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Hyers-Ulam stability and continuous dependence of the solution of a nonlocal stochastic-integral problem of an arbitrary (fractional) orders stochastic differential equation

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Abstract

Stochastic problems play a huge role in many applications including biology, chemistry, physics, economics, finance, mechanics, and several areas. In this paper, we are concerned with the nonlocal stochastic-integral problem of the arbitrary (fractional) orders stochastic differential equation

$$\frac{dX(t)}{dt} = f_1(t, D^{\alpha}X(t)) + f_2(t, B(t)), \quad t \in (0, T], \qquad X(0) = X_0 + \int_0^T f_3(s, D^{\beta}X(s))dW(s),$$

where B is any Brownian motion, W is a standard Brownian motion, and X_0 is a second order random variable. The Hyers - Ulam stability of the problem will be studied. The existence of solution and its continuous dependence on the Brownian motion B will be proved. The three spatial cases Brownian bridge process, the Brownian motion with drift and the Brownian motion started at A will be considered.

Keywords: Stochastic processes, stochastic differential equations, existence of solutions, continuous dependence, Brownian motion, Brownian bridge process, Brownian motion with drift .

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1. Introduction

Over the years, fractional differential equations and its applications have gotten extensive attention, it is widely used in various disciplines, interested researchers can see the work in [4, 9, 12, 17, 31, 34]. Many authors have been interested to study fractional stochastic differential equations (see [1, 2, 5, 8, 10, 11, 13, 15, 19, 21, 27, 28]). The existence and uniqueness of solutions to stochastic differential equations have been studied by many authors see [14, 16, 18, 25]. In [24], the author discussed a computational method

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to get an approximate solution of a stochastic beam equation. A simulation analysis of this problem is carried out with matlab, author constructed the stochastic partial differential equation

$$\alpha^2\frac{\partial^4 u(x,t)}{\partial x^4}+\frac{\partial^2 u(x,t)}{\partial t^2}+c\frac{\partial u(x,t)}{\partial t}-s\frac{\partial^2 u(x,t)}{\partial x^2}=G_1(x,t)+G_2(x,t)\dot{B}(t)$$

subject to some conditions. He referred that by employing the Hilbert space of all square-integrable functions, the problem is reduced to a first order of the form

$$dx_t=f(t,x_t)+g(t,x_t)dB_t.$$

So, interested researchers with numerical methods of stochastic problems (see [6, 24, 33]). Let (Ω, G, μ) be a probability space where Ω is a sample space, G is a σ-algebra of subsets of Ω and μ is the probability measure (see [7, 32, 36]). Let $I=[0,T]$ and $X(t;w)=\{X(t), t\in I, w\in\Omega\}$ be a second order stochastic process,

$$E(X^2(t))<\infty, t\in I.$$

Let $C=C(I, L_2(\Omega))$ be the class of all mean square second order continuous stochastic processes on I with the norm

$$\| X \|_C=\sup_{t\in I}\| X(t)\|_2,\quad \| X(t)\|_2=\sqrt{E(X^2(t))}.$$

The motivation of this work is to generalize the results of [14]. The authors, in [14], studied the stochastic differential equation

$$\frac{dX}{dt}=f(t,X(t))+W(t),\quad t\in(0,T]$$

with the nonlocal random initial condition

$$X(0)+\sum_{k=1}^{n}a_kX(\tau_k)=X_0,\quad \tau_k\in(0,T),$$

where X_0 is a second order random variable, $W(t)$ is the standard Brownian motion and a_k are positive real numbers. Let $B(t), t\in[0,T]$ be any Brownian motion, $W(t)$ is a standard brownian motion and $\alpha,\beta\in(0,1], \beta\leqslant\alpha$. Here, we are concerned with the nonlocal stochastic-integral problem of the arbitrary (fractional) orders stochastic differential equation

