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On the superstability of the p-power-radical sine functional equation related to Pexider type

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Abstract

In this paper, we investigate the superstability bounded by a function (Găvruta sense) for the p-power-radical sine functional equation from the p-power-radical Pexider type's functional equation:

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = h(\sqrt[p]{x})k(\sqrt[p]{y}),$$

where p is an odd positive integer and f, g, h, k are complex valued functions on $\mathbb{R}$. Furthermore, the obtained results are extended to Banach algebras.

Keywords: Stability, superstability, sine functional equation, p-radical functional equation, p-power-radical functional equation.

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1. Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [40]. Next year, Hyers [14] obtained a partial answer for the case of the additive mapping in this problem: if f satisfies $|f(x+y) - f(x) - f(y)| \leqslant \varepsilon$ for some fixed $\varepsilon > 0$, then f satisfies the additive mapping $f(x+y) = f(x) + f(y)$, which is called the Hyers-Ulam stability.

Thereafter this problem was improved by Bourgin [9] in 1949, Aoki [4] in 1950, Rassias [37] in 1978, and Găvruta [11].

In 1979, Baker et al. in [8] postulated that if f satisfies the inequality $|E_1(f) - E_2(f)| \leqslant \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is referred to as *superstability*. For more information on the stability of functional equations, see [1, 2, 15, 16, 33–36, 39].

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Baker [7] showed the superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{C}$$

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$\begin{aligned}
f(x+y)+f(x-y)&=2f(x)g(y), && (W)\\
f(x+y)+f(x-y)&=2g(x)f(y), && (K_{gf})\\
f(x+y)+f(x-y)&=2g(x)h(y),\\
f(x+y)-f(x-y)&=2f(x)f(y),\\
f(x+y)-f(x-y)&=2f(x)g(y),\\
f(x+y)-f(x-y)&=2g(x)f(y),\\
f(x+y)-f(x-y)&=2g(x)h(y),
\end{aligned}$$

in which (W) is called the Wilson equation, and all other equations have been raised in Kim's papers [20, 25].

The superstability of the cosine (C), Wilson (W), and Kim (K_{gf}) functional equations was founded in Badora [5], Ger [6], Kannappan [18], and Kim [20, 22, 25, 29] and in [17, 38].

In 1983, Cholewa [10] investigated the superstability of the sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2=f(x)f(y) \tag{S}$$

under the condition bounded by constant. This was improved to the condition bounded by a function in Badora and Ger [6]. This was also improved by Kim [21, 25], which are the superstability of the generalized sine functional equations

$$f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2=f(x)g(y), \qquad f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2=g(x)f(y),$$
$$f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2=g(x)g(y), \qquad f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2=g(x)h(y),$$

under the condition bounded by a constant or a function.

In 2009, Eshaghi Gordji and Parviz [12] introduced the radical functional equation

$$f\big(\sqrt{x^2+y^2}\big)=f(x)+f(y)$$

related to the quadratic function.

Almahalebi et al. [3] and Kim [27] obtained the superstability of p-radical functional equations related to Wilson equation (W), Kim's equation (K_{gf}), and $f(x+y)\pm f(x-y)=\lambda g(x)k(y)$.

The sine functional equation (S) is dued by the sine formula: $\big(\sin\frac{x+y}{2}\big)^2-\big(\sin\frac{x-y}{2}\big)^2=\sin x\sin y$. Similarly, the trigonometric formula: $\big(\sin\frac{x+y}{2}\big)^2-\big(\cos\frac{x-y}{2}\big)^2=-\cos x\cos y$ gives rise to the functional equation

$$f\left(\frac{x+y}{2}\right)^2-g\left(\frac{x-y}{2}\right)^2=h(x)k(y). \tag{S_{fghk}}$$

It means that (S_{fghk}) has the solution (f,g,h,k) with $f(x)=\sin x$, $g(x)=\cos x$, and $(h,k)\in\{(g,-g),(-g,g)\}$. Moreover, (S) and ($S_{fghk}$) are represented by the exponential function and the hyperbolic function.

Namely, (S) and (S_{fghk}) have simultaneously, as solution, an exponential function and sine function, an exponential function and hyperbolic function, respectively.

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+=[0,\infty)$, and $\mathbb{C}$ be the field of complex numbers.

