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Application of a matrix Mittag-Leffler function to the fractional partial integro-differential equation in $\mathbb{R}^n$

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Abstract

In this paper, we investigate the uniqueness of solutions to a new fractional partial integro-differential equation (abbreviated FPIDE) with a boundary condition by using a recently established matrix Mittag-Leffler function, Banach's contractive principle, and Babenko's approach. Furthermore, we supply an example that employs the results derived in the paper via a python code which computes an approximate value to the matrix Mittag-Leffler function.

Keywords: Banach's contractive principle, matrix Mittag-Leffler function, Babenko's approach, implicit integral equation.

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1. Background

Let $\tau \in \Omega = [0,1]^n \subset \mathbb{R}^n$ $(n \geqslant 1)$. The partial Riemann-Liouville fractional integral of order $\beta > 0$ with respect to τ_i is given by [3]

$$(I^{\beta}_{\tau_i}\chi)(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau_i} (\tau_i - \zeta)^{\beta-1} \chi(\tau_1, \dots, \tau_{i-1}, \zeta, \tau_{i+1}, \dots, \tau_n) d\zeta,$$

and the partial Liouville-Caputo fractional derivative of order $\alpha \in (0,1]$ with respect to τ_i is given by

$$\left(\frac{{}_c\partial^{\alpha}}{\partial \tau_i^{\alpha}} \chi \right)(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau_i} (\tau_i - s)^{-\alpha} \chi'_s(\tau_1, \dots, \tau_{i-1}, s, \tau_{i+1}, \dots, \tau_n) ds.$$

It follows from [8] that for $\alpha \in (0,1]$,

$$I^{\alpha}_{\tau_i} \left(\frac{{}_c\partial^{\alpha}}{\partial \tau_i^{\alpha}} \chi \right)(\tau) = \chi(\tau) - \chi(\tau_1, \dots, \tau_{i-1}, 0, \tau_{i+1}, \dots, \tau_n). \tag{1.1}$$

This paper will make use of the Banach space $C([0,1]^n)$ with the following norm:

$$\|\chi\| = \sup_{\tau \in [0,1]^n} |\chi(\tau)| \quad \text{for } \chi \in C([0,1]^n).$$

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We will be concerned primarily with the study of the uniqueness of bounded solutions to the following FPIDE with initial conditions, for $\alpha_i \in (0,1]$, $\beta_i > 0$, in the space $C([0,1]^n)$:

$$\begin{cases} \sum_{i=1}^{n} a_i \frac{{}_c\partial^{\alpha_i}}{\partial \tau_i^{\alpha_i}} \chi(\tau) + \sum_{i=1}^{n} b_i I_{\tau_i}^{\beta_i} \chi(\tau) = f(\tau, \chi(\tau)), \\ \forall k \in \{1, \dots, n-1\} : \tau_k = 0 \implies \chi(\tau) = 0, \\ \tau_n = 1 \implies \chi(\tau) = 0, \ \tau \in [0,1]^n, \end{cases} \tag{1.2}$$

where f satisfies certain conditions, a_i (not all zeros) and b_i are complex constants. Clearly without loss of generality, we can take $a_1 = 1$. Let $\alpha_{ij} \geqslant 0, \gamma_i > 0$ for all $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$, and

$$M = \begin{bmatrix} \alpha_{01} & \cdots & \alpha_{0m} & \gamma_0 \\ \alpha_1 & \cdots & \alpha_{1m} & \gamma_1 \\ \vdots & & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} & \gamma_n \end{bmatrix}.$$

In this paper we apply the following notation:

$$\prod_{i=1}^{n} I_{\tau_i}^{\alpha_i} = I_{\tau_n}^{\alpha_n} \cdots I_{\tau_1}^{\alpha_1} = I_{\tau_1}^{\alpha_1} \cdots I_{\tau_n}^{\alpha_n}.$$

Definition 1.1 ([6])**.** A matrix Mittag-Leffler function $E_M(z_1, \dots, z_m)$ is defined by the following series

$$E_M(z_1, \dots, z_m) = \sum_{l=0}^{\infty} \sum_{\substack{l_1 + \cdots + l_m = 1 \\ l_i \geqslant 0}} \binom{l}{l_1, \dots, l_m} \frac{z_1^{l_1} \cdots z_m^{l_m}}{\Gamma(\alpha_{01} l_1 + \cdots + \alpha_{0m} l_m + \gamma_0)}$$
$$\cdot \frac{1}{\Gamma(\alpha_{11} l_1 + \cdots + \alpha_{1m} l_m + \gamma_1) \cdots \Gamma(\alpha_{n1} l_1 + \cdots + \alpha_{nm} l_m + \gamma_n)},$$

where $z_i \in \mathbb{C}$ for $i \in \{1, \dots, m\}$ and

$$\binom{1}{l_1, \dots, l_m} = \frac{l!}{l_1! \cdots l_m!}.$$

In [6], Li et al. showed that the series is always convergent using the gamma function. This is a property that undoubtedly contributes to the famous Mittag-Leffler function and its generalizations' usefulness and is something that we will make use of later in this paper.

On the other hand, Babenko's approach [2], a technique introduced in 1986, is an effective means for dealing with both integral and differential equations that have initial conditions or boundary value problems as demonstrated in [4, 7], respectively. To demonstrate this method, we will solve the following equation with initial conditions in the space $C[0,1]$:

$$\begin{cases} {}_cD_0^{\alpha} y(x) + \lambda I_0^{\beta} y(x) = x^n, \ \beta > 0, \ \alpha \in (1,2], \\ y(0) = 0, y'(0) = 0, \end{cases}$$

where

$${}_cD_0^{\alpha} y(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{-\alpha+1} y''(s) ds, \ \alpha \in (1,2],$$

and

$$I_0^{\beta} y(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y(s) ds, \ \beta > 0.$$

It can be readily shown that

$$I_0^{\alpha}({}_cD_0^{\alpha} y(x)) = y(x) - y(0) - y'(0)x = y(x).$$

Applying I_0^α to both sides of the equation, we obtain

$$(1+\lambda I_0^{\alpha+\beta})y(x) = I_0^\alpha x^n = \frac{\Gamma(n+1)x^{n+\alpha}}{\Gamma(n+\alpha+1)}.$$

