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Partially equi-integral ϕ_0-stability of nonlinear differential systems

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Abstract

This paper introduces the notions of partially equi-integral stability and partially equi-integral ϕ_0-stability for two differential systems, and establishes some criteria on stability relative to the x-component by using the cone-valued Lyapunov functions and the comparison technique. An example is also given to illustrate our main results.

Keywords: Differential systems, partially integral stability, partially integral ϕ_0-stability, cone-valued Lyapunov functions, comparison technique.
2010 MSC: 34D20.

1. Introduction

In this paper, we discuss the partially equi-integral stability and partially equi-integral ϕ_0-stability relative to the x-component for two differential systems

$$\begin{cases} x' = F(t,x,y), & x(t_0) = x_0, \\ y' = H(t,x,y), & y(t_0) = y_0, \end{cases} \tag{1.1}$$

and the perturbed system

$$\begin{cases} x' = F(t,x,y) + h_1(t,x,y), & x(t_0) = x_0, \\ y' = H(t,x,y) + h_2(t,x,y), & y(t_0) = y_0, \end{cases} \tag{1.2}$$

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where $F, h_1 \in C[R_+ \times R^n \times R^m, R^n], H, h_2 \in C[R_+ \times R^n \times R^m, R^m]$, $F(t,0,0) = h_1(t,0,0) = 0$, $H(t,0,0) = h_2(t,0,0) = 0$, $t \in R_+ = [0,+\infty)$, R^n and R^m are n-dimensional and m-dimensional real Euclidean spaces, respectively, with any convenient norm $\|\cdot\|$ and scalar product $(,)$.

It is well-known that the stability of system is one the most important property which must be considered in system analysis and control system design. Owing to its complicated structure and many other factors, it is very difficult to analyze its stable property. With the development of science and technology, various notions of system stability were proposed based on Lyapunov stability theory. In many actual problems, people are only interested in part state variables of systems, or because of the technical difficulty, other state variables of the systems cannot be controlled or measured, this will oblige people to study the partial stability property of system. Thus, the study of partial stability for systems has its theoretical importance and practical values. Up till now, there are some results on the partial stability for various systems. For example, Lakshmikantham and Leela [7] discussed the partial stability of ordinary differential equations. El-Sheikh et al. [2] justified the partial stability of nonlinear differential systems. Ignatyev [6] studied the partial equi-asymptotical stability of functional differential equations.

As another development, there has been rapid development in the integral stability theory recently. Soliman and Abd Alla [9] gave the integral stability criteria of nonlinear differential systems, Hristova and Russinov [5] investigated the ϕ_0-integral stability in terms of two measures for differential equations, Hristova [3, 4] obtained the integral stability in terms of two measures for impulsive differential equations with "supremum" and impulsive functional differential equations, respectively. The main purpose of this paper is to discuss the notions of partially integral stability of two differential systems, and extend the notions to the so-called partially integral ϕ_0-stability relative to the x-component by employing the cone-valued Lyapunov functions that is used in [1, 2, 8] and the comparison technique. Finally, we give an example to illustrate our main results.

2. Partially integral stability

In this section, we extend partial stability to partially integral stability relative to the x-component. Firstly, we give the following class of functions and definitions:

$\mathcal{K} = \{a \in C[[0,\rho], R_+] : a(0) = 0,$ and $a(r)$ is strictly monotone increasing in r, $\rho > 0$ is a constant$\}$.

We say that function $V(t,x,y)$ belongs to the class $\mathcal{V}$ if $V \in C[R_+ \times R^n \times R^m, R_+]$, is locally Lipschitzian in x and y. Meanwhile, we define the upper right-hand derivative of $V(t,x,y)$ by

$$D^+V(t,x,y) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x+hF(t,x,y), y+hH(t,x,y)) - V(t,x,y)].$$

Consider the comparison equation

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0, \tag{2.1}$$

and its perturbed equation

$$u' = g(t,u) + p(t), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0, \tag{2.2}$$

where $g \in C[R_+ \times R_+, R], p \in C[R_+, R_+]$, $g(t,0) = 0$.

Definition 2.1. The zero solution of (1.1) is said to be partially equi-integral stable relative to the x-component, if for $\alpha \geq 0$, $t_0 \in R_+$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0

for $\alpha, \beta \in \mathcal{K}$ such that for every solution $x(t, t_0, x_0, y_0)$ of the perturbed system (1.2), the inequality $\|x(t, t_0, x_0, y_0)\| < \beta$, $t \geq t_0$ holds, provided that $\|x_0\| + \|y_0\| \leq \alpha$, and

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq \alpha, \quad T > 0.$$

Definition 2.2. The zero solution of (1.1) is said to be partially uniformly-integral stable relative to the x-component, if Definition 2.1 is satisfied, where β is independent of t_0.