1) Note that, for all $x, y \in \mathbb{C}$,

$$\begin{aligned} f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 &= \sin\left(\frac{x+y}{2}\right)^2 - \sin\left(\frac{x-y}{2}\right)^2 \\ &= \left(\frac{1}{2i}\left(e^{i\frac{x+y}{2}} - e^{-i\frac{x+y}{2}}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i\frac{x-y}{2}} - e^{-i\frac{x-y}{2}}\right)\right)^2 \\ &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{iy}-e^{-iy}}{2i}\right) = \sin(x)\sin(y) = f(x)f(y). \end{aligned}$$

So (S) has a solution as the sine function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x) = \sin x$ for all $x \in \mathbb{C}$. For a function $f : \mathbb{C} \to \mathbb{C}$ and a constant $a \in \mathbb{C} \setminus \{0\}$, consider the function $g : \mathbb{C} \to \mathbb{C}$ given by $g(x) := af(x)$ for all $x \in \mathbb{C}$. Then f satisfies (S) if and only if g satisfies (S). Since $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ for all $x \in \mathbb{C}$, (S) also has simultaneously the solution as the sum of the exponential functions, i.e., it has the solution $g(x) = (e^{ix} - e^{-ix})/2i$.

2) Note that $(\sin(x+y))^2 - (\cos(x-y))^2 = (\sin^2 - \cos^2)(x)(\cos^2 - \sin^2)(y)$. Since $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ for all $x \in \mathbb{C}$, we know that

$$\begin{aligned} f\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 &= \sin(x+y)^2 - \cos(x-y)^2 \\ &= \left(\frac{1}{2i}\left(e^{i(x+y)} - e^{-i(x+y)}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i(x-y)} + e^{-i(x-y)}\right)\right)^2 \\ &= \left(\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 - \left(\frac{e^{ix}+e^{-ix}}{2i}\right)^2\right)\left(\left(\frac{e^{iy}+e^{-iy}}{2i}\right)^2 - \left(\frac{e^{iy}-e^{-iy}}{2i}\right)^2\right) \\ &= (\sin^2 - \cos^2)(x)(\cos^2 - \sin^2)(y) = h(x)k(y). \end{aligned}$$

So (S_{fghk}) has a solution as the 4-tuple of the trigonometric functions $f(x) = \sin(2x)$, $g(x) = \cos(2x)$, $h(x) = (\sin^2 - \cos^2)(x)$, $k(x) = (\cos^2 - \sin^2)(x)$. For four functions $f, g, h, k : \mathbb{C} \to \mathbb{C}$ and a constant $a \in \mathbb{C} \setminus \{0\}$, consider the functions $F, G, H, K : \mathbb{C} \to \mathbb{C}$ given by $F(x) = af(x), G(x) = ag(x), H(x) = ah(x), K(x) = ak(x)$ for all $x \in \mathbb{C}$. Then (f, g, h, k) satisfies (S_{fghk}) if and only if (F, G, H, K) satisfies (S_{fghk}). (S_{fghk}) also has simultaneously the solution as the 4-tuple of the four sums of exponential functions, i.e., it has the solution (f, g, h, k) with $f(x/2) = \left(e^{ix} - e^{-ix}\right)/2i$, $g(x/2) = \left(e^{ix} + e^{-ix}\right)/2i$, $h(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 - \left(\frac{e^{ix}+e^{-ix}}{2i}\right)^2$, $k(x) = \left(\frac{e^{iy}+e^{-iy}}{2i}\right)^2 - \left(\frac{e^{iy}-e^{-iy}}{2i}\right)^2$, and it has the solution (F, G, H, K) with $F(x/2) = \left(e^{ix} - e^{-ix}\right)$, $G(x/2) = \left(e^{ix} + e^{-ix}\right)$, $H(x) = \left(e^{ix} - e^{-ix}\right)^2 - \left(e^{ix} + e^{-ix}\right)$, $K(x) = \left(e^{iy} + e^{-iy}\right)^2 - \left(e^{iy} - e^{-iy}\right)^2$.

Since $\sin x = -i\sinh(ix)$ and $\cos x = \cosh(ix)$, (S_{fghk}) also has simultaneously as the 4-tuple of the four hyperbolic functions $f(x) = -i\sinh(ix)$, $g(x) = \cosh(ix)$, $h(x) = -\cosh(ix)$, $k(x) = \cosh(ix)$.

3) Note that $\left(\sinh\frac{x+y}{2}\right)^2 - \left(\sinh\frac{x-y}{2}\right)^2 = \frac{\cosh(x+y)-1}{2} - \frac{\cosh(x-y)-1}{2} = \sinh x \sinh y$ for all $x, y \in \mathbb{C}$. So, (S) has a solution as the hyperbolic sine function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x) = \sinh x$ for all $x \in \mathbb{C}$. Since $\sinh x = \frac{e^x - e^{-x}}{2}$ for all $x \in \mathbb{C}$, (S) also has simultaneously the solution as the sum of the exponential functions, i.e., it has the solution $g(x) := e^x - (-e^{-x}) = f(x)/2$. Namely, we see as following: i.e.,

$$\begin{aligned} f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 &= \sinh\left(\frac{x+y}{2}\right)^2 - \sinh\left(\frac{x-y}{2}\right)^2 \\ &= \left(\frac{1}{2}\left(e^{\frac{x+y}{2}} - e^{-\frac{x+y}{2}}\right)\right)^2 - \left(\frac{1}{2}\left(e^{\frac{x-y}{2}} - e^{-\frac{x-y}{2}}\right)\right)^2 \\ &= \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)\left(\frac{1}{2}\left(e^y - e^{-y}\right)\right) = \sinh(x)\sinh(y) = f(x)f(y). \end{aligned}$$

Although all functional equations mentioned, may have arisen from sine or cosine, as shown in the previous, they have simultaneously solutions as the sine, hyperbolic sine, the exponential function.

