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Generalized interval valued fuzzy ideals in semigroups

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Abstract

In this paper, we give the concepts of interval valued fuzzy (m, n)-ideals in semigroups and investigate the properties of interval valued fuzzy (m, n)-ideals. We characterize the (m, n)-regular semigroup by using interval valued fuzzy (m, n)-ideals.

Keywords: Interval valued fuzzy (m, n)-ideals, (m, n)-regular, (m, n)-ideals.

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1. Introduction

The classical of interval valued fuzzy sets was conceptualized by Zadeh in 1975 [20]. This concept is not just used in mathematics and logic but also in medical science [3], image processing [8] and decision making method [21] etc. Biswas [2] used the ideal of interval valued fuzzy sets to interval valued subgroups in In 1994. In 2006, Narayanan and Manikantan [15] developed theory of an interval valued fuzzy set to interval valued fuzzy subsemigroups and types interval valued fuzzy ideals in semigroups. In 2012, Kim et al. [9] defined interval valued fuzzy quasi-ideals in semigroup and they studied of its properties. In 2013 Singaram and Kandasamy [16] characterized regular and intra-regular semigroup in terms of interval valued fuzzy ideals.

In 1961, Lajos [12] studied the concepts of (m, n)-ideals in semigroups which generalized ideals of semigroups. The research of (m, n)-ideals of semigroups has interested many such as Akram et al. [1], Tilidetzke [17], Yaqoob and Chinram [18], and many others. In 2019 Ahsan et al. [13] extended the ideals of (m, n)-ideals in semigroups to fuzzy sets in semigroup and they characterize the regular semigroup by using fuzzy (m, n)-ideals. In 2021, Gaketem [6] studied concept of interval valued fuzzy almost (m, n)-ideals in semigroups. Later, Gaketem [5] studied concept of interval valued fuzzy almost (m, n)-bi-ideals in semigroups. In 2022, Nakkhasen [14] gave concept picture fuzzy (m, n)-ideals of semigroups and investigated some basic properties of picture fuzzy (m, n)-ideals of semigroups. Later, Gaketem [7] discussed concept of interval valued fuzzy almost (m, n)-quasi-ideal in semigroups. Recently, Khamrot

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and Gaketem [10] defined spherical interval valued fuzzy ideals and find necessary and sufficient of kinds spherical interval valued fuzzy ideals.

In this section, we give the concepts of interval valued fuzzy (m,n)-ideals in semigroups and we investigate the properties of interval valued fuzzy (m,n)-ideals. Furthermore, we characterize the regular semigroup by using interval valued fuzzy (m,n)-ideals.

2. Preliminaries

In this topic, concepts of basic definitions are given.

A non-empty subset L of a semigroup E is called a *subsemigroup* of E if $L^2 \subseteq L$. A non-empty subset L of a semigroup E is called a *left* (right) ideal of E if $EL \subseteq L$ ($LE \subseteq L$). An *ideal* L of E is a non-empty subset which is both a left ideal and a right ideal of E. An ideal L of a semigroup E and m,n are positive integers. We called (m,n)-ideal of a semigroup E if $L^mEL^n \subseteq L$, a non-empty subset L of a semigroup E. We denote

$$[L](m,n) = \bigcup_{r=1}^{m+n} L^r \cap L^mEL^n \text{ is principal } (m,n)\text{-ideal},$$

$$[L](m,0) = \bigcup_{r=1}^{m} L^r \cap L^mE \text{ is principal } (m,0)\text{-ideal},$$

$$[L](0,n) = \bigcup_{r=1}^{n} L^r \cap EL^n \text{ is the principal } (0,n)\text{-ideal},$$

i.e., the smallest (m,n)-ideal, the smallest $(m,0)$-ideal, and the smallest $(0,n)$-ideal of E containing L, respectively.

Lemma 2.1 ([11]). *Let E be a semigroup and m,n positive integers, and $[\pi]_{(m,n)}$ the principal (m,n)-ideal generated by the element π. Then*

(1) $([\pi]_{(m,0)})^mE = \pi^mE$;
(2) $E([\pi]_{(0,n)})^n = E\pi^n$;
(3) $([\pi]_{(m,0)})^mE([\pi]_{(0,n)})^n = \pi^mE\pi^n$.