Definition 2.3. The zero solution of (1.1) is said to be partially equi-asymptotically integral stable relative to the x-component, if Definition 2.1 holds, and for every $\epsilon > 0$, $\alpha \geq 0$ and $t_0 \in R_+$, there exist positive numbers $T = T(t_0, \alpha, \epsilon)$ and $\gamma = \gamma(t_0, \alpha, \epsilon)$ such that for every solution $x(t, t_0, x_0, y_0)$ of the perturbed system (1.2), the inequality $\|x(t, t_0, x_0, y_0)\| < \epsilon$, $t \geq t_0 + T$ holds, provided that $\|x_0\| + \|y_0\| \leq \alpha$, and

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq \gamma, \quad T > 0.$$

Definition 2.4. The zero solution of (1.1) is said to be partially uniformly-asymptotically integral stable relative to the x-component, if Definition 2.3 is satisfied, where T, γ is independent of t_0.

Theorem 2.5. *Assume that there exists a function $V \in \mathcal{V}$, $V(t, 0, 0) = 0$, satisfying:*

(PS_1) $a(\|x\|) \leq V(t, x, y)$, $a \in \mathcal{K}$;
(PS_2) $D^+V(t, x, y) \leq g(t, V(t, x, y))$.

If the zero solution of (2.1) *is equi-integral stable, then the zero solution of* (1.1) *is partially equi-integral stable relative to the x-component.*

Proof. Since the zero solution of (2.1) is equi-integral stable, for $\alpha \geq 0$, there exists a positive function $\beta = \beta(t_0, \alpha)$ which is continuous in t_0 for $\alpha, \beta \in \mathcal{K}$ such that

$$u_0 \leq \alpha, \quad \int_{t_0}^{t_0+T} p(s) ds \leq \alpha, \quad T > 0,$$

implies $u(t, t_0, u_0) < a(\beta)$, where $u(t, t_0, u_0)$ is an arbitrary solution of (2.2).

Setting $\|h_1(t, x, y)\| = M_1 p(t)$, $\|h_2(t, x, y)\| = M_2 p(t)$, in which $M_1, M_2 > 0$ are constants, then

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq (M_1 + M_2)\alpha.$$

Since $V(t, 0, 0) = 0$ and the function $V(t, x, y)$ is continuous, there exists a $\delta = \delta(t_0, \epsilon)$, such that

$$V(t_0, x_0, y_0) < \alpha, \text{whenever } \|x_0\| + \|y_0\| \leq \delta.$$

By choosing $u_0 = V(t_0, x_0, y_0)$ and noting condition (PS_2), we can apply Theorem 3.1.1 of [7] to obtain

$$V(t, x, y) \leq u^*(t, t_0, u_0), \quad t \geq t_0, \tag{2.3}$$

where $u^*(t, t_0, u_0)$ is the maximal solution of (2.2).

From $\|x_0\| + \|y_0\| \leq \delta$, we can obtain

$$V(t, x, y) \leq u^*(t, t_0, u_0) < a(\beta), \tag{2.4}$$

thus from (2.4) and the condition (PS_1), we obtain $\|x_0\| + \|y_0\| \leq \delta$ implies $a(\|x\|) \leq V(t, x, y) < a(\beta),\ t \geq t_0$.

Furthermore, by choosing $\alpha^* = \min\{\delta, (M_1 + M_2)\alpha\}$, then

$$\|x_0\| + \|y_0\| \leq \alpha^*, \quad \int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq \alpha^*,$$

implies $\|x\| < \beta,\ t \geq t_0$, where $x(t, t_0, u_0)$ is any solution of (1.2). The proof is therefore complete. □

Theorem 2.6. *Let the conditions of Theorem* 2.5 *be satisfied except the conditions* (PS_1)*, is replaced by*

(PS_3) $a(\|x\|) \leq V(t, x, y) \leq b(\|x\| + \|y\|),\ a, b \in \mathcal{K}.$

If the solution of (2.1) *is uniformly-integral stable, then the zero solution of* (1.1) *is partially uniformly-integral stable relative to the x-component.*

Proof. Since the zero solution of (2.1) is uniformly-integral stable, then, for $\alpha \geq 0$, $\beta(\alpha) > 0$ are independent of t_0, and $\alpha, \beta \in \mathcal{K}$ such that

$$u_0 \leq \alpha, \quad \int_{t_0}^{t_0+T} p(s) ds \leq \alpha, \quad T > 0,$$

implies $u(t, t_0, u_0) < a(\beta), a \in \mathcal{K}$, where $u(t, t_0, u_0)$ is an arbitrary solution of (2.2).