$$\frac{dX(t)}{dt}=f_1(t,D^{\alpha}X(t))+f_2(t,B(t)),\quad t\in(0,T] \tag{1.1}$$

with the stochastic-integral condition

$$X(0)=X_0+\int_0^T f_3(s,D^{\beta}X(s))dW(s), \tag{1.2}$$

where X_0 is a second order random variable. The existence of solutions $X\in C$ of the problem (1.1)-(1.2) will be proved. The sufficient condition of the uniqueness of the solution will be given. The Hyers-Ulam stability of the problem (1.1)-(1.2) will be proved. The continuous dependence of the unique solution on the Brownian motion B and its three spatial cases, Brownian bridge process, the Brownian motion with drift and the Brownian motion started at A, will be studied.

2. Preliminaries

Here, we offer some fundamental definitions.

Definition 2.1. Let $X \in C(I, L_2(\Omega))$ and $\alpha, \beta \in (0, 1]$. The stochastic integral operator of order β is defined by

$$I^{\beta} X(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) ds$$

and the stochastic fractional order derivative is defined by

$$D^{\alpha} X(t) = I^{1-\alpha} \frac{dX}{dt}.$$

For properties of stochastic fractional calculus see [11, 20].

Definition 2.2 (Brownian motion with drift, [25, 30])**.** A Brownian motion B is called a Brownian motion with drift μ and volatility σ if it can be written as

$$B(t) = \mu t + \sigma W(t), \quad t \in R_+,$$

where $W(t)$ is a standard Brownian motion.

Definition 2.3 (Brownian motion started at A, [26])**.** A process $B(t)$ is called a Brownian motion started at $A, A \in L_2(\Omega)$ if it can be written as

$$B(t) = A + W(t),$$

where $W(t)$ is a standard Brownian motion.

Definition 2.4 (Brownian bridge, [29])**.** A Brownian motion B is called a Brownian bridge if it can be written as

$$B(t) = a(1-t) + bt + (1-t) \int_0^t \frac{dW(s)}{1-s}, \quad t \in [0, 1), \quad a, b \in R,$$

where $W(t)$ is a standard Brownian motion.

3. Solution of the problem

Throughout the paper we assume that the following assumptions hold.

i- The functions $f_i : I \times L_2(\Omega) \to L_2(\Omega), \; i = 1, 2, 3$ are measurable in $t \in I, \; \forall x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega), \; \forall t \in I$.

ii- There exists a constant $b > 0$, and a second order process $a(t) \in L_2(\Omega), \; a = \sup_{t \in I} \|a(t)\|_2$, such that

$$\| f_i(t, x(t)) \|_2 \leqslant a + b \| x(t) \|_2, \quad i = 1, 2, 3.$$

iii- $bT^{1-\alpha} < \Gamma(2-\alpha)$.

Now, we have the following lemma concerning the integral representation of the solution of the problem (1.1)-(1.2).

Lemma 3.1. *Let the solution of the initial value problem* (1.1)-(1.2) *be exists. Then it can be represented as*

$$X(t) = X_0 + \int_0^T f_3(s, I^{\alpha-\beta} U(s)) dW(s) + I^{\alpha} U(t), \quad t \in [0, T], \tag{3.1}$$

where $U(t)$ *is given by*

$$U(t) = I^{1-\alpha} \left[f_1(t, U(t)) + f_2(t, B(t)) \right]. \tag{3.2}$$

Proof. Let $X(t)$ be a solution of (1.1). Operating by $I^{1-\alpha}$ on equations (1.1), we obtain

$$D^{\alpha}X(t) = I^{1-\alpha}\frac{dX(t)}{dt} = I^{1-\alpha}\left[f_1(t, D^{\alpha}X(t)) + f_2(t, B(t))\right].$$

Let

$$D^{\alpha}X(t) = U(t) \in C([0,T], L_2(\Omega)),$$

then

$$X(t) = X(0) + I^{\alpha}U(t) = X_0 + \int_0^T f_3(s, D^{\beta}X(s))dW(s) + I^{\alpha}U(t).$$