In the concept of the p-radical, (S) is expressed as follows:

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = f(\sqrt[p]{x})f(\sqrt[p]{y}). \qquad (S^r)$$

In (S^r), applying $f(x) = F(x^p)$, then F satisfies the sine functional equation (S). Namely, $f : \mathbb{R} \to \mathbb{C}$ satisfies (S^r) if and only if $F : \mathbb{R} \to \mathbb{C}$ satisfies (S), where $F(x) = f(\sqrt[p]{x})$. Letting the p-power function $f(x) = x^p$ in (S^r), then we obtain

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy = f(\sqrt[p]{x})f(\sqrt[p]{y}).$$

This means that (S^r) has a solution as the p-power function.

Recently, Kim [27, 28, 30, 32] obtained the superstability for the p-radical functional equations related to the sine function. The target equation (S_{fghk}) in the concept of the p-radical is expressed as

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = h(\sqrt[p]{x})k(\sqrt[p]{y}), \qquad (S^r_{fghk})$$

where p is a positive odd integer and f, g, h, k are complex valued functions on $\mathbb{R}$.

Since the function $f(x) = \sin x^p$ is the solution of the equation (S^r), in this paper, this equation is reasonably called the p-power-radical sine functional equation. Also, the equation (S^r_{fghk}) will be called the p-power-radical functional equation of Pexider type.

The aim of this paper is to investigate the superstability bounded by a function (Găvruta sense) for the p-power-radical functional equation (S^r) from the p-power-radical functional equation (S^r_{fghk}) of Pexider type. Applying our target equation (S^r_{fghk}) with $f(x) = F(x^p), g(x) = G(x^p), h(x) = H(x^p)$, and $k(x) = K(x^p)$, it arises (S_{fghk}). Namely, f, g, h, k satisfy (S^r_{fghk}) if and only if F, G, H, K satisfy (S_{fghk}), where $F(x) = f(\sqrt[p]{x}), G(x) = g(\sqrt[p]{x}), H(x) = h(\sqrt[p]{x})$, and $K(x) = k(\sqrt[p]{x})$.

Furthermore, the obtained results are extended to Banach algebras. As a corollary, we obtain the superstability bounded by a constant (Hyers sense) and a function (Găvruta sense) for the p-power-radical sine functional equation from the p-power-radical functional equations:

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = h(\sqrt[p]{x})f(\sqrt[p]{y}), \qquad (S^r_{fghf})$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = h(\sqrt[p]{x})g(\sqrt[p]{y}), \qquad (S^r_{fghg})$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = h(\sqrt[p]{x})h(\sqrt[p]{y}), \qquad (S^r_{fghh})$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = f(\sqrt[p]{x})h(\sqrt[p]{y}),$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = g(\sqrt[p]{x})h(\sqrt[p]{y}),$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = f(\sqrt[p]{x})f(\sqrt[p]{y}), \qquad (S^r_{fgff})$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = g(\sqrt[p]{x})g(\sqrt[p]{y}), \tag{S^r_{fggg}}$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = f(\sqrt[p]{x})g(\sqrt[p]{y}), \tag{S^r_{fgfg}}$$

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 = g(\sqrt[p]{x})f(\sqrt[p]{y}). \tag{S^r_{fggf}}$$

In this paper, we assume that f, g, h, k are nonzero functions, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function, and p is a positive odd integer. The following equations will be used for notations:

$$f(\sqrt[p]{x+y}) + f(\sqrt[p]{x-y}) = 2f(\sqrt[p]{x})f(\sqrt[p]{y}), \tag{C^r}$$

$$f(\sqrt[p]{x+y}) + f(\sqrt[p]{x-y}) = 2f(\sqrt[p]{x})g(\sqrt[p]{y}), \tag{W^r}$$

$$f(\sqrt[p]{x+y}) - f(\sqrt[p]{x-y}) = 2g(\sqrt[p]{x})f(\sqrt[p]{y}). \tag{T^r_{gf}}$$

2. Superstability of the functional equation (S^r) from an approximate equation of (S^r_{fghk})

In this section, we investigate the superstability of the p-power-radical sine functional equation (S^r) from an approximate equation of the p-power-radical functional equation (S^r_{fghk}) of Pexider type related to (S).

