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# A qualitative investigation of some rational difference equations

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## Abstract

In this paper we study some qualitative properties of the solutions for the following difference equation

$$y_{n+1}=\frac{\alpha+\alpha_0 y_n^r+\alpha_1 y_{n-1}^r+\cdots+\alpha_k y_{n-k}^r}{\beta+\beta_0 y_n^r+\beta_1 y_{n-1}^r+\cdots+\beta_k y_{n-k}^r},\quad n\geqslant 0, \tag{I}$$

where $r,\alpha,\alpha_0,\alpha_1,\dots,\alpha_k,\beta,\beta_0,\beta_1,\dots,\beta_k\in(0,\infty)$ and $k$ is a non-negative integer number. We find the equilibrium points for the considered equation. Then classify these points in terms of local stability or not. We investigate the boundedness and the global stability of the solutions for the considered equation. Also we study the existence of periodic solutions of Eq. (I).

**Keywords:** Boundedness, local stability, periodicity, global stability.

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## 1. Introduction

Difference equations play a crucial role in mathematical modeling, providing a powerful framework to describe dynamic processes over discrete time intervals. Unlike differential equations that deal with continuous changes, difference equations focus on the evolution of a system at distinct time points. Their significance is evident in diverse fields, from physics and economics to biology and engineering. Engineers employ them to model discrete-time control systems, ensuring stability and performance. Economists use the difference equations to analyze economic trends and forecast future developments. In biology, these equations help describe population dynamics and ecological systems. Additionally in signal processing, difference equations contribute to the design of digital filters for audio and image processing. Overall difference equations serve as a versatile tool for understanding and predicting the behavior of systems in numerous real-world applications (see [1–6, 10, 12–14, 22, 26, 27]).

In this paper we study the boundedness and the global attractivity of the solutions of the difference equation

$$y_{n+1}=\frac{\alpha+\alpha_0 y_n^r+\alpha_1 y_{n-1}^r+\cdots+\alpha_k y_{n-k}^r}{\beta+\beta_0 y_n^r+\beta_1 y_{n-1}^r+\cdots+\beta_k y_{n-k}^r},\quad n\geqslant 0, \tag{1.1}$$

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where $r, \alpha, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta, \beta_0, \beta_1, \ldots, \beta_k \in (0, \infty)$ and $k$ is a non-negative integer number. Also, we investigate the periodicity character of the solutions of Eq. (1.1).

The study of the properties of the solutions for the difference equations such as periodicity, global stability and boundedness has been discussed by many authors. See for examples the following papers and the references therein. Cinar [2] studied the properties of the positive solution for the equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots.$$

Yang et al. [25] investigated the qualitative behavior of the recursive sequence

$$x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots.$$

Li et al. [19] studied the global asymptotic of the solutions for the following non-linear difference equation

$$x_{n+1} = \frac{a + x_{n-1} + x_{n-2} + x_{n-3} + x_{n-1}x_{n-2}x_{n-3}}{a + x_{n-1}x_{n-2} + x_{n-1}x_{n-3} + x_{n-2}x_{n-3} + 1}, \quad n = 0, 1, \ldots,$$

with $a \geqslant 0$. Kulenovic and Ladas [16] presented a summary of a recent work and a large number of open problems and conjectures on the third order rational recursive sequence of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots.$$

Xianyi and Deming [20] proved that the positive equilibrium of the difference equations

$$x_{n+1} = \frac{x_n x_{n-1} + x_{n-2} + a}{x_n + x_{n-1}x_{n-2} + a}, \quad x_{n+1} = \frac{x_{n-1} + x_n x_{n-2} + a}{x_n x_{n-1} + x_{n-2} + a}, \quad n = 0, 1, \ldots,$$

with positive initial values $x_{-2}, x_{-1}, x_0$ and non-negative parameter $a$, is globally asymptotically stable. Simsek et al. [23] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots.$$

Yalçınkaya et al. [24] investigated the dynamics of the difference equation

$$x_{n+1} = \frac{a x_{n-k}}{b + c x_n^p}, \quad n = 0, 1, \ldots.$$

For more related results see [7–9, 11] and [15, 17–19, 21]. In the following, we present some definitions and some known results that will be useful in the investigation of Eq. (1.1). Now consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

with $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$.