For any $\eta_i \in [0,1]$, where $i \in \mathcal{J}$, define $\bigvee_{i\in\mathcal{J}} J_i := \sup_{i\in\mathcal{J}}\{\eta_i\}$ and $\bigwedge_{i\in\mathcal{J}} \eta_i := \inf_{i\in\mathcal{J}}\{\eta_i\}$. We see that for any $\eta_1, \eta_2 \in [0,1]$, we have $\eta_1 \vee \eta_2 = \max\{\eta_1,\eta_2\}$ and $\eta_1 \wedge \eta_2 = \min\{\eta_1,\eta_2\}$.

Definition 2.2 ([19]). A fuzzy subset (fuzzy set) σ of a non-empty set T is a function from T into the closed interval $[0,1]$, i.e., $\sigma : T \to [0,1]$.

Definition 2.3. [13] A fuzzy subset σ of a semigroup E is said to be a *fuzzy subsemigroup* of E if $\sigma(uv) \geqslant \sigma(u) \wedge \sigma(v)$ for all $u,v \in E$.

Definition 2.4 ([13]). A fuzzy subset σ of a semigroup E is said to be a *fuzzy left (right) ideal* of E if $\sigma(uv) \geqslant \sigma(v)$ ($\sigma(uv) \geqslant \sigma(u)$) for all $u,v \in E$. An IVF subset σ of a semigroup E is called a *fuzzy ideal* of E if it is both a fuzzy left ideal and a fuzzy right ideal of E.

Definition 2.5 ([13]). A fuzzy subsemigorup σ of a semigroup E is said to be a *fuzzy (m,n)-ideal* of E if $\sigma(u_1u_2\cdots u_mzv_1v_2\cdots v_n) \geqslant \sigma(u_1) \wedge \sigma(u_2)\cdots \wedge \sigma(u_m) \wedge \sigma(v_1) \wedge \sigma(v_2)\cdots\sigma(v_n)$ for all $u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_n,z \in E$ and m,n are positive integers.

The set of all closed subintervals of $[0,1]$ is denoted by $\mathcal{C}$, that is,

$$\mathcal{C} = \{\overline{\eta} = [\eta^-,\eta^+] \mid 0 \leqslant \eta^- \leqslant \eta^+ \leqslant 1\}.$$

We note that $[\eta,\eta] = \{\eta\}$ for all $\eta \in [0,1]$. For $\eta = 0$ or 1 we shall denote $[0,0]$ by $\overline{0}$ and $[1,1]$ by $\overline{1}$. Let $\overline{\eta} = [\eta^-,\eta^+]$ and $\overline{\vartheta} = [\vartheta^-,\vartheta^+]$ in $\mathcal{C}$. Define the operations "$\preceq$", "$=$", "$\curlywedge$", and "$\curlyvee$" as follows:

(1) $\overline{\eta} \preceq \overline{\vartheta}$ if and only if $\eta^- \leqslant \vartheta^-$ and $\eta^+ \leqslant \vartheta^+$;
(2) $\overline{\eta} = \overline{\vartheta}$ if and only if $\eta^- = \vartheta^-$ and $\eta^+ = \vartheta^+$;
(3) $\overline{\eta} \curlywedge \overline{\vartheta} = [(\eta^- \wedge \vartheta^-), (\eta^+ \wedge \vartheta^+)]$;
(4) $\overline{\eta} \curlyvee \overline{\vartheta} = [(\eta^- \vee \vartheta^-), (\eta^+ \vee \vartheta^+)]$.

If $\overline{\eta} \succeq \overline{\vartheta}$, we write $\overline{\vartheta} \preceq \overline{\eta}$.