But

$$D^{\beta}X(t) = I^{1-\beta}\frac{d}{dt}X(t) = I^{\alpha-\beta}I^{1-\alpha}\frac{d}{dt}X(t) = I^{\alpha-\beta}U(t).$$

Then we obtain (3.1),

$$X(t) = X_0 + \int_0^T f_3(s, I^{\alpha-\beta}U(s))dW(s) + I^{\alpha}U(t), \quad t \in [0,T],$$

and the fractional-order integral equation

$$U(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s))ds. \tag{3.3}$$

Conversely, let $U(t)$ be a solution of (3.3). Then from (3.1) and (3.2) we obtain

$$\begin{aligned} X(t) &= X_0 + \int_0^T f_3(s, I^{\alpha-\beta}U(s))dW(s) + I^{\alpha}I^{1-\alpha}[f_1(t, U(t)) + f_2(t, B(t))] \\ &= X_0 + \int_0^T f_3(s, I^{\alpha-\beta}U(s))dW(s) + \int_0^t [f_1(s, D^{\alpha}X(s)) + f_2(s, B(s))]ds, \end{aligned}$$

$$\frac{d}{dt}X(t) = f_1(t, D^{\alpha}X(t)) + f_2(t, B(t)),$$

and

$$X(0) = X_0 + \int_0^T f_3(s, D^{\beta}X(s))dW(s).$$

Then we have proved the equivalence between the problem (1.1)-(1.2) and the equations (3.1) and (3.3). □

4. Existence of solution

Theorem 4.1. *Let the assumptions* (i)-(iii) *be satisfied, then the fractional-order integral equation* (3.3) *has at least one solution* $U(t) \in C$.

Proof. Consider the set Q such that

$$Q = \{U \in C : \|U\|_C \leqslant r\} \subset C.$$

Define the mapping $FU(t)$ where

$$FU(t) = I^{1-\alpha}\left[f_1(t, U(t)) + f_2(t, B(t))\right].$$

Let $U \in Q$, then

$$\|FU\|_2 \leqslant \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel f_1(s, U(s)) \parallel_2 ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel f_2(s, B(s)) \parallel_2 ds$$

$$\leqslant \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} [\|a(s)\|_2 + b \parallel U(s) \parallel_2] \, ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} [\|a(s)\|_2 + b \parallel B(s) \parallel_2] \, ds$$

$$\leqslant [2a + b \parallel U \parallel_C] \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel B \parallel_C \, ds$$

$$\leqslant [2a + b \parallel U \parallel_C + b \parallel B \parallel_C] \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} = r,$$

where

$$r = [2a + b \parallel U \parallel_C + b \parallel B \parallel_C] \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \leqslant [2a + br + b \parallel B \parallel_C] \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Thus

$$r \leqslant \frac{[2a + b \parallel B \parallel_C] \, T^{1-\alpha}}{\Gamma(2-\alpha) - [bT^{1-\alpha}]}.$$

That proves $F : Q \to Q$ and the class $\{FQ\}$ is uniformly bounded on Q. Now, considering $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
\|FU(t_2) - FU(t_1)\|_2 &\leqslant \| \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s)) ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s)) ds \|_2 \\
&\quad + \| \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds \|_2 \\
&\leqslant \| \int_0^{t_1} \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s)) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s)) ds \|_2 \\
&\quad + \| \int_0^{t_1} \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds \|_2.
\end{aligned}$$