**Definition 1.1** ([15]). Eq. (1.2) is said to be permanent and bounded if there exists numbers $m$ and $M$ with $0 < m < M < \infty$ such that for any initial condition $x_{-k}, x_{-k+1}, \ldots, x_0 \in (0, \infty)$ there exists a positive integer $N$ which depends on these initial condition such that $m < x_n < M$ for all $n \geqslant N$.

**Definition 1.2** ([15]).

(i) The equilibrium point $\bar{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$ for all $n \geqslant -k$.

(ii) The equilibrium point $\bar{x}$ of Eq. (1.2) is globally asymptotically stable if $\bar{x}$ is locally stable and there exists $\lambda > 0$, such that for all $x_{-k}, x_{-k+1}, \dots, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \lambda$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iii) The equilibrium point $\bar{x}$ of Eq. (1.2) is global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) The equilibrium point $\bar{x}$ of Eq. (1.2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (1.2).

(v) The equilibrium point $\bar{x}$ of Eq. (1.2) is unstable if $\bar{x}$ is not locally stable.

Observe that, the linearised equation of Eq. (1.2) about the equilibrium point $\bar{x}$ is

$$y_{n+1} = p_1 y_n + p_2 y_{n-1} + \cdots + p_{k+1} y_{n-k}, \tag{1.3}$$

where

$$p_1 = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x}, \dots, \bar{x}), \quad p_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x}, \dots, \bar{x}), \dots, p_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\bar{x}, \bar{x}, \dots, \bar{x}),$$

and the characteristic equation of Eq. (1.3) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} p_i \lambda^{k-i+1} = 0.$$

**Theorem 1.3** ([15])**.** *Assume that $p_1, p_2, \dots, p_{k+1} \in \mathbb{R}$. Then the condition*

$$\sum_{i=1}^{k+1} |p_i| < 1,$$

*is a sufficient condition for the locally stability of Eq.* (1.3).

**Theorem 1.4** ([15])**.** *Let $J$ be an interval of real numbers, $f \in C\left[J^{\upsilon+1}, J\right]$, and let $\{x_n\}_{n=-\upsilon}^{\infty}$ be a bounded solution of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-\upsilon}), \quad n = 0, 1, \dots, \tag{1.4}$$

*with*

$$I = \lim_{n\to\infty} \inf x_n, \quad S = \lim_{n\to\infty} \sup x_n, \quad I, S \in J.$$

*Then there exist two solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.* (1.4) *with*

$$I_0 = I, \quad S_0 = S, \quad I_n, S_n \in [I, S] \ \textit{for all} \ n \in \mathbb{Z},$$

*and such that for every $N \in \mathbb{Z}$, $I_N$ and $S_N$ are limit points of $\{x_n\}_{n=-\upsilon}^{\infty}$. Furthermore, for every $m \leqslant -\upsilon$, there exist two subsequences $\{x_{r_n}\}$ and $\{x_{l_n}\}$ of the solution $\{x_n\}_{n=-\upsilon}^{\infty}$ such that the following are true*

$$\lim_{n\to\infty} x_{r_n+N} = I_N \quad \textit{and} \quad \lim_{n\to\infty} x_{l_n+N} = S_N \quad \textit{for every} \ N \geqslant m.$$

*The solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ are called full limiting sequences of Eq.* (1.4).

## 2. Boundedness for the solutions of Eq. (1.1)

In this section we study the boundedness of the solutions for Eq. (1.1).

**Theorem 2.1.** *Every solution of Eq.* (1.1) *is bounded and persists.*

*Proof.* Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1) and assume that $\alpha^* = \min\{\alpha, \alpha_0, \dots, \alpha_k\}$, $\alpha^{**} = \max\{\alpha, \alpha_0, \dots, \alpha_k\}$, $\beta^* = \min\{\beta, \beta_0, \dots, \beta_k\}$, $\beta^{**} = \max\{\beta, \beta_0, \dots, \beta_k\}$. It follows from Eq. (1.1) that

$$y_{n+1} = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r} \leqslant \frac{\max\{\alpha, \alpha_0, \alpha_1, \dots, \alpha_k\}(1 + y_n^r + y_{n-1}^r + \cdots + y_{n-k}^r)}{\min\{\beta, \beta_0, \beta_1, \dots, \beta_k\}(1 + y_n^r + y_{n-1}^r + \cdots + y_{n-k}^r)} = \frac{\max\{\alpha, \alpha_0, \alpha_1, \dots, \alpha_k\}}{\min\{\beta, \beta_0, \beta_1, \dots, \beta_k\}} = \frac{\alpha^{**}}{\beta^*}.$$