As per Babenko's approach, we treat the operator $(1+\lambda I_0^{\alpha+\beta})$ as a variable:

$$\begin{aligned} y(x) &= (1+\lambda I_0^{\alpha+\beta})^{-1}\cdot\frac{\Gamma(n+1)x^{n+\alpha}}{\Gamma(n+\alpha+1)} \\ &= \sum_{l=0}^{\infty}(-1)^l\lambda^l I_0^{l(\alpha+\beta)}\frac{\Gamma(n+1)x^{n+\alpha}}{\Gamma(n+\alpha+1)} \\ &= \Gamma(n+1)x^{n+\alpha}\sum_{l=0}^{\infty}\frac{(-\lambda x^{\alpha+\beta})^l}{\Gamma(l(\alpha+\beta)+n+\alpha+1)} \\ &= \Gamma(n+1)x^{n+\alpha}E_{\alpha+\beta,n+\alpha+1}(-\lambda x^{\alpha+\beta}), \end{aligned}$$

where

$$E_{\mu,\nu}(z) = \sum_{l=0}^{\infty}\frac{z^l}{\Gamma(\mu l+\nu)}, \quad \mu,\nu>0,\ z\in\mathbb{C},$$

is the 2-parameter Mittag-Leffler function, and

$$I_0^{l(\alpha+\beta)}x^{n+\alpha} = \frac{\Gamma(n+\alpha+1)x^{n+\alpha+l(\alpha+\beta)}}{\Gamma(l(\alpha+\beta)+n+\alpha+1)}.$$

Thus we can infer that

$$\|y\| \leqslant \Gamma(n+1)E_{\alpha+\beta,n+\alpha+1}(|\lambda|) < \infty.$$

Hence the series solution

$$y(x) = \Gamma(n+1)x^{n+\alpha}\sum_{l=0}^{\infty}\frac{(-\lambda x^{\alpha+\beta})^l}{\Gamma(l(\alpha+\beta)+n+\alpha+1)} = \Gamma(n+1)x^{n+\alpha}E_{\alpha+\beta,n+\alpha+1}(-\lambda x^{\alpha+\beta})$$

is in $C[0,1]$. There are numerous intensive studies on both the existence as well as the uniqueness of solutions to fractional differential equations [1, 9, 10]. Let $T>0$, $a\in C[0,T]$, $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $f:C[0,T]\to\mathbb{R}$. Very recently, Li [5] studied the uniqueness of solutions to a nonlinear integro-differential equation with nonlocal boundary condition and variable coefficients for $l<\alpha\leqslant l+1$ and $l\in\mathbb{N}$ as given below:

$$\begin{cases} {}_C D^\alpha\chi(\tau)+a(\tau)\,I^\beta\chi(\tau) = g(\tau,\chi(\tau)), & \tau\in[0,T],\ \beta>0, \\ \chi(0) = -f(\chi), \ \ \chi''(0) = \cdots = \chi^{(l)}(0) = 0, \\ \int_0^T\chi(\tau)d\tau = \lambda, \end{cases} \tag{1.3}$$

with λ being a constant. In particular when $l=1$, equation (1.3) becomes

$$\begin{cases} {}_C D^\alpha\chi(\tau)+a(\tau)\,I^\beta\chi(\tau) = g(\tau,\chi(\tau)), & \tau\in[0,T], \\ \chi(0) = -f(\chi), \ \ \int_0^T\chi(\tau)d\tau = \lambda. \end{cases}$$

For the remainder of the paper we will study the uniqueness of solutions to (1.2) in Section 2 using the given matrix Mittag-Leffler function, Babenko's approach, as well as Banach's contractive principle. In Section 3, we will provide an example of an equation of the form given in (1.2) in which we will apply the results from Section 2.

2. Uniqueness of solutions

Theorem 2.1. $\forall i \in \{1, \dots, n\} : \alpha_i \in (0, 1]$ *and* $\beta_i > 0$, *defined below, are* M_1 *and* M_2. *For formatting purposes, we partition* M_1 *and* M_2 *into* M_a, M_b, M_{c_1}, *and* M_{c_2} *all given as:*

$$M_a := \begin{bmatrix} 0 & \underbrace{\alpha_1 \cdots \alpha_1}_{n-3} & \alpha_1 & \alpha_1 & 0 & \underbrace{\alpha_1 \cdots \alpha_1}_{n-3} & \alpha_1 \\ & \ddots & & & & \ddots & \\ \alpha_{n-1} & \alpha_{n-1} \cdots \alpha_{n-1} & 0 & \alpha_{n-1} & \alpha_{n-1} & \alpha_{n-1} \cdots \alpha_{n-1} & 0 \\ a_n & a_n \cdots a_n & a_n & 0 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 \cdots 0 & 0 & 0 & a_n & a_n \cdots a_n & a_n \end{bmatrix},$$

$$M_b := \begin{bmatrix} \alpha_1 + \beta_1 & \underbrace{\alpha_1 \cdots \alpha_1}_{n-3} & \alpha_1 & \alpha_1 & \alpha_1 + \beta_1 & \underbrace{\alpha_1 \cdots \alpha_1}_{n-3} & \alpha_1 \\ & \ddots & & & & \ddots & \\ \alpha_{n-1} & \alpha_{n-1} \cdots \alpha_{n-1} & \alpha_{n-1} + \beta_{n-1} & \alpha_{n-1} & \alpha_{n-1} & \alpha_{n-1} \cdots \alpha_{n-1} & \alpha_{n-1} + \beta_{n-1} \\ 0 & 0 \cdots 0 & 0 & 0 & \alpha_n & \alpha_n \cdots \alpha_n & \alpha_n \\ \alpha_n & \alpha_n \cdots \alpha_n & \alpha_n & \alpha_n + \beta_n & 0 & 0 \cdots 0 & 0 \end{bmatrix},$$