Then

$$\begin{aligned}
\|FU(t_2) - FU(t_1)\|_2 &\leqslant [2a + br] [\int_0^{t_1} | \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} | ds + \int_{t_1}^{t_2} | \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} | ds] \\
&\quad + b\|B\|_C [\int_0^{t_1} | \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} | ds + \int_{t_1}^{t_2} | \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} | ds] \\
&= [2a + br] [\int_0^{t_1} \frac{(t_2-s)^{\alpha} - (t_1-s)^{\alpha}}{(t_2-s)^{\alpha}(t_1-s)^{\alpha}\Gamma(1-\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} ds] \\
&\quad + b\|B\|_C [\int_0^{t_1} \frac{(t_2-s)^{\alpha} - (t_1-s)^{\alpha}}{(t_2-s)^{\alpha}(t_1-s)^{\alpha}\Gamma(1-\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} ds].
\end{aligned}$$

This proves the equi-continuity of the class $\{FQ\}$ on Q. Now, let $U_n \in Q$, $U_n \to U$ *w.p.1* (see [7]).

$$\begin{aligned}
\underset{n\to\infty}{\text{l.i.m}} \, FU_n &= \underset{n\to\infty}{\text{l.i.m}} \left[\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U_n(s)) ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds \right] \\
&= \underset{n\to\infty}{\text{l.i.m}} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U_n(s)) ds + \underset{n\to\infty}{\text{l.i.m}} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s)) ds
\end{aligned}$$

$$= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, \underset{n\to\infty}{\text{l.i.m}}\, U_n(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s))ds$$

$$= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_1(s, U(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, B(s))ds = FU.$$

This proves that $\{FU\}$ is continuous. Consequently, the closure of $\{FU\}$ is compact (see [7]). Thus, equation (3.3) has a solution $U \in C$. □

Now for the problem (1.1)-(1.2), we have the following theorem.

Theorem 4.2. *Let the assumptions* (i)-(iii) *be satisfied, then the problem* (1.1)-(1.2) *has at least one solution* $X \in C$ *given by* (3.1)*.*

Proof. From Lemma 3.1, the solution of the problem (1.1)-(1.2) is given by (3.1),

$$X(t) = X_0 + \int_0^T f_3(s, I^{\alpha-\beta}U(s))dW(s) + I^{\alpha}U(t), \quad t \in [0, T],$$

where U is given by (3.3). Now, let U be a solution of (3.3), then we have

$$\begin{aligned}
\|X(t)\|_2 &\leqslant \|X_0\|_2 + \sqrt{\int_0^T \|f_3(s, I^{\alpha-\beta}U(s)))\|_2^2 ds} + I^{\alpha}\|U(t)\|_2 \\
&\leqslant \|X_0\|_2 + \sqrt{\int_0^T \left(a + b\|I^{\alpha-\beta}U(s)\|_2\right)^2 ds} + I^{\alpha}\|U(t)\|_2 \\
&\leqslant \|X_0\|_2 + \sqrt{\int_0^T \left(a + b\|U\|_C I^{\alpha-\beta}(1)\right)^2 ds} + \|U\|_C I^{\alpha}(1) \\
&\leqslant \|X_0\|_2 + \sqrt{\int_0^T \left(a + b\|U\|_C \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)^2 ds} + \|U\|_C \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&\leqslant \|X_0\|_2 + \sqrt{\int_0^T \left(a + b\|U\|_C \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)^2 ds} + \|U\|_c \frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
&\leqslant \|X_0\|_2 + \left(a + b\|U\|_C \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\sqrt{T} + \|U\|_C \frac{T^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Then

$$\|X\|_C \leqslant \|X_0\|_2 + \left(a + br \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\sqrt{T} + r \frac{T^{\alpha}}{\Gamma(\alpha+1)}.$$

So, the solution X of the problem (1.1)-(1.2) exists and $X \in C([0, T], L_2(\Omega))$. □

4.1. Uniqueness theorem

For discussing the uniqueness of the solution $U \in C([0, T], L_2(\Omega))$ of fractional order integral equation (3.3), consider the following assumption.

iv- The functions $f_i : I \times L_2(\Omega) \to L_2(\Omega)$, $i = 1, 3$ are measurable in $t \in I$, $\forall x \in L_2(\Omega)$ and satisfy the Lipschitz condition

$$\| f_i(t, x(t)) - f_i(t, y(t) \|_2 \leqslant b \| x(t) - y(t) \|_2 \quad \text{and} \quad a(t) = f_i(t, 0), \quad i = 1, 3.$$

Theorem 4.3. *Let the assumptions* (ii)-(iv) *be satisfied, then the integral equation* (3.3) *has a unique solution* $U \in C$ *and consequently, the problem* (1.1)-(1.2) *has a unique solution* $X \in C$.