Similarly it is easy to see that

$$y_n \geqslant \frac{\min\{\alpha, \alpha_0, \alpha_1, \dots, \alpha_k\}}{\max\{\beta, \beta_0, \beta_1, \dots, \beta_k\}} = \frac{\alpha^*}{\beta^{**}}.$$

Thus we get

$$0 < \gamma := \frac{\alpha^*}{\beta^{**}} \leqslant y_n \leqslant \frac{\alpha^{**}}{\beta^*} := \delta < \infty, \quad \text{for all} \quad n \geqslant 1.$$

Therefore every solution of Eq. (1.1) is bounded and persists. Hence the result holds. □

**Theorem 2.2.** *Every solution of Eq. (1.1) is bounded and persists.*

*Proof.* Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Eq. (1.1). Then, it follows that

$$y_{n+1} = \frac{\alpha}{\beta + \beta_0 y_n^r + \cdots + \beta_k y_{n-k}^r} + \frac{\alpha_0 y_n^r}{\beta + \beta_0 y_n^r + \cdots + \beta_k y_{n-k}^r} + \cdots + \frac{\alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \cdots + \beta_k y_{n-k}^r} \leqslant \frac{\alpha}{\beta} + \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \cdots + \frac{\alpha_k}{\beta_k} = \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} := D.$$

Then $\{y_n\}_{n=-k}^{\infty}$ is bounded from above by D, that is $y_n \leqslant D$ for all $n \geqslant 1$. Now we can obtain the lower bound of $\{y_n\}_{n=-k}^{\infty}$ by two ways.

(I) By the change of variables $y_n = \frac{1}{z_n}$ for all $n \geqslant 1$, Eq. (1.1) can be rewritten in the form

$$z_{n+1} = \frac{\beta z_n^r z_{n-1}^r \cdots z_{n-k}^r + \beta_0 z_{n-1}^r z_{n-2}^r \cdots z_{n-k}^r + \cdots + \beta_k z_n^r z_{n-1}^r \cdots z_{n-k+1}^r}{\alpha z_n^r z_{n-1}^r \cdots z_{n-k}^r + \alpha_0 z_{n-1}^r z_{n-2}^r \cdots z_{n-k}^r + \cdots + \alpha_k z_n^r z_{n-1}^r \cdots z_{n-k+1}^r} \leqslant \frac{\beta}{\alpha} + \frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_k}{\alpha_k} = \frac{\beta}{\alpha} + \sum_{i=0}^{k} \frac{\beta_i}{\alpha_i} := d^*.$$

That is $y_n \geqslant \frac{1}{d^*} := d$ for all $n \geqslant 1$ and therefore

$$d = \frac{1}{\frac{\beta}{\alpha} + \sum_{i=0}^{k} \frac{\beta_i}{\alpha_i}} \leqslant y_n \leqslant \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} = D, \quad \text{for all} \quad n \geqslant 1,$$

this completes the proof.

(II) Since $y_n \leqslant D$ for all $n \geqslant 1$, we get from Eq. (1.1) that

$$y_{n+1} \geqslant \frac{\alpha}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r} \geqslant \frac{\alpha}{\beta + D^r \sum_{i=0}^{k} \beta_i} := d^{**}.$$

Then we see again that

$$d^{**} = \frac{\alpha}{\beta + D^r \sum_{i=0}^{k} \beta_i} \leqslant y_n \leqslant \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} = D, \quad \text{for all} \quad n \geqslant 1.$$

Thus the proof is completed. □

### 3. Stability analysis

In this section we investigate the global asymptotic stability of Eq. (1.1). Observe that the equilibrium points of Eq. (1.1) are given by $\overline{y} = \frac{\alpha + A\overline{y}^r}{\beta + B\overline{y}^r}$, where $A = \sum_{i=0}^{k} \alpha_i$ and $B = \sum_{i=0}^{k} \beta_i$.

**Lemma 3.1.** *Eq. (1.1) has a unique positive equilibrium point if one of the following is true*

(I) $A\beta \leqslant B\alpha$;
(II) $r < 1$;
(III) $rA\beta < A\beta + \alpha[B(r+1) + \beta\overline{y}^{-r}] + AB\overline{y}^r$;
(IV) $rA\beta < B(r\alpha + 2\beta\overline{y} + B\overline{y}^{r+1}] + \beta^2\overline{y}^{1-r}$.