Proposition 2.6 ([4]). *For any $\overline{\eta}, \overline{\vartheta}, \overline{\varpi} \in \mathcal{C}$, the following properties are true:*

(1) $\overline{\eta} \curlywedge \overline{\eta} = \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\eta} = \overline{\eta}$;
(2) $\overline{\eta} \curlywedge \overline{\vartheta} = \overline{\vartheta} \curlywedge \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\vartheta} = \overline{\vartheta} \curlyvee \overline{\eta}$;
(3) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlywedge \overline{\varpi} = \overline{\eta} \curlywedge (\overline{\vartheta} \curlywedge \overline{\varpi})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlyvee \overline{\varpi} = \overline{\eta} \curlyvee (\overline{\vartheta} \curlyvee \overline{\varpi})$;
(4) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlyvee \overline{\varpi} = (\overline{\eta} \curlyvee \overline{\varpi}) \curlywedge (\overline{\vartheta} \curlyvee \overline{\varpi})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlywedge \overline{\varpi} = (\overline{\eta} \curlywedge \overline{\varpi}) \curlyvee (\overline{\vartheta} \curlywedge \overline{\varpi})$;
(5) *if* $\overline{\eta} \preceq \overline{\vartheta}$, *then* $\overline{\eta} \curlywedge \overline{\varpi} \preceq \overline{\vartheta} \curlywedge \overline{\varpi}$ *and* $\overline{\eta} \curlyvee \overline{\varpi} \preceq \overline{\vartheta} \curlyvee \overline{\varpi}$.

Definition 2.7 ([20]). An interval valued fuzzy subset (shortly, IVF subset) of a non-empty set T is a function $\overline{\sigma} : T \to \mathcal{C}$.

For two IVF subsets $\overline{\sigma}$ and $\overline{\tau}$ of a non-empty set T, define

(1) $\overline{\sigma} \sqsubseteq \overline{g} \Leftrightarrow \overline{\sigma}(u) \preceq \overline{\tau}(u)$ for all $u \in T$;
(2) $\overline{\sigma} = \overline{\tau} \Leftrightarrow \overline{\sigma} \sqsubseteq \overline{\tau}$ and $\overline{\tau} \sqsubseteq \overline{\sigma}$;
(3) $(\overline{\sigma} \sqcap \overline{\tau})(u) = \overline{\sigma}(u) \curlywedge \overline{\tau}(u)$ for all $u \in T$.

For two IVF subsets $\overline{\sigma}$ and $\overline{\tau}$ of a semigroup E, define the product $\overline{\sigma} \circ \overline{\tau}$ as follows: for all $u \in E$,

$$(\overline{\sigma} \circ \overline{\tau})(u) = \begin{cases} \underset{(x,y) \in F_u}{\curlyvee} \{\overline{\sigma}(x) \curlywedge \overline{\tau}(y)\}, & \text{if } F_u \neq \emptyset, \\ \overline{0}, & \text{if } F_u = \emptyset, \end{cases}$$

where $F_u := \{(x, y) \in E \times E \mid u = xy\}$.

Definition 2.8 ([15]). An IVF subset $\overline{\sigma}$ of a semigroup E is said to be an *IVF subsemigroup* of E if $\overline{\sigma}(uv) \succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}(v)$ for all $u, v \in E$.

Definition 2.9 ([15]). An IVF subset $\overline{\sigma}$ of a semigroup E is said to be an *IVF left (right) ideal* of E if $\overline{\sigma}(uv) \succeq \overline{\sigma}(v)$ ($\overline{\sigma}(uv) \succeq \overline{\sigma}(u)$) for all $u, v \in E$. An IVF subset $\overline{\sigma}$ of a semigroup E is called an *IVF ideal* of E if it is both an IVF left ideal and an IVF right ideal of E.

Definition 2.10 ([15]). Let L be a subset of a semigroup E. An interval valued characteristic function $\overline{\chi}_L$ of L is defined to be a function $\overline{\chi}_L : E \to \mathcal{C}$ by

$$\overline{\chi}_L(u) = \begin{cases} \overline{1}, & \text{if } u \in L, \\ \overline{0}, & \text{if } u \notin L, \end{cases}$$

for all $u \in E$. In case if $L = E$ (resp. $L = \emptyset$), then $\overline{\chi}_L = \overline{\mathcal{E}}$ (resp. $\overline{\chi}_L = \emptyset$). Where $\overline{\mathcal{E}}$ is an IVF subset of E mapping every element of E to $\overline{1}$.