Similar to the above arguments in Theorem 2.5, by choosing $\|h_1(t, x, y)\| = M_1 p(t)$, $\|h_2(t, x, y)\| = M_2 p(t)$, in which $M_1, M_2 > 0$ are constants, we obtain

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq (M_1 + M_2)\alpha.$$

By setting $u_0 = b(\|x_0\| + \|y_0\|)$ and by (PS_3), we get

$$V(t_0, x_0, y_0) \leq b(\|x_0\| + \|y_0\|) = u_0.$$

Now, by using (PS_2), and applying Theorem 3.1.1 of [7], we obtain the inequality (2.3). Meanwhile, we choose $\alpha_1 > 0$ such that $b(\alpha_1) = \alpha$. Then the inequalities

$$\|x_0\| + \|y_0\| \leq \alpha_1 \quad \text{and} \quad b(\|x_0\| + \|y_0\|) \leq \alpha,$$

hold together. Therefore, by (PS_3) and (2.3) we get

$$a(\|x\|) \leq V(t, x, y) \leq u^*(t, t_0, u_0) < a(\beta).$$

Thus $\|x_0\| + \|y_0\| \leq \alpha_1$ implies $a(\|x\|) \leq V(t, x, y) < a(\beta),\ t \geq t_0$.
Set $\alpha^* = \min\{\alpha_1, (M_1 + M_2)\alpha\}$, then

$$\|x_0\| + \|y_0\| \leq \alpha^*, \quad \int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\} ds \leq \alpha^*$$

implies $\|x\| < \beta,\ t \geq t_0$, where $x(t, t_0, u_0)$ is any solution of (1.2). The proof is therefore complete. □

Theorem 2.7. *Let the hypotheses of Theorem* 2.5 *be satisfied. If the zero solution of* (2.1) *is equi-asymptotically integral stable, then the zero solution of* (1.1) *is partially equi-asymptotically integral stable relative to the x-component.*

Proof. Since the zero solution of (2.1) is equi-asymptotically integral stable, there exist $\alpha \geq 0$, positive numbers $T = T(t_0, \alpha, \epsilon)$ and $\gamma_1 = \gamma_1(t_0, \alpha, \epsilon)$ such that

$$u_0 \leq \alpha, \quad \int_{t_0}^{t_0+T} p(s)ds \leq \gamma_1, \quad T > 0,$$

implies $u(t, t_0, u_0) < a(\epsilon)$, $t \geq t_0 + T$, where $u(t, t_0, u_0)$ is any solution of (2.2).

Set $\|h_1(t, x, y)\| = M_1 p(t), \|h_2(t, x, y)\| = M_2 p(t)$, in which $M_1, M_2 > 0$ are constants. Similar to the arguments in Theorem 2.5, we can obtain

$$V(t, x, y) \leq u^*(t, t_0, u_0) < a(\epsilon),$$

where $u^*(t, t_0, u_0)$ is the maximal solution of (2.2).

Next, we show that $\|x\| < \epsilon$ whenever $\|x_0\| + \|y_0\| \leq \delta$, $t \geq t_0 + T$. Suppose that this is not true, then there exists a sequence $\{t_k\}$, $t_k \geq t_0 + T$, and $t_k \to \infty$ as $k \to \infty$, such that $\|x_0\| + \|y_0\| \leq \delta$ implies $\|x\| \geq \epsilon$. Then we get the following contradiction:

$$a(\epsilon) = a(\|x(t_k)\|) \leq V(t_k, x(t_k), y(t_k)) < a(\epsilon).$$

Hence $\|x_0\| + \|y_0\| \leq \delta$ implies $\|x\| < \epsilon$, $t \geq t_0 + T$. Furthermore, we have

$$\|x_0\| + \|y_0\| \leq \delta, \quad \int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x, y)\| + \|h_2(s, x, y)\|\}ds \leq \gamma,$$

which implies $\|x\| < \epsilon, t \geq t_0 + T$, where $x(t, t_0, u_0)$ is any solution of (1.2). The proof is therefore complete. □

3. Partially integral ϕ_0-stability

In this section, we extend the partial stability to the partially equi-integral ϕ_0-stable relative to the x-component. To obtain the main results, we give the following definitions.