*Proof.* Define the function $f(x) = \frac{\alpha + Ax^r}{\beta + Bx^r} - x$, $x \in \mathbb{R}$. Then

$$f'(x) = \frac{rx^{r-1}(A\beta - B\alpha)}{(\beta + Bx^r)^2} - 1,$$

so

$$f'(\overline{y}) = \frac{r\overline{y}^{(r-1)}(A\beta - B\alpha)}{(\beta + B\overline{y}^r)^2} - 1.$$

Also note that

$$f(0) = \frac{\alpha}{\beta} > 0, \quad \lim_{x \to \infty} f(x) = -\infty, \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty.$$

Now we discuss the following cases.

(1) If (I) holds we obtain that $f'(\overline{y}) < 0$ for all $\bar{y} \in \mathbb{R}^+$, thus Eq. (1.1) has a unique positive equilibrium point $\overline{y}$ that satisfies the relation $\overline{y} = \frac{\alpha + A\overline{y}^r}{\beta + B\overline{y}^r}$ and this completes the proof of (I).

(2) Note that

$$\begin{aligned}
f'(\overline{y}) < 0 \iff & r\overline{y}^{r-1}(A\beta - B\alpha) < (\beta + B\overline{y}^r)^2 \\
\iff & r\overline{y}^r(A\beta - B\alpha) < \overline{y}(\beta + B\overline{y}^r)^2 \\
\iff & r\overline{y}^r(A\beta - B\alpha) < (\alpha + A\overline{y}^r)(\beta + B\overline{y}^r) \\
\iff & rA\beta\overline{y}^r - rB\alpha\overline{y}^r < \alpha\beta + A\beta\overline{y}^r + \alpha B\overline{y}^r + AB\overline{y}^{2r} \\
\iff & A\beta\overline{y}^r(r-1) < \alpha\beta + \alpha B\overline{y}^r(r+1) + AB\overline{y}^{2r},
\end{aligned}$$

which is true by using (II), then the result follows.

(3) Again we see from case (2) that

$$\begin{aligned}
f(\overline{y}) < 0 \iff & A\beta(r-1) < \alpha B(r+1) + \alpha\beta\overline{y}^{-r} + AB\overline{y}^r \\
\iff & rA\beta < A\beta + \alpha[B(r+1) + \beta\overline{y}^{-r}] + AB\overline{y}^r.
\end{aligned}$$

Therefore again Eq. (1.1) has a unique positive equilibrium point $\overline{y}$.

(4) The proof of the case wherever (IV) holds, is similar to the case (II) as

$$\begin{aligned}
f'(\overline{y}) < 0 \iff & r\overline{y}^{r-1}(A\beta - B\alpha) < (\beta + B\overline{y}^r)^2 \\
\iff & r\overline{y}^{r-1}(A\beta - B\alpha) < \beta^2 + 2B\beta\overline{y}^r + B^2\overline{y}^{2r} \\
\iff & r(A\beta - B\alpha) < \beta^2\overline{y}^{1-r} + 2B\beta\overline{y} + B^2\overline{y}^{1+r} \\
\iff & rA\beta - rB\alpha < \beta^2\overline{y}^{1-r} + 2B\beta\overline{y} + B^2\overline{y}^{1+r} \\
\iff & rA\beta < B(r\alpha + 2\beta\overline{y} + B\overline{y}^{r+1}) + \beta^2\overline{y}^{1-r},
\end{aligned}$$

which is true by using (IV), then the result follows. Thus the proof of the theorem is completed. □

Define the following function

$$F(y_n, y_{n-1}, \dots, y_{n-k}) = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}. \tag{3.1}$$

Then

$$\frac{\partial F}{\partial y_{n-i}} = \frac{r y_{n-i}^{r-1}[(\alpha_i \beta - \alpha \beta_i) + \sum_{j=0,\, j\neq i}^{k} (\alpha_i \beta_j - \alpha_j \beta_i) y_{n-i}^r]}{(\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r)^2}, \quad i = 0, 1, \dots.$$