Theorem 2.1. *Assume that* $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})k(\sqrt[p]{y}) \right| \leqslant \varphi(y) \tag{2.1}$$

for all $x, y \in \mathbb{R}$. *Suppose that* $k(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *for all* $x \in \mathbb{R}$. *Then either* h *is bounded or* k *satisfies* (S^r). *In addition, if* h *satisfies* (C^r), *then* k *and* h *satisfy*

$$(T^r_{gf}) := k(\sqrt[p]{x+y}) - k(\sqrt[p]{x-y}) = 2h(\sqrt[p]{x})k(\sqrt[p]{y})$$

for all $x, y \in \mathbb{R}$.

Proof. Inequality (2.1) may equivalently be written as

$$|f(\sqrt[p]{x+y})^2 - g(\sqrt[p]{x-y})^2 - h(\sqrt[p]{2x})k(\sqrt[p]{2y})| \leqslant \varphi(2y) \tag{2.2}$$

for all $x, y \in \mathbb{R}$. Let h be unbounded. Then we can choose a sequence $\{x_n\}$ in $\mathbb{R}$ such that

$$0 \neq \left|h\left(\sqrt[p]{2x_n}\right)\right| \to \infty \quad \text{as} \quad n \to \infty. \tag{2.3}$$

Taking $x = x_n$ in (2.2), we obtain

$$\left| \frac{f(\sqrt[p]{x_n+y})^2 - g(\sqrt[p]{x_n-y})^2}{h(\sqrt[p]{2x_n})} - k(\sqrt[p]{2y}) \right| \leqslant \frac{\varphi(2y)}{|h(\sqrt[p]{2x_n})|}$$

for all $y \in \mathbb{R}$ and all $n \in \mathbb{N}$. By the above inequality, using (2.3), we obtain

$$k\left(\sqrt[p]{2y}\right) = \lim_{n\to\infty} \frac{f(\sqrt[p]{x_n+y})^2 - g(\sqrt[p]{x_n-y})^2}{h(\sqrt[p]{2x_n})} \tag{2.4}$$

for all $y \in \mathbb{R}$. Replacing x by $2x_n + x$ and $2x_n - x$ in (2.1), we have

$$\begin{aligned} 2\varphi(y) &\geqslant \left| h(\sqrt[p]{2x_n + x})k(\sqrt[p]{y}) - f\left(\sqrt[p]{\frac{2x_n + x + y}{2}}\right)^2 + g\left(\sqrt[p]{\frac{2x_n + x - y}{2}}\right)^2 \right| \\ &\quad + \left| h(\sqrt[p]{2x_n - x})k(\sqrt[p]{y}) - f\left(\sqrt[p]{\frac{2x_n - x + y}{2}}\right)^2 + g\left(\sqrt[p]{\frac{2x_n - x - y}{2}}\right)^2 \right| \\ &\geqslant \left| \left(h(\sqrt[p]{2x_n + x}) + h(\sqrt[p]{2x_n - x})\right) k(\sqrt[p]{y}) - \left(f\left(\sqrt[p]{x_n + \frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{x_n - \frac{x+y}{2}}\right)^2 \right) \right. \\ &\quad \left. - \left(f\left(\sqrt[p]{x_n + \frac{-x+y}{2}}\right)^2 - g\left(\sqrt[p]{x_n - \frac{-x+y}{2}}\right)^2 \right) \right| \end{aligned} \tag{2.5}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$. Consequently, we obtain

$$\begin{aligned} \frac{2\varphi(y)}{|h(\sqrt[p]{2x_n})|} &\geqslant \left| \frac{h(\sqrt[p]{2x_n + x}) + h(\sqrt[p]{2x_n - x})}{h(\sqrt[p]{2x_n})} k(\sqrt[p]{y}) - \frac{f\left(\sqrt[p]{x_n + \frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{x_n - \frac{x+y}{2}}\right)^2}{h(\sqrt[p]{2x_n})} \right. \\ &\quad \left. - \frac{f\left(\sqrt[p]{x_n + \frac{-x+y}{2}}\right)^2 - g\left(\sqrt[p]{x_n - \frac{-x+y}{2}}\right)^2}{h(\sqrt[p]{2x_n})} \right| \end{aligned} \tag{2.6}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ with the use of (2.4) and (2.6), we conclude that, for every $x \in \mathbb{R}$, there exists the limit function

$$L_1(\sqrt[p]{x}) := \lim_{n\to\infty} \frac{h(\sqrt[p]{2x_n + x}) + h(\sqrt[p]{2x_n - x})}{h(\sqrt[p]{2x_n})},$$