$$M_{c_1} := \begin{bmatrix} \alpha_1 & 0 & \alpha_1 + 1 \\ \vdots & \vdots & \vdots \\ \alpha_{n-1} & 0 & \alpha_{n-1} + 1 \\ \alpha_n + \beta_n & 0 & \alpha_n + 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_{c_2} := \begin{bmatrix} \alpha_1 & 0 & \alpha_1 + 1 \\ \vdots & \vdots & \vdots \\ \alpha_{n-1} & 0 & \alpha_{n-1} + 1 \\ \alpha_n + \beta_n & 0 & 1 \\ 0 & 0 & \alpha_n + 1 \end{bmatrix},$$

let

$$M_1 = (M_a | M_b | M_{c_1}),\ M_2 = (M_a | M_b | M_{c_2}).$$

Suppose $f : [0, 1]^n \times \mathbb{R} \to \mathbb{R}$ *is continuous and bounded. Then* $\chi(\tau)$ *is a solution to equation* (1.2) *if and only if it is both bounded and satisfies the following implicit integral equation in the space* $C([0, 1]^n)$*:*

$$\begin{aligned} \chi(\tau) = & \sum_{l=0}^{\infty} (-1)^l \sum_{l_1 + \cdots + l_{4n} = l} \binom{l}{l_1, \dots, l_{4n}} a_2^{l_2} \cdots a_n^{l_n} (-a_1)^{l_{n+1}} \cdots (-a_{n-1})^{l_{2n-1}} \\ & \cdot (-b_1)^{l_{2n}} \cdots (-b_n)^{l_{3n-1}} b_1^{l_{3n}} \cdots b_n^{l_{4n-1}} (-1)^{l_{4n}} \\ & \cdot \prod_{k=1}^{n-1} (I_{\tau_k})^{\sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k}) + \alpha_k} \\ & \cdot (I_{\tau_n})^{\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1} + \alpha_n} (I_{\tau_n = 1})^{\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1}} f(\tau, \chi(\tau)) \\ & - \sum_{l=0}^{\infty} (-1)^l \sum_{l_1 + \cdots + l_{4n} = l} \binom{l}{l_1, \dots, l_{4n}} a_2^{l_2} \cdots a_n^{l_n} (-a_1)^{l_{n+1}} \cdots (-a_{n-1})^{l_{2n-1}} \\ & \cdot (-b_1)^{l_{2n}} \cdots (-b_n)^{l_{3n-1}} b_1^{l_{3n}} \cdots b_n^{l_{4n-1}} (-1)^{l_{4n}} \\ & \cdot \prod_{k=1}^{n-1} (I_{\tau_k})^{\sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k}) + \alpha_k} \\ & \cdot (I_{\tau_n})^{\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1}} (I_{\tau_n = 1})^{\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1} + \alpha_n} f(\tau, \chi(\tau)). \end{aligned}$$

Proof. By equation (1.1), applying the operator $I_{\tau_1}^{\alpha_1}$ to $\sum_{i=1}^{n} a_i \frac{{}^c\partial^{\alpha_i}}{\partial \tau_i^{\alpha_i}} \chi(\tau)$ yields $a_1 \chi(\tau) + \sum_{i=2}^{n} a_i I_{\tau_1}^{\alpha_1} \frac{{}^c\partial^{\alpha_i}}{\partial \tau_i^{\alpha_i}} \chi(\tau)$.

Similarly, applying $I^{\alpha_2}_{\tau_2} I^{\alpha_1}_{\tau_1}$ to $\sum_{i=1}^{n} a_i \frac{{}_c\partial^{\alpha_i}}{\partial\tau_i^{\alpha_i}}\chi(\tau)$ yields

$$a_1 I^{\alpha_2}_{\tau_2}\chi(\tau) + a_2 I^{\alpha_1}_{\tau_1}\chi(\tau) + \sum_{i=3}^{n} a_i I^{\alpha_2}_{\tau_2} I^{\alpha_1}_{\tau_1} \frac{{}_c\partial^{\alpha_i}}{\partial\tau_i^{\alpha_i}}\chi(\tau).$$

If we continue this process, applying $\prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}$ to $\sum_{i=1}^{n} a_i \frac{{}_c\partial^{\alpha_i}}{\partial\tau_i^{\alpha_i}}\chi(\tau)$ will produce

$$\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I^{\alpha_k}_{\tau_k}\chi(\tau) + a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i} \frac{{}_c\partial^{\alpha_n}}{\partial\tau_n^{\alpha_n}}\chi(\tau).$$

Hence, applying the operator $\prod_{i=1}^{n} I^{\alpha_i}_{\tau_i}$ to $\sum_{i=1}^{n} a_i \frac{{}_c\partial^{\alpha_i}}{\partial\tau_i^{\alpha_i}}\chi(\tau)$ will yield

$$\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n} I^{\alpha_k}_{\tau_k}\chi(\tau) + a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}\big(\chi(\tau) - \chi(\tau_1,\dots,\tau_{n-1},0)\big).$$

Therefore, applying the operator $\prod_{i=1}^{n} I^{\alpha_i}_{\tau_i}$ to both sides of equation (1.2), gets us

$$\begin{aligned} &\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n} I^{\alpha_k}_{\tau_k}\chi(\tau) + a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}\big(\chi(\tau) - \chi(\tau_1,\dots,\tau_{n-1},0)\big) + \sum_{i=1}^{n} b_i\Big(\prod_{k=1}^{n} I^{\alpha_k}_{\tau_k}\Big) I^{\beta_i}_{\tau_i}\chi(\tau) \\ &= \prod_{i=1}^{n} I^{\alpha_i}_{\tau_i} f(\tau,\chi(\tau)). \end{aligned} \tag{2.1}$$

For $\tau_n = 1$, we come to

$$\begin{aligned} &\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n}_{\tau_n=1}\chi(\tau) - a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}\chi(\tau_1,\dots,\tau_{n-1},0) + \sum_{i=1}^{n-1} b_i\Big(\prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k}\Big) I^{\alpha_n}_{\tau_n=1} I^{\beta_i}_{\tau_i}\chi(\tau) \\ &\quad + b_n \prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n+\beta_n}_{\tau_n=1}\chi(\tau) = \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i} I^{\alpha_n}_{\tau_n=1} f(\tau,\chi(\tau)). \end{aligned}$$