Thus the linearised equation of Eq. (1.1) about the equilibrium point $\overline{y}$ is the linear difference equation

$$w_{n+1} - \frac{r\overline{y}^{r-1}}{(\beta + B\overline{y}^r)^2} \sum_{i=0}^{k} [(\alpha_i \beta - \alpha \beta_i) + \overline{y}^r \sum_{j=0}^{k} (\alpha_i \beta_j - \alpha_j \beta_i)] w_{n-i} = 0,$$

whose characteristic equation is

$$\phi^{n+1} - \frac{r\overline{y}^{r-1}}{(\beta + B\overline{y}^r)^2} \sum_{i=0}^{k} [(\alpha_i \beta - \alpha \beta_i) + \overline{y}^r \sum_{j=0}^{k} (\alpha_i \beta_j - \alpha_j \beta_i)] \phi^{n-i} = 0.$$

Then it follows by Theorem 1.3 that the equilibrium point $\overline{y}$ of Eq. (1.1) is locally asymptotically stable if

$$r\overline{y}^r \sum_{i=0}^{k} \left| (\alpha_i \beta - \alpha \beta_i) + \overline{y}^r \sum_{j=0}^{k} (\alpha_i \beta_j - \alpha_j \beta_i) \right| < (\alpha + A\overline{y}^r)(\beta + B\overline{y}^r).$$

*Remark* 3.2. For any partial order of the quotients $\frac{\alpha}{\beta}, \frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_k}{\beta_k}$, the function $F(y_n, y_{n-1}, \dots, y_{n-k})$ defined by relation (3.1) has the monotonicity character in some of its arguments.

Define the set $T = \{t \in \{0, 1, \dots, k\} : F \text{ is increasing in its argument } x_{n-t}\}$ and the set $J = \{j \in \{0, 1, \dots, k\} : F \text{ is decreasing in its argument } x_{n-j}\}$. Also assume the two quantities $G = \sum_{t\in T} \alpha_t$ and $L = \sum_{j\in J} \beta_j$.

**Theorem 3.3.** *Every solution of Eq.* (1.1) *is globally asymptotically stable if one of the following holds*

(1.) $A > 2G$ *and* $B > 2L$.

(2.) $A < 2G$, $B < 2L$, *and*

$$\beta + L(\gamma^r + \overline{y}^r) > (2G - A)\delta^{r-1} + \left(2L - B + \frac{2G - A}{\overline{y}}\right) \sum_{j=0}^{r-1} \overline{y}^{2r-i} \delta^i. \tag{3.2}$$

(3.) $A < 2G$, $B > 2L$, *and*

$$\beta + L(\gamma^r + \overline{y}^r) + (B - 2L) \sum_{i=0}^{r-1} \gamma^{r-i} \overline{y}^i > (2G - A) \left( \delta^{r-1} + \sum_{i=0}^{r-1} \overline{y}^{r-i} \delta^i \right).$$

(4.) $A > 2G$, $B < 2L$, *and*

$$\beta + L(\gamma^r + \overline{y}^r) + (A - 2G) \left( \delta^{r-1} + \sum_{i=0}^{r-1} \gamma^{r-i} \overline{y}^i \right) > (2L - B) \sum_{i=0}^{r-1} \overline{y}^{r-i} \delta^i.$$

*Proof.* Assume that $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1). Observe that it was proven in Theorem 2.1 that every solution of Eq. (1.1) is bounded and therefore it follows by Theorem 1.4 (method of full limiting sequences [18]) that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq. (1.1) with

$$\gamma \leqslant I = I_0 = \lim_{n \longrightarrow \infty} \inf y_n \leqslant \lim_{n \longrightarrow \infty} \sup y_n = S_0 = S \leqslant \delta,$$

where

$$I_n, S_n \in [I, S], \quad n = \dots, -1, 0, 1, \dots .$$

Now since $S \geqslant I$, it suffices to show that $I \geqslant S$. We obtain from Eq. (1.1) that

$$I = I_0 = \frac{\alpha + \alpha_0 I_{-1}^r + \alpha_1 I_{-2}^r + \cdots + \alpha_k I_{-k}^r}{\beta + \beta_0 I_{-1}^r + \beta_1 I_{-2}^r + \cdots + \beta_k I_{-k}^r} \geqslant \frac{\alpha + GI^r + HS^r}{\beta + KI^r + LS^r},$$