Proof: From assumption (iv) we can deduce that

$$\|f_i(t,X)\|_2 - \|f_i(t,0)\|_2 \leqslant \|f_i(t,X) - f_i(t,0)\|_2 \leqslant b\|x(t)\|_2$$

and

$$\|f_i(t,X)\|_2 \leqslant a + b\|X(t)\|_2.$$

Then the assumptions of Theorem 4.1 are satisfied and (3.3) has at least one solution. Let U_1 and U_2 be two solutions of (3.3), then

$$\begin{aligned}\|U_1(t) - U_2(t)\|_2 &\leqslant \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\|f_1(s,U_1(s)) - f_1(s,U_2(s))\|_2 ds\\ &\leqslant b\|U_1 - U_2\|_C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \leqslant b\|U_1 - U_2\|_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.\end{aligned}$$

Then

$$(1 - b\frac{T^{1-\alpha}}{\Gamma(2-\alpha)})\|U_1 - U_2\|_C \leqslant 0 \quad \Rightarrow \quad \|U_1 - U_2\|_C \leqslant 0$$

and this implies that

$$\|U_1 - U_2\|_C = 0 \quad \Rightarrow \quad U_1(t) = U_2(t).$$

Then the solution of fractional order integral equation (3.3) is unique. Let X_1, X_2 be two solutions of (3.1), then

$$X_1(t) - X_2(t) = \int_0^T [f_3(s, I^{\alpha-\beta}U_1(s)) - f_3(s, I^{\alpha-\beta}U_2(s))]dW(s) + I^\alpha(U_1(t) - U_2(t)),$$

then

$$\begin{aligned}\|X_1(t) - X_2(t)\|_2 &\leqslant \|\int_0^T [f_3(s, I^{\alpha-\beta}U_1(s)) - f_3(s, I^{\alpha-\beta}U_2(s))]dW(s)\|_2 + I^\alpha\|U_1(t) - U_2(t)\|_2\\ &\leqslant \sqrt{\int_0^T \|f_3(s, I^{\alpha-\beta}U_1(s)) - f_3(s, I^{\alpha-\beta}U_2(s))\|_2^2 ds} + I^\alpha\|U_1(t) - U_2(t)\|_2\\ &\leqslant b\sqrt{\int_0^T (I^{\alpha-\beta}\|U_1(s) - U_2(s))\|_2)^2 ds} + I^\alpha\|U_1(t) - U_2(t)\|_2.\end{aligned}$$

So

$$\|X_1 - X_2\|_C \leqslant \sqrt{T}b\frac{T^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}\|U_1 - U_2\|_C + \frac{T^\alpha}{\Gamma(1+\alpha)}\|U_1 - U_2\|_C.$$

Hence from the uniqueness of U, we obtain

$$\|X_1 - X_2\|_C = 0.$$

Consequently, the solution (3.1) of the initial value problem (1.1)-(1.2),

$$X(t) = X_0 + \int_0^T f_3(s, I^{\alpha-\beta}U(s))dW(s) + I^\alpha U(t) \in C(I, L_2(\Omega)),$$

is unique one.