Definition 3.1. A proper subset K_1 of R^n is called a cone if

(i) $\lambda K_1 \subseteq K_1,\ \lambda \geq 0$;
(ii) $K_1 + K_1 \subseteq K$;
(iii) $K_1 = \overline{K_1}$;
(iv) $K_1^0 \neq \emptyset$;
(v) $K_1 \cap (-K_1) = \{0\}$;

where $\overline{K_1}$, K_1^0 and ∂K_1 denote the closure, interior and boundary of K_1, respectively.

Definition 3.2. The set $K_1^* = \{\phi \in R^n : (\phi, x) \geq 0 \text{ for all } x \in K_1\}$ is called the adjoint cone if it satisfies the properties (i)-(v).

$$x \in \partial K_1 \quad \text{if } (\phi, x) = 0, \quad \text{for some } \phi \in K_0^*, \quad K_0 = K_1 \backslash \{0\}.$$

Definition 3.3. A function $g: D \to R^n$, $D \subset R^n$ is called quasi-monotone relative to the cone K_1 if $x, y \in D$ and $y - x \in \partial K_1$, then there exists $\phi_0 \in K_0^*$ such that $(\phi_0, y-x) = 0$ and $(\phi_0, g(y)-g(x)) \geq 0$.

Let $K_1 \subset R^n$ be a cone in R^n, $K_2 \subset R^m$ be a cone in R^m satisfying the properties (i)-(v) of Definition 3.1, then it follows that $K = K_1 \bigcup K_2 \subset R^n \bigcup R^m$ is a cone in $R^n \bigcup R^m$.

Let $K^* = \{\phi \in R^n \bigcup R^m : (\phi, x+y) \geq 0 \text{ for } x \in K_1 \subset K, y \in K_2 \subset K\}$ and satisfies the properties (i)-(v) of Definition 3.1, where $(\phi, x+y) \leq \|\phi\|(\|x\| + \|y\|)$. For $m > n$ and $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_m)$, thus

$$\begin{aligned} x + y &= (x_1, x_2, \cdots, x_n, 0, 0, \cdots, 0) + (y_1, y_2, \cdots, y_m) \\ &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n, y_{n+1}, y_{n+2}, \cdots, y_m). \end{aligned}$$

Definition 3.4. The zero solution of (1.1) is said to be partially equi-integral ϕ_0-stable relative to the x-component, if for every $\alpha > 0$ and $t_0 \in R_+$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0 for $\alpha, \beta \in \mathcal{K}$ such that for the maximal solution $x^*(t, t_0, x_0, y_0)$ of the perturbed system (1.2), the inequality $(\phi_0, x^*(t, t_0, x_0, y_0)) < \beta,\ t \geq t_0$ holds, provided that $(\phi_0, x_0 + y_0) \leq \alpha$, and

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x^*, y^*)\| + \|h_2(s, x^*, y^*)\|\} ds \leq \alpha, \quad T > 0.$$

Definition 3.5. The zero solution of (1.1) is said to be partially uniformly-integral ϕ_0-stable relative to the x-component, if Definition 3.4 is satisfied, where β is independent of t_0.

Definition 3.6. The zero solution of (1.1) is said to be partially equi-asymptotically integral ϕ_0-stable relative to the x-component, if Definition 3.4 holds, and for every $\epsilon > 0, \alpha \geq 0$ and $t_0 \in R_+$, there exist positive numbers $T = T(t_0, \alpha, \epsilon)$ and $\gamma = \gamma(t_0, \alpha, \epsilon)$ such that for the maximal solution $x^*(t, t_0, x_0, y_0)$ of the perturbed system (1.2) and $\phi_0 \in K_0^*$, the inequality $(\phi_0, x^*(t, t_0, x_0, y_0)) < \epsilon,\ t \geq t_0 + T$ holds, provided that $(\phi_0, x_0 + y_0) \leq \alpha$,

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s, x^*, y^*)\| + \|h_2(s, x^*, y^*)\|\} ds \leq \gamma, \quad T > 0.$$

Definition 3.7. The zero solution of (1.1) is said to be partially uniformly-asymptotically integral ϕ_0-stable relative to the x-component, if Definition 3.6 is satisfied, where T and γ are independent of t_0.