Definition 2.11 ([15]). An IVF subsemigroup $\overline{\sigma}$ of a semigroup E is said to be an *IVF bi-ideal* of E if $\overline{\sigma}(uvw) \succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}(w)$ for all $u, v, w \in E$.

Lemma 2.12 ([15]). *Let I and J be a non-empty subset of a semigroup E. Then the following statements hold:*

(1) $(\overline{\chi}_I) \curlywedge (\overline{\chi}_J) = (\overline{\chi}_{I \cap J})$;
(2) $(\overline{\chi}_I) \circ (\overline{\chi}_J) = (\overline{\chi}_{IJ})$;
(3) *if* $I \subseteq J$, *then* $\overline{\chi}_I \sqsubseteq \overline{\chi}_J$.

3. Interval valued fuzzy (m,n)-ideal in semigroups

In this section, we will study concepts of IVF (m,n)- ideal in a semigroup and we study properties of those.

Definition 3.1. An IVF subsemigorup σ of a semigroup E is said to be an *IVF* (m,n)*-ideal* of E if $\overline{\sigma}(u_1u_2\cdots u_mzv_1v_2\cdots v_n) \succeq \overline{\sigma}(u_1) \curlywedge \overline{\sigma}(u_2)\cdots \curlywedge \overline{\sigma}(u_m) \curlywedge \overline{\sigma}(v_1) \curlywedge \overline{\sigma}(v_2)\cdots \overline{\sigma}(v_n)$ for all $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots, v_n \in E$ and m, n are positive integers.

Theorem 3.2. *Let* $\{\overline{\sigma}_i \mid i \in \mathcal{J}\}$ *be a family of IVF* (m,n)*-ideals of a semigroup* E*. Then* $\underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i$ *is an IVF* (m,n)*-ideal of* E*.*

Proof. Let $u, v \in E$. Then,

$$\underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i(uv) \succeq \underset{i\in\mathcal{J}}{\curlywedge}\{\overline{\sigma}_i(u) \curlywedge \overline{\sigma}_i(v)\} = \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i(u) \curlywedge \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i(v).$$

Thus, $\underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i$ is an IVF subsemigroup of E. Let $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots, v_n \in E$. Then,

$$\begin{aligned}\underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i(u_1u_2\cdots u_mzv_1v_2\cdots v_n) &\succeq \underset{i\in\mathcal{J}}{\curlywedge}\{\overline{\sigma}(u_1) \curlywedge \overline{\sigma}(u_2)\cdots \curlywedge \overline{\sigma}(u_m) \curlywedge \overline{\sigma}(v_1) \curlywedge \overline{\sigma}(v_2)\cdots \overline{\sigma}(v_n)\}\\ &= \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(u_1) \curlywedge \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(u_2)\cdots \curlywedge \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(u_m) \curlywedge \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(v_1) \curlywedge \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(v_2)\cdots \underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}(v_n).\end{aligned}$$

Thus, $\underset{i\in\mathcal{J}}{\curlywedge}\overline{\sigma}_i$ is an IVF (m,n)-ideal of E. □

Theorem 3.3. *Let* L *is an ideal of a semigroup* E *and* m, n *are positive integers. Then* L *is an* (m,n)*-ideal of* E *if and only if the interval valued characteristic function* $\overline{\chi}_L$ *is an IVF* (m,n)*-ideal of* E*.*

Proof. Suppose that L is an (m,n)-ideal of E and let $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots, v_n \in E$. Then we have following cases.

Case 1. If $u_i \notin L$ for some $i \in \{1, 2, \dots, m\}$, then $\overline{\chi}_L(u_i) = \overline{0}$ for some $i \in \{1, 2, \dots, m\}$. Thus, $\overline{\chi}_L(u_1u_2\cdots c_mzv_1, v_2\cdots v_n) \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_L(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_1) \curlywedge \overline{\chi}_L(v_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_n)$.