Thus

$$\begin{aligned} -a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}\chi(\tau_1,\dots,\tau_{n-1},0) &= \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i} I^{\alpha_n}_{\tau_n=1} f(\tau,\chi(\tau)) - \sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n}_{\tau_n=1}\chi(\tau) \\ &\quad - \sum_{i=1}^{n-1} b_i\Big(\prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k}\Big) I^{\alpha_n}_{\tau_n=1} I^{\beta_i}_{\tau_i}\chi(\tau) - b_n \prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n+\beta_n}_{\tau_n=1}\chi(\tau). \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1) we get

$$\begin{aligned} &\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n} I^{\alpha_k}_{\tau_k}\chi(\tau) + a_n \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i}\chi(\tau) + \prod_{i=1}^{n-1} I^{\alpha_i}_{\tau_i} I^{\alpha_n}_{\tau_n=1} f(\tau,\chi(\tau)) - \sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n}_{\tau_n=1}\chi(\tau) \\ &- \sum_{i=1}^{n-1} b_i\Big(\prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k}\Big) I^{\alpha_n}_{\tau_n=1} I^{\beta_i}_{\tau_i}\chi(\tau) - b_n \prod_{k=1}^{n-1} I^{\alpha_k}_{\tau_k} I^{\alpha_n+\beta_n}_{\tau_n=1}\chi(\tau) + \sum_{i=1}^{n} b_i\Big(\prod_{k=1}^{n} I^{\alpha_k}_{\tau_k}\Big) I^{\beta_i}_{\tau_i}\chi(\tau) = \prod_{i=1}^{n} I^{\alpha_i}_{\tau_i} f(\tau,\chi(\tau)). \end{aligned}$$

Using Babenko's approach,

$$
\begin{aligned}
\chi(\tau) &= \Bigg(1+\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n} I_{\tau_k}^{\alpha_k} + a_n \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} - \sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} - \sum_{i=1}^{n-1} b_i \Big(\prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k}\Big) I_{\tau_n=1}^{\alpha_n} I_{\tau_i}^{\beta_i} \\
&\quad - b_n \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n+\beta_n} + \sum_{i=1}^{n} b_i \Big(\prod_{k=1}^{n} I_{\tau_k}^{\alpha_k}\Big) I_{\tau_i}^{\beta_i} - 1\Bigg)^{-1} \Bigg(\prod_{i=1}^{n} I_{\tau_i}^{\alpha_i} f(\tau,\chi(\tau)) - \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} I_{\tau_n=1}^{\alpha_n} f(\tau,\chi(\tau))\Bigg) \\
&= \sum_{l=0}^{\infty} (-1)^l \Bigg(\sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n} I_{\tau_k}^{\alpha_k} + a_n \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} - \sum_{i=1}^{n-1} a_i \prod_{\substack{k=1\\k\neq i}}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} - \sum_{i=1}^{n-1} b_i \Big(\prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k}\Big) I_{\tau_n=1}^{\alpha_n} I_{\tau_i}^{\beta_i} \\
&\quad - b_n \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n+\beta_n} + \sum_{i=1}^{n} b_i \Big(\prod_{k=1}^{n} I_{\tau_k}^{\alpha_k}\Big) I_{\tau_i}^{\beta_i} - 1\Bigg)^{l} \Bigg(\prod_{i=1}^{n} I_{\tau_i}^{\alpha_i} f(\tau,\chi(\tau)) - \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} I_{\tau_n=1}^{\alpha_n} f(\tau,\chi(\tau))\Bigg) \\
&= \sum_{l=0}^{\infty} (-1)^l \sum_{l_1+\cdots+l_{4n}=l} \binom{l}{l_1,\ldots,l_{4n}} \Bigg(\prod_{k=2}^{n} I_{\tau_k}^{\alpha_k}\Bigg)^{l_1} \cdots \Bigg(a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n} I_{\tau_k}^{\alpha_k}\Bigg)^{l_{n-1}} \Bigg(a_n \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i}\Bigg)^{l_n} \\
&\quad \cdot \Bigg(-a_1 \prod_{k=2}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Bigg)^{l_{n+1}} \cdots \Bigg(-a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Bigg)^{l_{2n-1}} \\
&\quad \cdot \Bigg(-b_1 \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_1}^{\beta_1}\Bigg)^{l_{2n}} \cdots \Bigg(-b_{n-1} \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_{n-1}}^{\beta_{n-1}}\Bigg)^{l_{3n-2}} \Bigg(-b_n \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n+\beta_n}\Bigg)^{l_{3n-1}} \\
&\quad \cdot \Bigg(b_1 \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_1}^{\beta_1}\Bigg)^{l_{3n}} \cdots \Bigg(b_n \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_n}^{\beta_n}\Bigg)^{l_{4n-1}} (-1)^{l_{4n}} \Bigg(\prod_{i=1}^{n} I_{\tau_i}^{\alpha_i} f(\tau,\chi(\tau)) - \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} I_{\tau_n=1}^{\alpha_n} f(\tau,\chi(\tau))\Bigg) \\
&= \sum_{l=0}^{\infty} (-1)^l \sum_{l_1+\cdots+l_{4n}=l} \binom{l}{l_1,\ldots,l_{4n}} a_2^{l_2} \cdots a_n^{l_n} (-a_1)^{l_{n+1}} \cdots (-a_{n-1})^{l_{2n-1}} \\
&\quad \cdot (-b_1)^{l_{2n}} \cdots (-b_n)^{l_{3n-1}} b_1^{l_{3n}} \cdots b_n^{l_{4n-1}} (-1)^{l_{4n}} \\
&\quad \cdot \prod_{k=1}^{n-1} (I_{\tau_k})^{\sum_{\substack{i=1\\i\neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1\\i\neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k})} \\
&\quad \cdot (I_{\tau_n})^{\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1}} (I_{\tau_n=1})^{\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1}} \\
&\quad \cdot \Bigg(\prod_{i=1}^{n} I_{\tau_i}^{\alpha_i} f(\tau,\chi(\tau)) - \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i} I_{\tau_n=1}^{\alpha_n} f(\tau,\chi(\tau))\Bigg).
\end{aligned}
$$

We should note that the power of I_{τ_k} for $k = 1, 2, \ldots, n-1$ is

$$
\sum_{\substack{i=1\\i\neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1\\i\neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k}).
$$