We will say that function $V(t, x, y)$ belongs to the class $\mathcal{W}$, if $V \in C[R_+ \times S_\rho^n \times S_\rho^m, K]$, $S_\rho^n = \{x \in R^n : \|x\| \leq \rho\}$, $S_\rho^m = \{y \in R^m : \|y\| \leq \rho\}$, and $V(t, x, y)$ is locally Lipschitzian in x and y. Meanwhile, we define the upper right-hand derivative of $V(t, x, y)$ by

$$D^+V(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + hF(t, x, y), y + hH(t, x, y)) - V(t, x, y)].$$

We can consider the comparison system

$$u' = G(t, u), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0. \tag{3.1}$$

$$u' = G(t, u) + P(t), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0, \tag{3.2}$$

where $G \in C[R_+ \times K_1, R^n], P \in C[R_+, R_+^n]$, $G(t, 0) = 0$.

Theorem 3.8. *Assume that there exists a function $V \in \mathcal{W}$, $V(t,0,0)=0$, satisfying*

(PS$_4$) $a((\phi_0, x^*)) \leq (\phi_0, V(t,x,y)), a \in \mathcal{K}$;

(PS$_5$) $F(t,x,y)$ *is quasi-monotone in x relative to K_1;*

(PS$_6$) $D^+V(t,x,y) \leq G(t, V(t,x,y))$.

If the solution of (3.1) *is equi-integral stable, then the zero solution of* (1.1) *is partially equi-integral ϕ_0-stable relative to the x-component.*

Proof. Since the zero solution of (3.1) is equi-integral stable, for given $\alpha, \beta \in \mathcal{K}$ such that

$$\|u_0\| \leq \alpha, \quad \int_{t_0}^{t_0+T} \|p(s)\| ds \leq \alpha, \quad T > 0,$$

implies $\|u(t,t_0,u_0)\| < a_1(\beta)$, $a_1 \in \mathcal{K}$, where $u(t,t_0,u_0)$ is any solution of (3.2).

Set $\|h_1(t,x^*,y^*)\| = M_1\|p(t)\|, \|h_2(t,x^*,y^*)\| = M_2\|p(t)\|$, in which $M_1, M_2 > 0$ are constants, then

$$\int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s,x^*,y^*)\| + \|h_2(s,x^*,y^*)\|\} ds \leq (M_1+M_2)\alpha.$$

Since $V(t,0,0) = 0$ and the function $V(t,x,y)$ is continuous, then there exists a $\delta_1 = \delta_1(t_0,\epsilon)$, such that

$$\|V(t_0,x_0,y_0)\| < \alpha, \quad \text{whenever } \|x_0\| + \|y_0\| \leq \delta_1.$$

By choosing $u_0 = V(t_0,x_0,y_0)$ and condition (PS$_6$), therefore we can apply Theorem 1.4.1 of [7] to obtain

$$V(t,x,y) \leq u^*(t,t_0,u_0), \quad t \geq t_0, \tag{3.3}$$

where $u^*(t)$ is maximal solution of (3.2). Now for some $\phi_0 \in K_0^*$,

$$(\phi_0, x_0+y_0) \leq \|\phi_0\|(\|x_0\| + \|y_0\|) \leq \|\phi_0\|\delta_1 = \delta,$$

implies

$$(\phi_0, V(t,x,y)) \leq \|\phi_0\|\|V(t,x,y)\| \leq \|\phi_0\|\|u^*(t,t_0,u_0)\| < \|\phi_0\|a_1(\beta) = a(\beta). \tag{3.4}$$

Thus from (3.4) and the condition (PS$_4$), we obtain $(\phi_0, x_0+y_0) \leq \delta$ implies $a((\phi_0,x^*)) \leq (\phi_0, V(t,x,y)) < a(\beta)$, $t \geq t_0$. Set $\alpha^* = \min\{\delta, (M_1+M_2)\alpha\}$, then

$$(\phi_0, x_0+y_0) \leq \alpha^*, \quad \int_{t_0}^{t_0+T} \sup_{x \in S_\beta^n,\ y \in S_\beta^m} \{\|h_1(s,x^*,y^*)\| + \|h_2(s,x^*,y^*)\|\} ds \leq \alpha^*,$$

implies $(\phi_0, x^*) < \beta$, $t \geq t_0$, where $x^*(t,t_0,u_0)$ is the maximal solution of (1.2). The proof is therefore complete. □

Theorem 3.9. *Let the conditions of Theorem* 3.8 *be satisfied except the conditions* (PS$_4$) *is replaced by*

(PS$_7$) $a(\|x^*\|) \leq (\phi_0, V(t,x,y)) \leq b(\|x\| + \|y\|), a, b \in \mathcal{K}$, *where* $\phi_0 \in K_0^*$.