where $H = A - G$ and $K = B - L$. Then we obtain

$$\alpha + GI^r + HS^r \leqslant \beta I + KI^{r+1} + LIS^r,$$

or equivalently

$$\alpha \leqslant \beta I + KI^{r+1} + LIS^r - GI^r - HS^r. \tag{3.3}$$

Similarly it is easy to see that

$$\alpha \geqslant \beta S + KS^{r+1} + LI^r S - GS^r - HI^r. \tag{3.4}$$

Therefore it follows from Eqs. (3.3) and (3.4) that

$$\beta S + KS^{r+1} + LI^r S - GS^r - HI^r \leqslant \beta I + KI^{r+1} + LIS^r - GI^r - HS^r,$$

or

$$\beta(I - S) + K(I^{r+1} - S^{r+1}) - LIS(I^{r-1} - S^{r-1}) - (G - H)(I^r - S^r) \geqslant 0,$$

or equivalently

$$\begin{aligned} \beta(I - S) + K(I^r + I^{r-1}S + \cdots + IS^{r-1} + S^r) - LIS(I^{r-2} + I^{r-3}S \\ + I^{r-4}S^2 + \cdots + IS^{r-3} + S^{r-2}) - (G - H)(I^{r-1} + I^{r-2}S + \cdots + IS^{r-2} + S^{r-1}) \geqslant 0, \end{aligned}$$

or

$$(I - S)\left[\beta + K(I^r + S^r) + (H - G)S^{r-1} + (K - L - \frac{G - H}{S})(I^{r-1}S + I^{r-2}S^{r-1} + \cdots + IS^{r-1}) \geqslant 0.\right.$$

Note that $H = A - G$ and $K = B - L$, then

$$(I - S)\left[\beta + L(I^r + S^r) + (A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i\right] \geqslant 0. \tag{3.5}$$

Now if (3.2) is true then we obtain that

$$\beta + L(I^r + S^r) + (A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i > 0.$$

Thus it follows from (3.5) that $I \geqslant S$ and this completes the proof of (1.). Note that $\gamma \leqslant I \leqslant \overline{y} \leqslant S \leqslant \delta$, therefore we see that

$$\beta + L(\gamma^r + \overline{y}^r) < \beta + L(I^r + S^r), \tag{3.6}$$

and

$$(A-2G)S^{r-1}+\left(B-2L+\frac{A-2G}{S}\right)\sum_{i=0}^{r-1}I^{r-i}S^{i}<(2G-A)\delta^{r-1}+(2L-B+\frac{2G-A}{\overline{y}})\sum_{i=0}^{r-1}\overline{y}^{r-i}\delta^{i}. \qquad (3.7)$$

Then we get from (3.2), (3.6), and (3.7) that

$$\beta+L(I^{r}+S^{r})+(A-2G)S^{r-1}+\left(B-2L+\frac{A-2G}{S}\right)\sum_{i=0}^{r-1}I^{r-i}S^{i}>0.$$

Thus it follows again from (3.5) that $I \geqslant S$ and this completes the proof of (2.).

The proofs of cases (3.) and (4.) are similar to the proofs of the previous two cases and will be omitted. Hence, the proof of the theorem is completed. □

## 4. Existence of periodic solutions of Eq. (1.1)

In this section, we investigate the existence of periodic solutions of prime period two of Eq. (1.1). In fact to achieve the existence of periodic solutions of Eq. (1.1) we need some very complicated computations. So we consider the case whenever $r=1$. The cases when $r>1$ are similar. Let $D=\sum_{i=0,i\text{ odd}}^{k}\alpha_i$, $E=\sum_{j=0,j\text{ even}}^{k}\alpha_j$, $F=\sum_{i=0,i\text{ odd}}^{k}\beta_i$, and $R=\sum_{j=0,j\text{ even}}^{k}\beta_j$.

**Theorem 4.1.** *Assume that $r=1$, $D>E+\beta$, and $R>F$, then Eq.* (1.1) *has periodic solutions of prime period two if and only if*

$$(R-F)(D-E-\beta)^2>4F[\alpha F+E(D-E-\beta)]. \qquad (4.1)$$

*Proof.* First suppose that there exists a periodic solution $\{\ldots,\phi,\psi,\phi,\psi,\ldots\}$ of Eq. (1.1), where $\phi$ and $\psi$ are distinct positive real numbers. Then it follows from Eq. (1.1) that $\phi$, $\psi$ satisfy the following