Indeed, (i) the factor

$$
\Bigg(\prod_{k=2}^{n} I_{\tau_k}^{\alpha_k}\Bigg)^{l_1} \cdots \Bigg(a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n} I_{\tau_k}^{\alpha_k}\Bigg)^{l_{n-1}} \Bigg(a_n \prod_{i=1}^{n-1} I_{\tau_i}^{\alpha_i}\Bigg)^{l_n}
$$

generates the term $\sum_{\substack{i=1\\i\neq k}}^{n} \alpha_k l_i$; (ii) the factor

$$\Big(-a_1 \prod_{k=2}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Big)^{l_{n+1}} \cdots \Big(-a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Big)^{l_{2n-1}}$$

generates the term $\sum_{\substack{i=1\\i\neq k}}^{n-1} \alpha_k l_{n+i}$; (iii) the factor

$$\Big(-b_1 \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_1}^{\beta_1}\Big)^{l_{2n}} \cdots \Big(-b_{n-1} \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_{n-1}}^{\beta_{n-1}}\Big)^{l_{3n-2}} \Big(-b_n \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n+\beta_n}\Big)^{l_{3n-1}}$$
$$\cdot \Big(b_1 \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_1}^{\beta_1}\Big)^{l_{3n}} \cdots \Big(b_n \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_n}^{\beta_n}\Big)^{l_{4n-1}}$$

generates the term $\sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k(l_{2n-1+k} + l_{3n-1+k})$. As for the power of I_{τ_n}, we get

$$\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1}.$$

Indeed, (i) the factor

$$\Big(\prod_{k=2}^{n} I_{\tau_k}^{\alpha_k}\Big)^{l_1} \cdots \Big(a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n} I_{\tau_k}^{\alpha_k}\Big)^{l_{n-1}}$$

generates the term $\sum_{i=1}^{n-1} \alpha_n l_i$; (ii) the factor

$$\Big(b_1 \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_1}^{\beta_1}\Big)^{l_{3n}} \cdots \Big(b_n \prod_{k=1}^{n} I_{\tau_k}^{\alpha_k} I_{\tau_n}^{\beta_n}\Big)^{l_{4n-1}}$$

generates the term $\sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1}$. Finally, for the power of $I_{\tau_n=1}$,

$$\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1}$$

is similarly deduced by

$$\Big(-a_1 \prod_{k=2}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Big)^{l_{n+1}} \cdots \Big(-a_{n-1} \prod_{\substack{k=1\\k\neq n-1}}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n}\Big)^{l_{2n-1}}$$
$$\cdot \Big(-b_1 \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_1}^{\beta_1}\Big)^{l_{2n}} \cdots \Big(-b_{n-1} \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n} I_{\tau_{n-1}}^{\beta_{n-1}}\Big)^{l_{3n-2}} \Big(-b_n \prod_{k=1}^{n-1} I_{\tau_k}^{\alpha_k} I_{\tau_n=1}^{\alpha_n+\beta_n}\Big)^{l_{3n-1}}.$$

Hence,

$$\chi(\tau) = \sum_{l=0}^{\infty} (-1)^l \sum_{l_1+\cdots+l_{4n}=l} \binom{l}{l_1,\ldots,l_{4n}} a_2^{l_2} \cdots a_n^{l_n} (-a_1)^{l_{n+1}} \cdots (-a_{n-1})^{l_{2n-1}}$$
$$\cdot (-b_1)^{l_{2n}} \cdots (-b_n)^{l_{3n-1}} b_1^{l_{3n}} \cdots b_n^{l_{4n-1}} (-1)^{l_{4n}}$$

$$
\begin{aligned}
&\cdot \prod_{k=1}^{n-1} (I_{\tau_k})^{\sum_{\substack{i=1\\ i\neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1\\ i\neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k}) + \alpha_k} \\
&\cdot (I_{\tau_n})^{\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1} + \alpha_n} (I_{\tau_n=1})^{\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1}} f(\tau, \chi(\tau)) \\
&- \sum_{l=0}^{\infty} (-1)^l \sum_{l_1+\cdots+l_{4n}=l} \binom{l}{l_1, \ldots, l_{4n}} a_2^{l_2} \cdots a_n^{l_n} (-a_1)^{l_{n+1}} \cdots (-a_{n-1})^{l_{2n-1}} \\
&\cdot (-b_1)^{l_{2n}} \cdots (-b_n)^{l_{3n-1}} b_1^{l_{3n}} \cdots b_n^{l_{4n-1}} (-1)^{l_{4n}} \\
&\cdot \prod_{k=1}^{n-1} (I_{\tau_k})^{\sum_{\substack{i=1\\ i\neq k}}^{n} \alpha_k l_i + \sum_{\substack{i=1\\ i\neq k}}^{n-1} \alpha_k l_{n+i} + \sum_{i=2n}^{4n-1} \alpha_k l_i + \beta_k (l_{2n-1+k} + l_{3n-1+k}) + \alpha_k} \\
&\cdot (I_{\tau_n})^{\sum_{i=1}^{n-1} \alpha_n l_i + \sum_{i=3n}^{4n-1} \alpha_n l_i + \beta_n l_{4n-1}} (I_{\tau_n=1})^{\sum_{i=n+1}^{3n-1} \alpha_n l_i + \beta_n l_{3n-1} + \alpha_n} f(\tau, \chi(\tau)).
\end{aligned}
$$