Case 2. If $v_j \notin L$ for some $j \in \{1, 2, \dots, n\}$, then $\overline{\chi}_L(v_j) = \overline{0}$ for some $j \in \{1, 2, \dots, n\}$. Thus, $\overline{\chi}_L(u_1u_2\cdots u_mzv_1, v_2\cdots v_n) \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_I(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_1) \curlywedge \overline{\chi}_I(v_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_n)$.

Case 3. If $u_i, v_j \in L$ for each $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$, then $u_1u_2\cdots u_mzv_1v_2\cdots v_n \in L^mEL^n \subseteq L$. Thus $\overline{\chi}_L(u_i) = \overline{1} = \overline{\chi}_I(u_j)$ for each $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. Hence, $\overline{\chi}_K(u_1u_2\cdots u_mzv_1v_2\cdots v_n) = \overline{1} \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_I(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_1) \curlywedge \overline{\chi}_I(v_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_n)$.

Therefore, $\overline{\chi}_L$ is an IVF (m,n)-ideal of E. Conversely, suppose that $\overline{\chi}_L$ is an IVF (m,n)-ideal of E. Let $u_1, u_2, \dots, u_m, z, v_1, v_2, \dots, v_n \in L$ and $z \in E$. Then, $\overline{\chi}_L(u_1u_2\cdots u_mzv_1v_2\cdots v_n) \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_L(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_1) \curlywedge \overline{\chi}_L(v_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(v_n) = \overline{1}$. It follows that $\overline{\chi}_L(u_1u_2\cdots u_mzv_1v_2\cdots v_n) = \overline{1}$. Hence, $u_1u_2\cdots u_mzv_1v_2\cdots v_n \in L$. This implies that, $L^mEL^n \subseteq L$. Therefore, L is an (m,n)-ideal of E. □

Theorem 3.4. *Let* $\overline{\sigma}$ *an IVF subsemigroup of* E *and* m, n *positive integers. Then,* $\overline{\sigma}$ *is an IVF* (m,n)*-ideal of* E *if and only if* $\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \sqsubseteq \overline{\sigma}$*, where* $\overline{\mathcal{E}}$ *is an IVF subset of* E *mapping every element of* E *to* $\overline{1}$*.*

Proof. Suppose that $\overline{\sigma}$ is an IVF (m,n)-ideal of E and $u \in E$. If $F_u = \emptyset$, then it is easy to verify that $(\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)(u) \preceq \overline{\sigma}(u)$. If $F_u \neq \emptyset$, then $(\overline{\sigma}^m \circ \overline{\mathcal{E}})(u) \neq 0$ and $\overline{\sigma}^n(y) \neq \overline{0}$. Thus there exist $x, y \in E$ such that $u = xy$, $(\overline{\sigma}^m \circ \overline{\mathcal{E}})(x) \neq \overline{0}$, and $\overline{\sigma}^n(y) \neq \overline{0}$. Since $(\overline{\sigma}^m \circ \overline{\mathcal{E}})(x) \neq \overline{0}$, there exist $x_1, y_1 \in E$ such that $x = x_1y_1$, $\overline{\sigma}^m(x_1) \neq \overline{0}$ and $\overline{\mathcal{E}}(y_1) \neq \overline{0}$. By induction, it is easy to prove that there exist $x_2, y_2, \dots, x_m, y_m \in E$ such that for any $i \in \{2, \dots, m\}$ so $x_{i-1} = x_iy_i$, $\overline{\sigma}^{m-i+1}(x_i) \neq \overline{0}$ and $\overline{\sigma}(y_i) \neq \overline{0}$. Similarly, for case $\overline{\sigma}^n(y) \neq \overline{0}$ there exist $x_2', y_2', \dots, x_{n-1}', y_{n-1}' \in E$ such that , $x_{j-1}' = x_j'y_j'$, for all $j \in \{2, \dots, n-1\}$, $\overline{\sigma}^{n-j}(x_j') \neq \overline{0}$ and $\overline{\sigma}(y_j') \neq \overline{0}$. Thus,