5. Continuous dependence on the Brownian motions

Definition 5.1. The solution $X \in C$ of the problem (1.1)-(1.2) depends continuously on the Brownian motion B if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$||B(t) - B^*(t)||_2 \leqslant \delta \quad \Rightarrow \quad ||X - X^*||_C \leqslant \epsilon,$$

where X^* is the solution of

$$X(t) = X_0 + \int_0^T f_3(s, I^{\alpha-\beta} U^*(s)) dW(s) + I^\alpha U^*(t), \quad U^*(t) = I^{1-\alpha}[f_1(t, U^*(t)) + f_2(t, B^*(t))].$$

Consider now the following theorem.

Theorem 5.2. *The unique solution of the problem* (1.1)-(1.2) *depends continuously on* $B(t)$*.*

Proof. First of all we have

$$\begin{aligned} ||U(t) - U^*(t)||_2 &\leqslant \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||f_1(s, U(s)) - f_1(s, U^*(s))||_2 ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \| f_2(s, B(s)) - f_2(s, B^*(s))||_2 ds \\ &\leqslant b||U - U^*||_C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||B(s) - B^*(s)||_2 ds. \end{aligned}$$

Thus, we get

$$\begin{aligned} ||U(t) - U^*(t)||_2 &\leqslant b||U - U^*||_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||B(t) - B^*(t)||_2 ds \\ &\leqslant bT^*||U - U^*||_C + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \delta ds = bT^*||U - U^*||_C + bT^*\delta, \end{aligned}$$

then

$$(1 - bT^*)||U - U^*||_C \leqslant bT^*\delta$$

and

$$||U - U^*||_C \leqslant \frac{bT^*\delta}{(1 - bT^*)} = \epsilon_1.$$

Now

$$\begin{aligned} ||X(t) - X^*(t)||_2 &\leqslant || \int_0^T [f_3(s, I^{\alpha-\beta} U(s)) - f_3(s, I^{\alpha-\beta} U^*(s))] dW(s)||_2 + I^\alpha ||U(t) - U^*(t)||_2 \\ &\leqslant \sqrt{\int_0^T ||f_3(s, I^{\alpha-\beta} U^*(s)) - f_3(s, I^{\alpha-\beta} U^*(s))||_2^2 ds} + I^\alpha ||U^*(t) - U^*(t)||_2 \\ &\leqslant b\sqrt{\int_0^T (I^{\alpha-\beta} ||U^*(s) - U^*(s)||_2)^2 ds} + I^\alpha ||U^*(t) - U^*(t)||_2. \end{aligned}$$

Then

$$\begin{aligned} ||X - X^*||_C &\leqslant b\sqrt{T} \frac{T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} ||U - U^*||_C + \frac{T^\alpha}{\Gamma(1+\alpha)} ||U - U^*||_C \\ &\leqslant \epsilon_1 (b\sqrt{T} \frac{T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}) \leqslant \epsilon \end{aligned}$$

and the result follows. □

5.1. Examples

(I) Let $B(t) = \mu t + \sigma W(t)$ be the Brownian motion with drift, $B^*(t) = \mu^* t + \sigma^* W(t)$ and W is a standard Brownian motion, then $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$\max\{|\mu - \mu^*|, |\sigma - \sigma^*|\} \leqslant \delta,$$

then

$$||B(t) - B^*(t)||_2 = t|\mu - \mu^*| + ||W(t)||_2 |\sigma - \sigma^*| \leqslant \delta(T + \sqrt{T}) = \delta.$$

Then our results in Theorems 4.1-4.3 and 5.2 can be applied for the Brownian motion with drift.

(II) Let W be a standard Brownian motion and

$$B(t) = a(1-t) + bt + (1-t)\int_0^t \frac{dW(s)}{1-s}, \quad t \in [0,1),$$

and

$$B^*(t) = a^*(1-t) + b^* t + (1-t)\int_0^t \frac{dW(s)}{1-s}, \quad t \in [0,T),$$

where

$$\max\{a - a^*, b - b^*\} \leqslant \delta.$$

So, we can get

$$||B - B^*||_2 = |(a - a^*)(1-t) + (b - b^*)t| \leqslant \delta|(1-t) + t| = \delta.$$

Then our results in Theorems 4.1-4.3 and 5.2 can be applied for the Brownian bridge.