All of the above steps are reversible. We must now show that $\chi \in C([0,1]^n)$. Indeed,

$$
\begin{aligned}
\|\chi\| \leqslant &\sum_{l=0}^{\infty} \sum_{l_1+\ldots+l_{4n}=l} \binom{l}{l_1, \ldots, l_{4n}} |a_1|^{l_1} \cdots |a_n|^{l_n} |a_1|^{l_{n+1}} \cdots |a_{n-1}|^{l_{2n-1}} \quad (\text{note that } a_1 = 1) \\
&\cdot |b_1|^{l_{2n}} \cdots |b_n|^{l_{3n-1}} |b_1|^{l_{3n}} \cdots |b_n|^{l_{4n-1}} 1^{l_{4n}} \\
&\cdot \|(I_{\tau_1})^{\alpha_1 l_2 + \cdots + \alpha_1 l_n + \alpha_1 l_{n+2} + \cdots + \alpha_1 l_{2n-1} + (\alpha_1+\beta_1) l_{2n} + \alpha_1 l_{2n+1} + \cdots + \alpha_1 l_{3n-1} + (\alpha_1+\beta_1) l_{3n} + \alpha_1 l_{3n+1} + \cdots + \alpha_1 l_{4n-1} + \alpha_1}\| \\
&\cdots \|(I_{\tau_{n-1}})^{\alpha_{n-1} l_1 + \cdots + \alpha_{n-1} l_{n-2} + \alpha_{n-1} l_n + \cdots + \alpha_{n-1} l_{2n-2} + \alpha_{n-1} l_{2n} + \cdots + \alpha_{n-1} l_{3n-3} + (\alpha_{n-1}+\beta_{n-1}) l_{3n-2}} \\
&\cdot (I_{\tau_{n-1}})^{\alpha_{n-1} l_{3n-1} + \cdots + \alpha_{n-1} l_{4n-3} + (\alpha_{n-1}+\beta_{n-1}) l_{4n-2} + \alpha_{n-1} l_{4n-1} + \alpha_{n-1}}\| \\
&\cdot \|(I_{\tau_n})^{\alpha_n l_1 + \cdots + \alpha_n l_{n-1} + \alpha_n l_{3n} + \cdots + \alpha_n l_{4n-2} + (\alpha_n+\beta_n) l_{4n-1} + \alpha_n}\| \\
&\cdot \|(I_{\tau_n=1})^{\alpha_n l_{n+1} + \cdots + \alpha_n l_{3n-2} + (\alpha_n+\beta_n) l_{3n-1}}\| \cdot \|f\| \\
&+ \sum_{l=0}^{\infty} \sum_{l_1+\ldots+l_{4n}=l} \binom{l}{l_1, \ldots, l_{4n}} |a_1|^{l_1} \cdots |a_n|^{l_n} |a_1|^{l_{n+1}} \cdots |a_{n-1}|^{l_{2n-1}} \\
&\cdot |b_1|^{l_{2n}} \cdots |b_n|^{l_{3n-1}} |b_1|^{l_{3n}} \cdots |b_n|^{l_{4n-1}} 1^{l_{4n}} \\
&\cdot \|(I_{\tau_1})^{\alpha_1 l_2 + \cdots + \alpha_1 l_n + \alpha_1 l_{n+2} + \cdots + \alpha_1 l_{2n-1} + (\alpha_1+\beta_1) l_{2n} + \alpha_1 l_{2n+1} + \cdots + \alpha_1 l_{3n-1} + (\alpha_1+\beta_1) l_{3n} + \alpha_1 l_{3n+1} + \cdots + \alpha_1 l_{4n-1} + \alpha_1}\| \\
&\cdots \|(I_{\tau_{n-1}})^{\alpha_{n-1} l_1 + \cdots + \alpha_{n-1} l_{n-2} + \alpha_{n-1} l_n + \cdots + \alpha_{n-1} l_{2n-2} + \alpha_{n-1} l_{2n} + \cdots + \alpha_{n-1} l_{3n-3} + (\alpha_{n-1}+\beta_{n-1}) l_{3n-2}} \\
&\cdot (I_{\tau_{n-1}})^{\alpha_{n-1} l_{3n-1} + \cdots + \alpha_{n-1} l_{4n-3} + (\alpha_{n-1}+\beta_{n-1}) l_{4n-2} + \alpha_{n-1} l_{4n-1} + \alpha_{n-1}}\| \\
&\cdot \|(I_{\tau_n})^{\alpha_n l_1 + \cdots + \alpha_n l_{n-1} + \alpha_n l_{3n} + \cdots + \alpha_n l_{4n-2} + (\alpha_n+\beta_n) l_{4n-1}}\| \\
&\cdot \|(I_{\tau_n=1})^{\alpha_n l_{n+1} + \cdots + \alpha_n l_{3n-2} + (\alpha_n+\beta_n) l_{3n-1} + \alpha_n}\| \cdot \|f\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\chi\| \leqslant &\sum_{l=0}^{\infty} \sum_{l_1+\ldots+l_{4n}=l} \binom{l}{l_1, \ldots, l_{4n}} |a_1|^{l_1} \cdots |a_n|^{l_n} |a_1|^{l_{n+1}} \cdots |a_{n-1}|^{l_{2n-1}} \\
&\cdot |b_1|^{l_{2n}} \cdots |b_n|^{l_{3n-1}} |b_1|^{l_{3n}} \cdots |b_n|^{l_{4n-1}} 1^{l_{4n}} \\
&\cdot \big(\Gamma(0 l_1 + \alpha_1 l_2 + \cdots + \alpha_1 l_n + 0 l_{n+1} + \alpha_1 l_{n+2} \cdots + \alpha_1 l_{2n-1} + (\alpha_1 + \beta_1) l_{2n} \\
&+ \alpha_1 l_{2n+1} + \cdots + \alpha_1 l_{2n-1} + (\alpha_1 + \beta_1) l_{3n} + \alpha_1 l_{3n+1} + \cdots + \alpha_1 l_{4n-1} + \alpha_1 + 1)\big)^{-1} \\
&\cdots \big(\Gamma(\alpha_{n-1} l_1 + \cdots + \alpha_{n-1} l_{n-2} + 0 l_{n-1} + \alpha_{n-1} l_n + \cdots \alpha_{n-1} l_{2n-2} + 0 l_{2n-1} \\
&+ \alpha_{n-1} l_{2n} + \cdots + \alpha_{n-1} l_{3n-3} + (\alpha_{n-1} + \beta_{n-1}) l_{3n-2} + \alpha_{n-1} l_{3n-1} + \cdots + \alpha_{n-1} l_{4n-3} \\
&+ (\alpha_{n-1} + \beta_{n-1}) l_{4n-2} + \alpha_{n-1} l_{4n-1} + \alpha_{n-1} + 1)\big)^{-1} \\
&\cdot \big(\Gamma(\alpha_n l_1 + \cdots + \alpha_n l_{n-1} + 0 l_n + \cdots + 0 l_{3n-2} + 0 l_{3n-1} + \alpha_n l_{3n} + \cdots + \alpha_n l_{4n-2}
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha_n+\beta_n)l_{4n-1}+\alpha_n+1)\Big)^{-1}\\
&\cdot\Big(\Gamma(0l_1+\ldots+0l_n+\alpha_n l_{n+1}+\cdots+\alpha_n l_{3n-2}+(\alpha_n+\beta_n)l_{3n-1}+0l_{3n}+\cdots+0l_{4n}+1)\Big)^{-1}\\
&\cdot\|f\|+\sum_{l=0}^{\infty}\sum_{l_1+\ldots+l_{4n}=l}\binom{l}{l_1,\ldots,l_{4n}}|a_1|^{l_1}\cdots|a_n|^{l_n}|a_1|^{l_{n+1}}\cdots|a_{n-1}|^{l_{2n-1}}\\
&\cdot|b_1|^{l_{2n}}\cdots|b_n|^{l_{3n-1}}|b_1|^{l_{3n}}\cdots|b_n|^{l_{4n-1}}1^{l_{4n}}\\
&\cdot\Big(\Gamma(0l_1+\alpha_1 l_2+\cdots+\alpha_1 l_n+0l_{n+1}+\alpha_1 l_{n+2}\cdots+\alpha_1 l_{2n-1}+(\alpha_1+\beta_1)l_{2n}\\
&+\alpha_1 l_{2n+1}+\cdots+\alpha_1 l_{2n-1}+(\alpha_1+\beta_1)l_{3n}+\alpha_1 l_{3n+1}+\cdots+\alpha_1 l_{4n-1}+\alpha_1+1)\Big)^{-1}\\
&\cdots\Big(\Gamma(\alpha_{n-1}l_1+\cdots+\alpha_{n-1}l_{n-2}+0l_{n-1}+\alpha_{n-1}l_n+\cdots\alpha_{n-1}l_{2n-2}+0l_{2n-1}\\
&+\alpha_{n-1}l_{2n}+\cdots+\alpha_{n-1}l_{3n-3}+(\alpha_{n-1}+\beta_{n-1})l_{3n-2}+\alpha_{n-1}l_{3n-1}+\cdots+\alpha_{n-1}l_{4n-3}\\
&+(\alpha_{n-1}+\beta_{n-1})l_{4n-2}+\alpha_{n-1}l_{4n-1}+\alpha_{n-1}+1)\Big)^{-1}\\
&\cdot\Big(\Gamma(\alpha_n l_1+\cdots+\alpha_n l_{n-1}+0l_n+\cdots+0l_{3n-2}+0l_{3n-1}+\alpha_n l_{3n}+\cdots+\alpha_n l_{4n-2}\\
&+(\alpha_n+\beta_n)l_{4n-1}+1)\Big)^{-1}\\
&\cdot\Big(\Gamma(0l_1+\ldots+0l_n+\alpha_n l_{n+1}+\cdots+\alpha_n l_{3n-2}+(\alpha_n+\beta_n)l_{3n-1}+0l_{3n}+\cdots+0l_{4n}+\alpha_n+1)\Big)^{-1}\|f\|,
\end{aligned}
$$