$$(\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)(u) = \underset{(x,y)\in F_u}{\curlyvee}(\overline{\sigma}^m \circ \overline{\mathcal{E}})(x) \curlywedge \overline{\sigma}^n(y)$$

$$
\begin{aligned}
&= \curlyvee_{(x,y)\in F_u}\Big(\curlyvee_{(x_1,y_1)\in F_x} \overline{\sigma}^m(x_1) \curlywedge \overline{\mathcal{E}}(y_1)\Big) \curlyvee_{(x_1',y_1')\in F_y} \overline{\sigma}^{n-1}(x_1') \curlywedge \overline{\sigma}(y_1')\\
&= \curlyvee_{(x,y)\in F_u} \curlyvee_{(x_1,y_1)\in F_x} \curlyvee_{(x_1',y_1')\in F_y} (\overline{\sigma}^m(x_1) \curlywedge \overline{\mathcal{E}}(y_1) \curlywedge \overline{\sigma}^{n-1}(x_1') \curlywedge \overline{\sigma}(y_1'))\\
&= \curlyvee_{(x,y)\in F_u} \curlyvee_{(x_1,y_1)\in F_x} \curlyvee_{(x_1',y_1')\in F_y} (\overline{\sigma}^m(x_1) \curlywedge \overline{\sigma}^{n-1}(x_1') \curlywedge \overline{\sigma}(y_1'))\\
&= \curlyvee_{(x,y)\in F_u} \curlyvee_{(x_1,y_1)\in F_x} \curlyvee_{(x_1',y_1')\in F_y} \Big(\curlyvee_{(x_2',y_2')\in F_{y_1}} \overline{\sigma}^{m-1}(x_2) \curlywedge \overline{\sigma}^(y_2)\\
&\quad \curlyvee_{(x_2',y_2')\in F_{x_1'}} \overline{\sigma}^{n-2}(x_2') \curlywedge \overline{\sigma}(y_2') \curlywedge \overline{\sigma}(y_1')\Big)\\
&= \curlyvee_{(x,y)\in F_u} \curlyvee_{(x_1,y_1)\in F_x} \curlyvee_{(x_1',y_1')\in F_y} \curlyvee_{(x_2',y_2')\in F_{y_1}} \curlyvee_{(x_2',y_2')\in F_{x_1'}} (\overline{\sigma}(x_m) \curlywedge \overline{\sigma}(y_m) \curlywedge \overline{\sigma}(y_{m-1})\\
&\quad \curlywedge \cdots \curlywedge \overline{\sigma}(y_2) \curlywedge \overline{\sigma}(x_{n-1}') \curlywedge \overline{\sigma}(y_{n-1}') \curlywedge \overline{\sigma}(y_{n-2}') \curlywedge \cdots \curlywedge \overline{\sigma}(y_1'))\\
&= \curlyvee_{(x,y)\in F_u} \curlyvee_{(x_1,y_1)\in F_x} \curlyvee_{(x_1',y_1')\in F_y} \curlyvee_{(x_2',y_2')\in F_{y_1}} \curlyvee_{(x_2',y_2')\in F_{x_1'}} \cdots \curlyvee_{(x_m,y_m)\in F_{x_{m-1}}}\\
&\quad \curlyvee_{(x_{n-1}',y_{n-1}')\in F_{x_{n-2}'}} (\overline{\sigma}(x_m) \curlywedge \overline{\sigma}(y_m) \curlywedge \overline{\sigma}(y_{m-1}) \curlywedge \cdots \curlywedge \overline{\sigma}(x_{n-1}')\\
&\quad \curlywedge \overline{\sigma}(y_{n-1}') \curlywedge \overline{\sigma}(y_{n-2}') \curlywedge \cdots \overline{\sigma}(y_1')) = \curlyvee_{(x,y)\in F_u} \overline{\sigma}(xy) = \overline{\sigma}(u).
\end{aligned}
$$