$$\phi=\frac{\alpha+D\phi^{r}+E\psi^{r}}{\beta+F\phi^{r}+R\psi^{r}} \quad\text{and}\quad \psi=\frac{\alpha+D\psi^{r}+E\phi^{r}}{\beta+F\psi^{r}+R\phi^{r}},$$

which are equivalent to

$$\beta\phi+F\phi^{r+1}+R\phi\psi^{r}=\alpha+D\phi^{r}+E\psi^{r}, \qquad (4.2)$$

and

$$\beta\psi+F\psi^{r+1}+R\phi^{r}\psi=\alpha+D\psi^{r}+E\phi^{r}. \qquad (4.3)$$

Subtracting (4.3) from (4.2) gives

$$\beta(\phi-\psi)+F(\phi^{r+1}-\psi^{r+1})+R\phi\psi(\psi^{r-1}-\phi^{r-1})=(D-E)(\phi^{r}-\psi^{r}).$$

Wherever $r=1$ we see that

$$\beta(\phi-\psi)+F(\phi-\psi)(\phi+\psi)=(D-E)(\phi-\psi).$$

Since $\phi\neq\psi$, we have that

$$\phi+\psi=\frac{D-E-\beta}{F}.$$

By adding (4.2) and (4.3) we obtain

$$\beta(\phi+\psi)+F(\phi^2+\psi^2)+2R\phi\psi=2\alpha+(D+E)(\phi+\psi),$$

and therefore

$$\phi\psi=\frac{\alpha F+E(D-E-\beta)}{F(R-F)}.$$

Thus $\phi$ and $\psi$ are the roots of the following quadratic equation from which

$$u^2 - \frac{D-E-\beta}{F}u + \frac{\alpha F + E(D-E-\beta)}{F(R-F)} = 0. \tag{4.4}$$

Again, since $\phi \neq \psi$, we obtain

$$\left(\frac{D-E-\beta}{F}\right)^2 > \frac{4[\alpha F + E(D-E-\beta)]}{F(R-F)},$$

which implies that $(R-F)(D-E-\beta)^2 > 4F[\alpha F + E(D-E-\beta)]$. Thus Eq. (4.1) holds.

Secondly suppose that the condition (4.1) is true. We will show that Eq. (1.1) has positive prime period two solutions. Now assume that $k$ is odd (the case wherever $k$ is even is similar and will be left to the reader) and choose

$$x_{-k} = \cdots = x_{-3} = x_{-1} = \phi = \frac{\frac{D-E-\beta}{F} + \sqrt{\left(\frac{D-E-\beta}{F}\right)^2 - \frac{4[\alpha F+E(D-E-\beta)]}{F(R-F)}}}{2},$$

and

$$x_{-k+1} = \cdots = x_{-2} = x_0 = \psi = \frac{\frac{D-E-\beta}{F} - \sqrt{\left(\frac{D-E-\beta}{F}\right)^2 - \frac{4[\alpha F+E(D-E-\beta)]}{F(R-F)}}}{2}.$$

It is easy by direct substitution in Eq. (1.1) to prove that

$$x_1 = x_{-1} \quad \text{and} \quad x_2 = x_0.$$

Then it follows by mathematical induction that

$$x_{2n} = \phi \quad \text{and} \quad x_{2n+1} = \psi \quad \text{for all} \quad n \geqslant -1.$$

Thus Eq. (1.1) has the positive prime period two solution

$$\ldots, \phi, \psi, \phi, \psi, \ldots,$$

where $\phi$ and $\psi$ are the distinct roots of the quadratic Eq. (4.4) and the proof is completed. □

## 5. Conclusion

This paper investigated the qualitative properties of solutions for a specific difference equation, characterized by a complex interplay of coefficients and terms. The exploration includes the identification of equilibrium points, their classification in relation to local stability, and an in-depth analysis of the boundedness and global stability of solutions. The investigation extends to the intriguing realm of periodic solutions for the considered equation. Through rigorous examination, the study sheds light on the dynamic behavior of the system governed by the given difference equation. The findings contribute to our understanding of discrete-time processes, offering valuable insights into the stability and periodicity aspects of solutions within the specified parameter space. This research not only expands the theoretical foundation of the discussed difference equation but also provides a basis for further exploration and application in diverse fields where discrete modeling plays a crucial role. In the future we hope that we or others study the considered equation with real coefficients not only with positive ones.

## Author's contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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