which follows from the fact that

$$\|I_\tau^\alpha f\|\leqslant\frac{\|f\|}{\Gamma(\alpha+1)}.$$

Hence,

$$
\begin{aligned}
\|\chi\|=&E_{M_1}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\|f\|\\
&+E_{M_2}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\|f\|<\infty,
\end{aligned}
$$

since f is bounded and

$$\|f\|=\sup_{\tau\in[0,1]^n,\,w\in\mathbb{R}}|f(\tau,w)|.$$

Hence $\chi\in C[0,1]^n$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f:[0,1]^n\times\mathbb{R}\to\mathbb{R}$ be a bounded and continuous function satisfying the following Lipschitz condition for a constant $\mathcal{L}>0$:*

$$|f(\tau,x)-f(\tau,y)|\leqslant\mathcal{L}|x-y|,\ x,\ y\in\mathbb{R}.$$

Additionally, suppose $\alpha_i\in(0,1]$ for all $i\in\{1,\ldots,n\}$ and

$$
\begin{aligned}
W:=\mathcal{L}\big[&E_{M_1}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\\
&+E_{M_2}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\big]<1.
\end{aligned}
$$

Then equation (1.2) has a unique solution in the space $C([0,1]^n)$.

Proof. We define a mapping $\mathcal{S}$ over the space $C([0,1]^n)$ given by

$$
\begin{aligned}
(\mathcal{S}\chi)(\tau)=&\sum_{l=0}^{\infty}(-1)^l\sum_{l_1+\cdots+l_{4n}=l}\binom{l}{l_1,\ldots,l_{4n}}a_2^{l_2}\cdots a_n^{l_n}(-a_1)^{l_{n+1}}\cdots(-a_{n-1})^{l_{2n-1}}\\
&\cdot(-b_1)^{l_{2n}}\cdots(-b_n)^{l_{3n-1}}b_1^{l_{3n}}\cdots b_n^{l_{4n-1}}(-1)^{l_{4n}}\\
&\cdot\prod_{k=1}^{n-1}(I_{\tau_k})^{\sum_{i=1,i\neq k}^{n}\alpha_k l_i+\sum_{i=1,i\neq k}^{n-1}\alpha_k l_{n+i}+\sum_{i=2n}^{4n-1}\alpha_k l_i+\beta_k(l_{2n-1+k}+l_{3n-1+k})+\alpha_k}\\
&\cdot(I_{\tau_n})^{\sum_{i=1}^{n-1}\alpha_n l_i+\sum_{i=3n}^{4n-1}\alpha_n l_i+\beta_n l_{4n-1}+\alpha_n}(I_{\tau_n=1})^{\sum_{i=n+1}^{3n-1}\alpha_n l_i+\beta_n l_{3n-1}}f(\tau,\chi(\tau))
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{l=0}^{\infty}(-1)^{l} \sum_{l_1+\cdots+l_{4n}=l} \binom{l}{l_1,\ldots,l_{4n}} a_2^{l_2}\cdots a_n^{l_n}(-a_1)^{l_{n+1}}\cdots(-a_{n-1})^{l_{2n-1}} \\
&\cdot (-b_1)^{l_{2n}}\cdots(-b_n)^{l_{3n-1}} b_1^{l_{3n}}\cdots b_n^{l_{4n-1}}(-1)^{l_{4n}} \\
&\cdot \prod_{k=1}^{n-1}(I_{\tau_k})^{\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_k l_i+\sum_{\substack{i=1\\ i\neq k}}^{n-1}\alpha_k l_{n+i}+\sum_{i=2n}^{4n-1}\alpha_k l_i+\beta_k(l_{2n-1+k}+l_{3n-1+k})+\alpha_k} \\
&\cdot (I_{\tau_n})^{\sum_{i=1}^{n-1}\alpha_n l_i+\sum_{i=3n}^{4n-1}\alpha_n l_i+\beta_n l_{4n-1}} (I_{\tau_n=1})^{\sum_{i=n+1}^{3n-1}\alpha_n l_i+\beta_n l_{3n-1}+\alpha_n} f(\tau,\chi(\tau)).
\end{aligned}
$$