Conversely, suppose that $\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \sqsubseteq \overline{\sigma}$. For any $u_1, u_2, \ldots, u_m, z, v_1, v_2, \ldots, v_n \in E$, let $u = u_1u_2 \cdots u_m z v_1 v_2 \cdots v_n$. Then

$$
\begin{aligned}
\overline{\sigma}(u_1u_2\cdots u_m z v_1 v_2 \cdots v_n) &= \overline{\sigma}(u) \succeq (\overline{\sigma}^m \circ \overline{\mathcal{E}} \overline{\sigma}^n)(u)\\
&= \curlyvee_{(p,q)\in F_u} (\overline{\sigma}^m \circ \overline{\mathcal{E}})(p) \curlywedge \overline{\sigma}^n(q)\\
&\succeq \curlyvee_{(p,q)\in F_u} (\overline{\sigma}^m \circ \overline{\mathcal{E}})(u_1u_2\cdots u_m z) \curlywedge \overline{\sigma}^n(v_1v_2\cdots v_n)\\
&\succeq \curlyvee_{(u_1u_2\cdots u_m z)\in F_{uv}} \overline{\sigma}^m(u) \curlywedge \overline{\sigma}^m(v) \curlywedge \Big(\curlyvee_{(v_1v_2\cdots v_n)\in F_{u'v'}} \overline{\sigma}^{n-1}(u') \curlywedge \overline{\sigma}(v')\Big)\Big)\\
&\succeq \{\overline{\sigma}^m(u_1u_2\cdots u_m) \curlywedge \overline{\mathcal{E}}(z)\} \curlywedge \{\overline{\sigma}^{n-1}(v_1v_2\cdots v_{n-1}) \curlywedge \overline{\sigma}(v_n)\}\\
&\succeq \overline{\sigma}^m(u_1u_2\cdots u_m) \curlywedge \overline{\sigma}^{n-1}(v_1v_2\cdots v_{n-1}) \curlywedge \overline{\sigma}(v_n)\\
&\;\;\vdots\\
&\succeq \overline{\sigma}(u_1) \curlywedge \overline{\sigma}(u_2) \curlywedge \cdots \curlywedge \overline{\sigma}(u_m) \curlywedge \overline{\sigma}(v_1) \curlywedge \overline{\sigma}(v_2) \curlywedge \cdots \curlywedge \overline{\sigma}(v_n).
\end{aligned}
$$

Hence, $\overline{\sigma}$ is an IVF (m, n)-ideal of E. □

Definition 3.5. Let E be a semigroup and m, n be positive integers. Then E is called (m, n)-*regular* if for each $u \in E$ there exists $z \in E$ such that $\omega = \omega^m z \omega^n$ equivalently for each subset L of E if $L \subseteq L^m E L^n$ or for each element u of E, $\omega \in \omega^m E \omega^n$.

Lemma 3.6. *Let E be an (m, n)-regular of semigroup and m, n be positive integers. Then every IVF (m, n)-ideal of E is an IVF bi-ideal of E.*

Proof. Suppose that $\overline{\sigma}$ is an IVF (m, n)-ideal of E and $i, j, k \in E$. By assumption there exist $x, y \in E$ such that $ijk = i^m xjk^m yj^n$. Thus $\overline{\sigma}(ijk) = \overline{\sigma}(i^m xjk^m yj^n) = \overline{\sigma}(i^m(xjk^m y)j^n) \succeq \overline{\sigma}(i^m) \curlywedge \overline{\sigma}(j^n) \succeq \overline{\sigma}(i) \curlywedge \overline{\sigma}(j)$. Hence, $\overline{\sigma}$ is an IVF bi-ideal of E. □

Theorem 3.7. *Let $\overline{\sigma}$ be an IVF subsets of a semigroup E. Then $\overline{\sigma}(u) \preceq \overline{\sigma}^m(u^m)$ for any positive integer m and $u \in E$.*