where $L_1 : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation

$$k(\sqrt[p]{x+y}) + k(\sqrt[p]{-x+y}) = L_1(\sqrt[p]{x})k(\sqrt[p]{y}) \tag{2.7}$$

for all $x, y \in \mathbb{R}$. First, let us consider the case $k(0) = 0$. Then it forces by (2.7) that k is odd. So (2.7) implies

$$k(\sqrt[p]{x+y}) - k(\sqrt[p]{x-y}) = L_1(\sqrt[p]{x})k(\sqrt[p]{y}) \tag{2.8}$$

for all $x, y \in \mathbb{R}$. Keeping this in mind, by means of (2.8), we infer the equality

$$\begin{aligned} k(\sqrt[p]{x+y})^2 - k(\sqrt[p]{x-y})^2 &= \left[k(\sqrt[p]{x+y}) + k(\sqrt[p]{x-y})\right] L_1(\sqrt[p]{x})k(\sqrt[p]{y}) \\ &= \left[k(\sqrt[p]{2x+y}) + k(\sqrt[p]{2x-y})\right] k(\sqrt[p]{y}) \\ &= \left[k(\sqrt[p]{y+2x}) - k(\sqrt[p]{y-2x})\right] k(\sqrt[p]{y}) \\ &= L_1(\sqrt[p]{y})k(\sqrt[p]{2x})k(\sqrt[p]{y}) \end{aligned} \tag{2.9}$$

for all $x, y \in \mathbb{R}$. The oddness of k forces it vanish at 0. Putting $x = y$ in (2.8) we conclude with the above result that

$$k(\sqrt[p]{2y}) = L_1(\sqrt[p]{y})k(\sqrt[p]{y})$$

for all $y \in \mathbb{R}$. The equation (2.9), in return, leads to the equation

$$k(\sqrt[p]{x+y})^2 - k(\sqrt[p]{x-y})^2 = k(\sqrt[p]{2x})k(\sqrt[p]{2y})$$

valid for all $x, y \in \mathbb{R}$, which, in the light of the unique $\sqrt[r]{2}$-divisibility of $\mathbb{R}$, states nothing else but (S^r). In addition, if h satisfies (C^r), then L_1 forces $2h$ and so (2.8) forces that k and h satisfy (T^r_{gf}) with $f = k$ and $g = h$.

For the next case $f(\sqrt[r]{x})^2 = g(\sqrt[r]{x})^2$ for all $x \in \mathbb{R}$, it is enough to show that $k(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $k(0) = c$: constant. Putting $y = 0$ in (2.1), from the above assumption, we obtain the inequality

$$|h(x)| \leqslant \frac{\varphi(0)}{c}$$

for all $x \in \mathbb{R}$. This inequality means that h is globally bounded, which is a contradiction to the unboundedness assumption. Thus $k(0) = 0$ holds, and so the proof is completed. □

Theorem 2.2. *Assume that* $f, g, h, k : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[r]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[r]{\frac{x-y}{2}}\right)^2 - h(\sqrt[r]{x})k(\sqrt[r]{y}) \right| \leqslant \varphi(x) \tag{2.10}$$

for all $x, y \in \mathbb{R}$*. Suppose that* $h(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{-x})^2$ *for all* $x \in \mathbb{R}$*. Then, either* k *is bounded or* h *satisfies* (S^r)*. In addition, if* k *satisfies* (C^r)*, then* h *and* k *satisfy the Wilson equation*

$$(W^r) := h\left(\sqrt[r]{x+y}\right) + h\left(\sqrt[r]{x-y}\right) = 2h(\sqrt[r]{x})k(\sqrt[r]{y})$$

for all $x, y \in \mathbb{R}$.

Proof. Let k be unbounded. Then we can choose a sequence $\{y_n\}$ in $\mathbb{R}$ such that $\left|k\left(\sqrt[r]{2y_n}\right)\right| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in the start of Theorem 2.1 gives us

$$h\left(\sqrt[r]{2x}\right) = \lim_{n\to\infty} \frac{f\left(\sqrt[r]{x+y_n}\right)^2 - g\left(\sqrt[r]{x-y_n}\right)^2}{k(\sqrt[r]{2y_n})} \tag{2.11}$$

for all $x \in \mathbb{R}$. Replacing y by $y + 2y_n$ and $-y + 2y_n$ in (2.10), by the same procedure of (2.5) and (2.6) with (2.11), one states the existence of a limit function

$$L_2(\sqrt[r]{y}) := \lim_{n\to\infty} \frac{k(\sqrt[r]{y+2y_n}) + k(\sqrt[r]{-y+2y_n})}{k(\sqrt[r]{2y_n})}$$

for all $y \in \mathbb{R}$, where $L_2 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$h(\sqrt[r]{x+y}) + h(\sqrt[r]{x-y}) = h(\sqrt[r]{x})L_2(\sqrt[r]{y}) \tag{2.12}$$