Consider for $\chi_1, \chi \in C([0,1]^n)$,

$$
\begin{aligned}
\|\mathcal{S}\chi_1-\mathcal{S}\chi_2\| &\leqslant \big(E_{M_1}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1) \\
&\qquad + E_{M_2}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\big)\|f(\tau,\chi_1)-f(\tau,\chi_2)\| \\
&\leqslant \mathcal{L}\big(E_{M_1}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1) \\
&\qquad + E_{M_2}(|a_1|,\ldots,|a_n|,|a_1|,\ldots,|a_{n-1}|,|b_1|,\ldots,|b_n|,|b_1|,\ldots,|b_n|,1)\big)\|\chi_1-\chi_2\| = W\|\chi_1-\chi_2\|.
\end{aligned}
$$

Since $W < 1$, $\mathcal{S}$ is contractive. Therefore, by Banach's fixed point theorem, there is a unique solution to equation (1.2) in the space $C([0,1]^n)$. □

3. Example

Consider the following nonlinear FPIDE with the boundary condition for $n = 3$:

$$
\begin{cases}
\sum_{i=1}^{3} i\,{}_c\frac{\partial^{\alpha_i}}{\partial\tau_i^{\alpha_i}}\chi(\tau)+\sum_{i=1}^{3}0.1 I_{\tau_i}^{\beta_i}\chi(\tau)=\frac{1}{7500000}\cos(\tau_1\chi(\tau))-\tau_2, \\
\text{where } \alpha_i=\frac{1}{2^i} \text{ and } \beta_i=\frac{1}{5^i},\ \tau_1=0\vee\tau_2=0 \implies \chi(\tau)=0, \\
\tau_3=1 \implies \chi(\tau)=0,\ \tau\in[0,1]^3.
\end{cases}
\tag{3.1}
$$

Then equation (3.1) has a unique solution in $C([0,1]^3)$.

Proof. We begin by noting that

$$
f(\tau,\chi)=\frac{1}{7500000}\cos(\tau_1\chi)-\tau_2
$$

is a continuous and bounded function that satisfies the following Lipschitz condition:

$$
|f(\tau,\chi_1)-f(\tau,\chi_2)| \leqslant \frac{1}{7500000}|\cos(\tau_1\chi_1)-\cos(\tau_1\chi_2)| \leqslant \frac{1}{7500000}|\chi_1-\chi_2|,
$$

for all $\chi_1, \chi_2 \in \mathbb{R}$, by noting that $\tau \in [0,1]^3$. Thus $\mathcal{L} = \frac{1}{7500000}$. The matrices M_1 and M_2 used in the matrix Mittag-Leffler functions in the calculation of W are given explicitly for this example as follows:

$$
M_1=\begin{bmatrix}
0 & \frac12 & \frac12 & 0 & \frac12 & \frac{7}{10} & \frac12 & \frac12 & \frac{7}{10} & \frac12 & \frac12 & 0 & \frac32 \\
\frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac{29}{100} & \frac14 & \frac14 & \frac{29}{100} & \frac14 & 0 & \frac54 \\
\frac18 & \frac18 & 0 & 0 & 0 & 0 & 0 & 0 & \frac18 & \frac18 & \frac{133}{1000} & 0 & \frac98 \\
0 & 0 & 0 & \frac18 & \frac18 & \frac18 & \frac18 & \frac{133}{1000} & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

$$
M_2=\begin{bmatrix}
0 & \frac12 & \frac12 & 0 & \frac12 & \frac{7}{10} & \frac12 & \frac12 & \frac{7}{10} & \frac12 & \frac12 & 0 & \frac32 \\
\frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac{29}{100} & \frac14 & \frac14 & \frac{29}{100} & \frac14 & 0 & \frac54 \\
\frac18 & \frac18 & 0 & 0 & 0 & 0 & 0 & 0 & \frac18 & \frac18 & \frac{133}{1000} & 0 & 1 \\
0 & 0 & 0 & \frac18 & \frac18 & \frac18 & \frac18 & \frac{133}{1000} & 0 & 0 & 0 & 0 & \frac98
\end{bmatrix}.
$$

Using a Python code, we now evaluate W from Theorem 2.2:

$$W = \frac{1}{7500000}\Big[\mathrm{E}_{M_1}(1,2,3,1,2,0.1,0.1,0.1,0.1,0.1,0.1,1) + \mathrm{E}_{M_2}(1,2,3,1,2,0.1,0.1,0.1,0.1,0.1,0.1,1)\Big] \approx 0.998 < 1.$$

Thus, by Theorem 2.2, equation (3.1) has a unique solution in the space $C([0,1]^3)$. □

4. Conclusion

Further expounding upon the utility of the recently introduced matrix Mittag-Leffler function, in conjunction with Babenko's approach and Banach's contractive principle, we have derived the sufficient condition for the uniqueness of solutions to the boundary value problem of the FPIDE presented in this paper. In order to demonstrate our main results, we provided a clarifying example derived via a Python code in order to approximate the value of the matrix Mittag-Leffler function.

Author contributions

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