(III) Finally, let W be a standard Brownian motion, A be a second order random variable $A \in L_2(\Omega)$ and

$$B(t) = A + W(t)$$

be the Brownian motion started at $A \in L_2(\Omega)$. Let

$$B^*(t) = A^* + W(t), \quad ||A - A^*||_2 \leqslant \delta,$$

then we can get

$$||B - B^*||_2 = ||A - A^*||_2 \leqslant \delta.$$

Then our results in Theorems 4.1-4.3 and 5.2 can be applied for the Brownian motion started at $A \in L_2(\Omega)$.

6. Hyers-Ulam stability

The functional equation

$$F_1(\phi(x)) = F_2(\phi(x))$$

is said to have the Hyers-Ulam stability if for an approximate solution ϕ_s such that

$$|F_1(\phi_s(x)) - F_2(\phi_s(x))| \leqslant \delta$$

for some fixed constant $\delta \geqslant 0$, there exists a solution ϕ such that

$$|\phi(x) - \phi_s(x)| \leqslant \epsilon$$

for some positive constant ϵ. Sometimes we call ϕ a δ-approximate solution (see [3, 22, 23]).

In this section, we have the following definition.

Definition 6.1. Problem (1.1)-(1.2) is said to be Hyers-Ulam stable if for an approximate (δ-approximate) solution $X_s \in C([0,T], L_2(\Omega))$ of (1.1)-(1.2) such that

$$\left|\left|\frac{d}{dt}X_s(t) - [f_1(t, D^\alpha X_s(t)) + f_2(t, B(t))]\right|\right|_2 \leqslant \delta$$

for some fixed constant $\delta > 0$, there exists a solution $X \in C([0,T], L_2(\Omega))$ of (1.1)-(1.2) such that

$$\| X - X_s \|_C < \epsilon$$

for some $\epsilon > 0$.

Now, we have the following theorem.

Theorem 6.2. *Let the assumptions of Theorem 4.1 be satisfied. Then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof. Firstly, from Lemma 3.1, we have

$$\begin{aligned}
&||U_s(t) - I^{1-\alpha}[f_1(t, U_s(t)) + f_2(t, B(t))]||_2 \\
&= ||I^{1-\alpha}\frac{d}{dt}X_s(t) - I^{1-\alpha}[f_1(t, D^\alpha X_s(t)) + f_2(t, B(t))]||_2 \\
&\leqslant I^{1-\alpha}||\frac{d}{dt}X_s(t) - [f_1(t, D^\alpha X_s(t)) + f_2(t, B(t))]||_2 \leqslant I^{1-\alpha}\delta \leqslant \frac{\delta t^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Now

$$\begin{aligned}
\| U(t) - U_s(t) \|_2 &= \left|\left|I^{1-\alpha}[f_1(t, U(t)) + f_2(t, B(t))] - I^{1-\alpha}[f_1(t, U_s(t)) + f_2(t, B(t))]\right|\right|_2 \\
&= \left|\left|I^{1-\alpha}[f_1(t, U(t)) + f_2(t, B(t))] - I^{1-\alpha}[f_1(t, U_s(t)) + f_2(t, B(t))]\right.\right. \\
&\quad \left.\left.+ I^{1-\alpha}[f_1(t, U_s(t)) + f_2(t, B(t))] - U_s(t)\right|\right|_2 \\
&\leqslant I^{1-\alpha}\, ||f_1(t, U(t)) - f_1(t, U_s(t))||_2 + \left|\left|I^{1-\alpha}[f_1(t, U_s(t)) + f_2(t, B(t))] - U_s(t)\right|\right|_2 \\
&\leqslant I^{1-\alpha}\, \| f_1(t, U(t)) - f_1(t, U_s(t)) \|_2 + \delta\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\leqslant b \parallel U - U_s \parallel_C \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\delta t^{1-\alpha}}{\Gamma(2-\alpha)} \leqslant bT^*||U - U_s||_C + \delta T^*, \quad T^* = \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Thus