for all $x, y \in \mathbb{R}$. From the definition of L_2, it is even and $L_2(0) = 2$. First, let us consider the case $h(0) = 0$. Then it forces by (2.12) that h is odd. Putting $y = x$ in (2.12), we get

$$h\left(\sqrt[r]{2x}\right) = h\left(\sqrt[r]{x}\right)L_2\left(\sqrt[r]{x}\right) \tag{2.13}$$

for all $x \in \mathbb{R}$. From (2.12), the oddness of h and (2.13), for all $x, y \in \mathbb{R}$, we obtain that

$$\begin{aligned} h(\sqrt[r]{x+y})^2 - h(\sqrt[r]{x-y})^2 &= h(\sqrt[r]{x})L_2(\sqrt[r]{y})[h(\sqrt[r]{x+y}) - h(\sqrt[r]{x-y})] \\ &= h(\sqrt[r]{x})\left[h\left(\sqrt[r]{x+2y}\right) - h\left(\sqrt[r]{x-2y}\right)\right] \\ &= h(\sqrt[r]{x})\left[h\left(\sqrt[r]{2y+x}\right) + h\left(\sqrt[r]{2y-x}\right)\right] \\ &= h(\sqrt[r]{x})h\left(\sqrt[r]{2y}\right)L_2(\sqrt[r]{x}) = h\left(\sqrt[r]{2x}\right)h\left(\sqrt[r]{2y}\right), \end{aligned}$$

which with $\sqrt[p]{2}$-divisibility of $\mathbb{R}$, states nothing else but (S^r).

For the next case $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$, it is enough to show that $h(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $h(0) = c$: constant. Putting $x = 0$ in (2.10), from the above assumption, we obtain the inequality

$$|k(y)| \leqslant \frac{\varphi(0)}{c}$$

for all $y \in \mathbb{R}$. This inequality means that k is globally bounded, which is a contradiction to the unboundedness assumption. Thus $h(0) = 0$ holds and so the proof of theorem is completed.

In addition, if k satisfies (C^r), then L_2 forces $2k$ and so (2.12) forces that h and k satisfy (W^r) with $f = h$ and $g = k$. □

From Theorems 2.1 and 2.2, we obtain the following result as a corollary.

Corollary 2.3. *Suppose that* $f, g, h, k : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})k(\sqrt[p]{y}) \right| \leqslant \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in \mathbb{R}$. *Then*

(i) *either* h *under the cases* $k(0) = 0$, $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *is bounded or* k *satisfies* (S^r), *in addition, if* h *satisfies* (C^r), *then* k *and* h *satisfy* (T^r_{gf});

(ii) *either* k *under the cases* $h(0) = 0$, $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *is bounded or* h *satisfies* (S^r), *in addition, if* k *satisfies* (C^r), *then* h *and* k *satisfy the Wilson equation* (W^r).

3. Application of the reduced equations

By replacing the functions k and h by f, g, and h in Theorems 2.1 and 2.2 and Corollary 2.3, as corollaries, we obtain the superstability of the p-power-radical sine functional equation (S^r) from the functional equations (S^r_{fghh}), (S^r_{fghf}), (S^r_{fgfk}), (S^r_{fgfg}), (S^r_{fgff}), and (S^r). The proofs follow from Theorems 2.1 and 2.2 and Corollary 2.3 immediately.

Corollary 3.1. *Suppose that* $f, g, h : \mathbb{R} \to \mathbb{C}$ *satisfy, for all* $x, y \in \mathbb{R}$,

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})h(\sqrt[p]{y}) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

which satisfies one of the conditions $h(0) = 0$, $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *for all* $x \in \mathbb{R}$. *Then either* h *is bounded or* h *satisfies* (S^r).

Corollary 3.2. *Suppose that* $f, g, h : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})f(\sqrt[p]{y}) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then*

(i) *either* h *under the cases* $f(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{x})^2$ *is bounded or* f *satisfies* (S^r), *respectively, in addition, if* h *satisfies* (C^r), *then* f *and* h *satisfy*

$$(T^r_{gf}) := f\big(\sqrt[r]{x+y}\big) - f\big(\sqrt[r]{x-y}\big) = 2h\big(\sqrt[r]{x}\big)f\big(\sqrt[r]{y}\big)$$

for all $x, y \in \mathbb{R}$;

(ii) *either* f *under the cases* $h(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{-x})^2$ *is bounded or* h *satisfies* (S^r), *in addition, if* f *satisfies* (C^r), *then* h *and* f *satisfy the Wilson equation*

$$(W^r) := h\big(\sqrt[r]{x+y}\big) + h\big(\sqrt[r]{x-y}\big) = 2h\big(\sqrt[r]{x}\big)f\big(\sqrt[r]{y}\big)$$

for all $x, y \in \mathbb{R}$;