If the solution of (3.1) *is equi-integralϕ_0-stable, then the zero solution of* (1.1) *is partially equi-integral stable relative to the x-component.*

Proof. Since the zero solution of (3.1) is equi-integral ϕ_0-stable, for given $\alpha, \beta \in \mathcal{K}$ such that

$$(\phi_0, u_0) \leq \alpha, \quad \int_{t_0}^{t_0+T} \|p(s)\| ds \leq \alpha, \quad T > 0,$$

implies $(\phi_0, u^*(t,t_0,u_0)) < a(\beta), a \in \mathcal{K}$, where $u^*(t,t_0,u_0)$ is the maximal solution of (3.2).

Similar to the above arguments in Theorem 3.8, by choosing

$$\|h_1(t,x^*,y^*)\| = M_1\|p(t)\|,\ \|h_2(t,x^*,y^*)\| = M_2\|p(t)\|,$$

in which $M_1, M_2 > 0$ are constants, thus

$$\int_{t_0}^{t_0+T} \sup_{x\in S_\beta^n,\ y\in S_\beta^m} \{\|h_1(s,x^*,y^*)\| + \|h_2(s,x^*,y^*)\|\}ds \le (M_1+M_2)\alpha.$$

Choose $(\phi_0, u_0) = b(\|x_0\| + \|y_0\|)$, then by ($PS_7$), we get

$$(\phi_0, V(t_0,x_0,y_0)) \le b(\|x_0\| + \|y_0\|) = (\phi_0, u_0).$$

Thus $V(t_0,x_0,y_0) \le u_0$. Now, by using (PS_6), and applying Theorem 1.4.1 of [7], we obtain the inequality (3.3). Set $\alpha_1 > 0$ such that $b(\alpha_1) = \alpha$, then the inequalities

$$\|x_0\| + \|y_0\| \le \alpha_1 \quad \text{and} \quad b(\|x_0\| + \|y_0\|) \le \alpha,$$

hold together. Therefore by (PS_7) and (3.3), we get

$$a(\|x^*\|) \le (\phi_0, V(t,x,y)) \le (\phi_0, u^*(t,t_0,u_0)) < a(\beta).$$

Thus $\|x_0\| + \|y_0\| \le \alpha_1$ implies $\|x^*\| < \beta, t \ge t_0$. Choose $\alpha^* = \min\{\alpha_1, (M_1+M_2)\alpha\}$, then

$$\|x_0\| + \|y_0\| \le \alpha^*, \quad \int_{t_0}^{t_0+T} \sup_{x\in S_\beta^n,\ y\in S_\beta^m} \{\|h_1(s,x^*,y^*)\| + \|h_2(s,x^*,y^*)\|\}ds \le \alpha^*,$$

implies $\|x^*\| < \beta$, $t \ge t_0$, where $x^*(t,t_0,u_0)$ is the maximal solution of (1.2). The proof is therefore complete. □

4. Example

Consider the two differential systems

$$\begin{cases} x' = y - x(x^2+y^2), & x(t_0) = x_0, \\ y' = -x - y(x^2+y^2), & y(t_0) = y_0, \end{cases} \tag{4.1}$$

and the perturbed system

$$\begin{cases} x' = y - x(x^2+y^2) + e^{-t}, & x(t_0) = x_0, \\ y' = -x - y(x^2+y^2) + e^{-t}, & y(t_0) = y_0. \end{cases}$$

Let $V(t,x,y) = \frac{1}{2}(x^2+y^2)$, $a(\|x\|) = \frac{1}{4}x^2$, $b(\|x\| + \|y\|) = x^2+y^2$, then $a(\|x\|) \le V(t,x,y) \le b(\|x\| + \|y\|)$, $V(t,x,y)$ is locally Lipschitzian in x and y, and $D^+_{(4.1)}V \le 0$.

Consider the comparison equation $u' = g(t,u) \equiv 0$ and its perturbed equation $u' = 0 + u_0e^{-t}$. We can easily know that the zero solution of the differential equation $u' = g(t,u) = 0$ is uniform-integral stable. According to Theorem 2.6, the zero solution of (4.1) is partially uniformly-integral stable relative to the x-component.

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