$$\| U - U_s \|_C \leqslant \frac{\delta T^*}{(1 - bT^*)} = \epsilon_1$$

and

$$\begin{aligned}
||X(t) - X_s(t)||_2 &\leqslant b\sqrt{T}(\frac{T^{\alpha-\beta}}{\Gamma(1-\alpha+\beta)}||U - U_s||_C) + \frac{T^\alpha}{\Gamma(1+\alpha)}||U - U_s||_C \\
&\leqslant ||U - U_s||_C(b\sqrt{T}\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(1+\alpha)}||U - U_s||_C) \\
&\leqslant \epsilon_1(b\sqrt{T}\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}) = \epsilon.
\end{aligned}$$

Then we obtain our result

$$||X - X_s||_C \leqslant \epsilon.$$

□

7. Example

Consider

$$\frac{dX(t)}{dt} = \frac{[k(t) + D^{\frac{3}{4}}X(t)]}{9(1+ \parallel X(t) \parallel_2)} + \frac{B(t)\sin t}{(1+ \parallel B(t) \parallel_2)} \tag{7.1}$$

with the stochastic-integral condition

$$X(0) = X_0 + \int_0^1 \frac{e^{-s}D^{\frac{1}{2}}X(s)}{(36+s^2)} dW(s), \quad t \in (0, 1]. \tag{7.2}$$

The solution of the initial value problem (7.1)-(7.2) can be represent as

$$X(t) = X_0 + \int_0^T \frac{e^{-s}I^{\frac{1}{4}}U(s)}{(1+s^2)} dW(s) + I^{\frac{3}{4}}U(t), \quad t \in [0, T], \tag{7.3}$$

where $U(t)$ is given by

$$U(t) = I^{\frac{1}{4}} \left[\frac{[k(t) + U(t)]}{9(1+ \parallel X(t) \parallel_2)} + \frac{B(t)\sin t}{(1+ \parallel B(t) \parallel_2)} \right].$$

In the basic problem of this paper, let $f_1(s, D^{\frac{3}{4}}X(s)) = \frac{[k(t)+D^{\frac{3}{4}}X(t)]}{9(1+\|X(t)\|_2)}$, $f_2(s, B(s)) = \frac{B(t)\sin t}{6(1+\|B(t)\|_2)}$, and $f_3(s, D^{\frac{1}{2}}X(s)) = \frac{e^{-s}D^{\frac{1}{2}}X(s)}{(36+s^2)}$. Let also $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{2}$. Easily, the problem (7.1) with nonlocal integral condition (7.2) satisfies all the assumptions (i)-(iii) of Theorem 4.1, then there exists at least one solution to the problem (7.1)-(7.2) on [0, 1], given by (7.3). It also satisfies condition (iv), so using Theorem 4.3, there exists a unique solution.

8. Conclusions

In this paper, in Theorem 4.1, we proved the existence of solutions $x \in C([0, T], L_2(\Omega))$ of the nonlocal stochastic-integral problem of the arbitrary (fractional) orders stochastic differential equation

$$\frac{dX(t)}{dt} = f_1(t, D^{\alpha}X(t)) + f_2(t, B(t)), \quad t \in (0, T], \qquad X(0) = X_0 + \int_0^T f_3(s, D^{\beta}X(s))dW(s),$$

where B is any Brownian motion, W is a standard Brownian motion, and X_0 is a second order random variable. The sufficient condition for the uniqueness of the solution have been given in Theorem 4.3. The Hyers-Ulam stability of the problem have been proved in Theorem 6.2. The continuous dependence of the unique solution on the Brownian motion B is proved. The three spatial cases Brownian bridge process, the Brownian motion with drift and the Brownian motion started at A have been considered.

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