(iii) (i) *and* (ii) *hold.*

Corollary 3.3. *Suppose that* $f, g, h : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[r]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[r]{\frac{x-y}{2}}\right)^2 - f\big(\sqrt[r]{x}\big)h\big(\sqrt[r]{y}\big) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then*

(i) *either* f *under the cases* $h(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{x})^2$ *is bounded or* h *satisfies* (S^r), *respectively, in addition, if* f *satisfies* (C^r), *then* h *and* f *satisfy*

$$(T^r_{gf}) := h\big(\sqrt[r]{x+y}\big) - h\big(\sqrt[r]{x-y}\big) = 2f\big(\sqrt[r]{x}\big)h\big(\sqrt[r]{y}\big)$$

for all $x, y \in \mathbb{R}$;

(ii) *either* h *under the cases* $f(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{-x})^2$ *is bounded or* f *satisfies* (S^r), *in addition, if* h *satisfies* (C^r), *then* f *and* h *satisfy the Wilson equation*

$$(W^r) := f\big(\sqrt[r]{x+y}\big) + f\big(\sqrt[r]{x-y}\big) = 2f\big(\sqrt[r]{x}\big)h\big(\sqrt[r]{y}\big)$$

for all $x, y \in \mathbb{R}$;

(iii) (i) *and* (ii) *hold.*

Corollary 3.4. *Suppose that* $f, g : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\sqrt[r]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[r]{\frac{x-y}{2}}\right)^2 - f(\sqrt[r]{x})g(\sqrt[r]{y}) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then*

(i) *either* f *under the cases* $g(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{x})^2$ *is bounded or* g *satisfies* (S^r), *respectively, in addition, if* f *satisfies* (C^r), *then* g *and* f *satisfy*

$$(T^r_{gf}) := g\big(\sqrt[r]{x+y}\big) - g\big(\sqrt[r]{x-y}\big) = 2f\big(\sqrt[r]{x}\big)g\big(\sqrt[r]{y}\big)$$

for all $x, y \in \mathbb{R}$;

(ii) *either* g *under the cases* $f(0) = 0$ *or* $f(\sqrt[r]{x})^2 = g(\sqrt[r]{-x})^2$ *is bounded or* f *satisfies* (S^r), *in addition, if* g *satisfies* (C^r), *then* f *and* g *satisfy the Wilson equation* (W^r);

(iii) (i) *and* (ii) *hold.*

Corollary 3.5. *Suppose that* $f, g : \mathbb{R} \to \mathbb{C}$ *satisfy, for all* $x, y \in \mathbb{R}$,

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - f(\sqrt[p]{x})f(\sqrt[p]{y}) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

which satisfies one of the cases $f(0) = 0$, $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$, *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *for all* $x \in \mathbb{R}$, *respectively. Then either* f *is bounded or* f *satisfies* (S^r).

Consequently, as a corollary of all mentioned results, we obtain the stability of the p-power-radical sine functional equation (S^r).

Corollary 3.6. *Assume that* $f : \mathbb{R} \to \mathbb{C}$ *satisfies the inequality*

$$\left| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - f(\sqrt[p]{x})f(\sqrt[p]{y}) \right| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then either* f *is bounded or* f *satisfies* (S^r).

Proof. Replacing g, h, k by f in (2.4) and (2.11) in Theorems 2.1 and 2.2, we get $f(0) = 0$. □

Remark 3.7.

(i) As the same process, we obtain similar results of superstability for the following additional functional equations (S^r_{fghg}), (S^r_{fggk}), (S^r_{fggf}), (S^r_{fggg}), and also (S^r_{ffgh}), (S^r_{ffgg}), (S^r_{ffgf}), (S^r_{fffg}). Some of them are found papers [13, 26–28, 30, 32].

(ii) For all of the obtained results and all the results obtained from above (i), by applying $\varphi(x) = \varphi(y) = \varepsilon$ in each stability inequality, each equation could be the superstability result bounded by constant (Hyers' sense).

(iii) By applying "$p = 1$" to all of the p-power-radical functional equations of (i) and (ii), it follows the superstability of the functional equations removed the p-power-radical property. Some of them are founded in Cholewa [10], Badora [5], Badora and Ger [6], Kannappan [18], and Kim [18, 19, 21–25, 29, 31].

4. Extension of the stability results to Banach algebras

All results in Section 2 also can be extended to the stability in Banach algebras. The following theorem is due to Theorems 2.1 and 2.2 and Corollary 2.3. The stability for the remainder all equations also are represented as similar as Theorem 4.1, respectively.

