad-\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}Z_2(\tau,\aleph,\eta_1(\aleph))d\tau d\aleph\Big|+|\vartheta_1(\zeta,\eta_1(\zeta))-\vartheta_2(\zeta,\eta_2(\zeta))|\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\varphi(\aleph)|\,|\eta_1(\aleph)-\eta_2(\aleph)|d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}|\vartheta_2(\aleph,\eta_1(\aleph))\\
&\quad-\vartheta_2(\aleph,\eta_2(\aleph))|d\aleph+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{\zeta}|Z_1(\tau,\aleph,\eta_1(\aleph))-Z_1(\tau,\aleph,\eta_2(\aleph))|d\tau d\aleph\\
&\quad+\frac{1}{\Gamma(\delta)}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\zeta_0}^{\zeta}(\zeta-\aleph)^{\delta-1}\int_{\aleph}^{b}|Z_2(\tau,\aleph,\eta_1(\aleph))-Z_2(\tau,\aleph,\eta_2(\aleph))|d\tau d\aleph,
\end{aligned}
$$

$$
\|\eta_1-\eta_2\|\leqslant\frac{\varepsilon}{\Gamma(\delta+1)}+\left[\lambda^*_{\vartheta_1}+\left(\frac{\|\varphi\|_\infty+\lambda^*_{\vartheta_2}}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right)b^\delta\right]\|\eta_1-\eta_2\|,
$$

$$
\|\eta_1-\eta_2\|\leqslant\frac{\varepsilon}{\Gamma(\delta+1)}+\epsilon\|\eta_1-\eta_2\|,
$$

$$
\|\eta_1-\eta_2\|\leqslant\frac{\varepsilon}{\Gamma(\delta+1)(1-\epsilon)}=\varepsilon M_1;\ \text{ where, } M_1=\frac{1}{\Gamma(\delta+1)(1-\epsilon)}.
$$

Hence, we can conclude that the problem (4.1)-(4.2) exhibits Ulam-Hyers stability. This concludes the proof of the theorem. □

Example 4.5. We investigate the CF Volterra-Fredholm IDE (4.1)-(4.2) under the following parameters: $\delta=0.5$, $b=0.5$, $\lambda^*_{\vartheta_1}=0.2$, $\lambda^*_{\vartheta_2}=0.4$, $\lambda^*_{Z_1}=0.3$, $\lambda^*_{Z_2}=0.3$, and $\|\varphi\|_\infty=0.3$. It follows that

$$
\epsilon=\left[\lambda^*_{\vartheta_1}+\left(\frac{\|\varphi\|_\infty+\lambda^*_{\vartheta_2}}{\Gamma(\delta+1)}+\frac{(\lambda^*_{z_1}+\lambda^*_{z_2})b}{(\delta+1)\Gamma(\delta)}\right)b^\delta\right]
$$

$$= \left[0.2 + \frac{0.3+0.4}{\Gamma(\frac{1}{2}+1)} + \frac{(0.3+0.3)(0.5)}{(\frac{1}{2}+1)\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}}$$
$$= \left[0.2 + \frac{0.7}{\Gamma(\frac{3}{2})} + \frac{(0.6)(0.5)}{(\frac{3}{2})\Gamma(\frac{1}{2})}\right](0.5)^{\frac{1}{2}} = \left[0.2 + \frac{0.7}{0.886} + \frac{0.30}{(1.5)(1.772)}\right](0.707) = 0.7791 < 1.$$

Since all the conditions of Theorem 4.2 are satisfied, $\exists$ a unique and stable solution to the provided equation.

5. Conclusion

In this paper, a comprehensive investigation into Volterra-Fredholm IDEs with CF derivatives has been presented, emphasizing their significance in modeling real-world phenomena with fractional-order behaviors. Employing the principles of the Banach and Krasnoselskii fixed-point theorems, the existence and uniqueness of solutions have been firmly established, enhancing the robustness of the findings. The exploration of Ulam stability within the context of fractional calculus has illuminated the long-term behavior of fractional-order systems, a pivotal consideration for understanding the dynamics of complex phenomena. Additionally, the extension to Volterra-Fredholm neutral IDEs has broadened the relevance of the research to a wide range of systems and practical challenges, further demonstrated by a practical illustrative example. This work offers a valuable contribution to the field, advancing both the theoretical foundations and practical tools for modeling and analyzing complex systems in the domain of fractional calculus and integro-differential equations. It is anticipated that these insights will find utility among researchers and practitioners, further stimulating exploration in this evolving field.

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