Theorem 4.1. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g, h, k : \mathbb{R} \to E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})k(\sqrt[p]{y}) \right\| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then, for an arbitrary linear multiplicative functional* $x^* \in E^*$,

(i) *either the superposition* $x^* \circ h$ *under the cases* $k(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *is bounded or* k *satisfies* (S^r), *in addition, if* h *satisfies* (C^r), *then* k *and* h *satisfy* (T^r_{gf});

(ii) *either the superposition* $x^* \circ k$ *under the cases* $h(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *is bounded or* h *satisfies* (S^r), *in addition, if* k *satisfies* (C^r), *then* h *and* k *satisfy the Wilson equation* (W^r);

(iii) (i) *and* (ii) *hold.*

Proof. Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known, we have $\|x^*\| = 1$ and hence, for every $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\varphi(y) &\geqslant \left\| h(\sqrt[p]{x})k(\sqrt[p]{y}) - f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 + g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(h(\sqrt[p]{x})k(\sqrt[p]{y}) - f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 + g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 \right) \right| \\ &\geqslant \left| x^*(g(x)) \cdot x^*(k(y)) - x^*\left(f\left(\sqrt[p]{\frac{2}{x+y}2}\right)\right) + x^*\left(g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 \right) \right|,\end{aligned}$$

which states that the superpositions $x^* \circ h$ and $x^* \circ k$ yield a solution of stability inequality (2.1) of Theorem 2.1. Since, by assumption, the superposition $x^* \circ h$ is unbounded. Theorem 2.1 forces that the function $x^* \circ k$ solves the p-power-radical sine functional equations (S^r). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in \mathbb{R}$, the difference $\mathcal{D}S^r : \mathbb{R} \times \mathbb{R} \to E$ defined by

$$\mathcal{D}S^r(x, y) := k\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - k\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - k(\sqrt[p]{x})k(\sqrt[p]{y})$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}S^r(x, y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in \mathbb{R}$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}S^r(x, y) = 0 \quad \text{for all } \ x, y \in \mathbb{R},$$

as claimed. The cases (ii) and (iii) also are similar. □

Corollary 4.2. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g, h : \mathbb{R} \to E$ *satisfy, for all* $x, y \in \mathbb{R}$,

$$\left\| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})h(\sqrt[p]{y}) \right\| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

which satisfies one of the conditions $h(0) = 0$, $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *for all* $x \in \mathbb{R}$. *Then, for an arbitrary linear multiplicative functional* $x^* \in E^*$, *either the superposition* $x^* \circ h$ *is bounded or* h *satisfies* (S^r).

Corollary 4.3. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g, h : \mathbb{R} \to E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - h(\sqrt[p]{x})f(\sqrt[p]{y}) \right\| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then, for an arbitrary linear multiplicative functional* $x^* \in E^*$,

(i) *either the superposition* $x^* \circ h$ *under the cases* $f(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *is bounded or* f *satisfies* (S^r), *in addition, if* h *satisfies* (C^r), *then* f *and* h *satisfy* (T^r_{gf});

(ii) *either the superposition* $x^* \circ f$ *under the cases* $h(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *is bounded or* h *satisfies* (S^r), *in addition, if* f *satisfies* (C^r), *then* h *and* f *satisfy the Wilson equation* (W^r);

(iii) (i) *and* (ii) *hold.*

Corollary 4.4. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g : \mathbb{R} \to E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - f(\sqrt[p]{x})h(\sqrt[p]{y}) \right\| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *Then, for an arbitrary linear multiplicative functional* $x^* \in E^*$,

(i) *either the superposition* $x^* \circ f$ *under the cases* $h(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{x})^2$ *is bounded or* h *satisfies* (S^r), *in addition, if* f *satisfies* (C^r), *then* h *and* f *satisfy* (T^r_{gf});

(ii) *either the superposition* $x^* \circ h$ *under the cases* $f(0) = 0$ *or* $f(\sqrt[p]{x})^2 = g(\sqrt[p]{-x})^2$ *is bounded or* f *satisfies* (S^r), *in addition, if* h *satisfies* (C^r), *then* f *and* h *satisfy the Wilson equation* (W^r);

(iii) (i) *and* (ii) *hold.*

Corollary 4.5. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f : \mathbb{R} \to E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x-y}{2}}\right)^2 - f(\sqrt[p]{x})f(\sqrt[p]{y}) \right\| \leqslant \begin{cases} \text{(i) } \varphi(y), \\ \text{(ii) } \varphi(x), \\ \text{(iii) } \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

for all $x, y \in \mathbb{R}$. *For an arbitrary linear multiplicative functional* $x^* \in E^*$, *either the superposition* $x^* \circ f$ *is bounded or* f *satisfies* (S^r).

Remark 4.6. As the same logic as all (i), (ii), and (iii) of Remark 3.7, all stability results in Section 2 also lead to the results for the four types stability for each functional equation.

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