Proof. Let $u \in E$ and m be positive integer. Then $u^m = uu^{m-1}$. Thus

$$\begin{aligned}
\overline{\sigma}^m(u^m) &= \curlyvee_{(p,q)\in F_{u^m}} \overline{\sigma}(p) \curlywedge \overline{\sigma}^{m-1}(q) \\
&\succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}^{m-1}(u^{m-1}) \\
&= \overline{\sigma}(u) \curlywedge \curlyvee_{(p',q')\in F_{u^{m-1}}} \overline{\sigma}(p') \curlywedge \overline{\sigma}^{m-2}(q') \\
&\succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}(u) \curlywedge \overline{\sigma}^{m-1}(u^{m-1}) \\
&\vdots \\
&\succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}(u) \curlywedge \cdots \curlywedge \overline{\sigma}(u) = \overline{\sigma}(u).
\end{aligned}$$

□

Theorem 3.8. *Let E be a semigroup and m, n be positive integers. Then E is (m,n)-regular if and only if $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$ for any positive integer m and $u \in E$.*

Proof. Let E be (m,n)-regular and $u \in E$. Then $u = u^m p u^n$ for some $p \in E$. Thus

$$\begin{aligned}
(\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)(u) &= \curlyvee_{(i,j)\in F_u} (\overline{\sigma}^m \circ \overline{\mathcal{E}})(i) \curlywedge \overline{\sigma}^n(j) \\
&\succeq (\overline{\sigma}^m \circ \overline{\mathcal{E}})(u^m p) \curlywedge \overline{\sigma}^n(u^n) \\
&= \curlyvee_{(k,l)\in F_{u^m p}} (\overline{\sigma}^m(k) \curlywedge \overline{\mathcal{E}}(l)) \curlywedge \overline{\sigma}^n(u^n) \\
&\succeq \overline{\sigma}^m(u^m) \curlywedge \overline{\mathcal{E}}(p) \curlywedge \overline{\sigma}^n(u^n) \\
&= \overline{\sigma}^m(u^m) \curlywedge \overline{\sigma}^n(u^n) \succeq \overline{\sigma}(u) \curlywedge \overline{\sigma}(u) = \overline{\sigma}(u).
\end{aligned}$$

Hence, $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$. Conversely, suppose that $u \in E$. Since $\overline{\chi}_K$ is an IVF subset of E, by assumption and Lemma 2.12 we have $\overline{\sigma}_u \sqsubseteq \overline{\sigma}_u^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}_u^n = \overline{\sigma}_{u^m E u^n}$. Thus, $u \in u^m E u^n$. Hence, E is (m,n)-regular. □

Theorem 3.9 ([11])**.** *Let E be a semigroup. Then E is (m,n)-regular if and only if for all $L \in \mathcal{L}_{(m,n)}$ (the set of all (m,n)-ideals of E), $L = L^m E L^n$.*

Theorem 3.10. *Let E be a semigroup and m, n positive integers. Then E is (m,n)-regular if and only if $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$ for each IVF (m,n)-ideal $\overline{\sigma}$ of E.*

Proof. Let E be an (m,n)-regular semigroup and $\overline{\sigma}$ be an IVF (m,n)-ideal of E. Then, by Theorems 3.4 and 3.8, $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$. Conversely, suppose that $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$ for each IVF (m,n)-ideal $\overline{\sigma}$ of E and let $u \in E$. Then, $[u]_{(m,n)}$ is an (m,n)-ideal of E, by Theorem 3.3, $\overline{\chi}_{[u](m,n)}$ is an IVF (m,n)-ideal of E. Thus, by hypothesis, we have

$$\overline{\chi}_{[u](m,n)} = \overline{\chi}^m_{[u](m,n)} \circ \overline{\mathcal{E}} \circ \overline{\chi}^n_{[u](m,n)} = \overline{\chi}_{([u]_{(m,n)})^m E ([u]_{(m,n)})^n}.$$

So, $[u]_{(m,n)} = ([u]_{(m,n)})^m E ([u]_{(m,n)})^n$. By Lemma 2.1, $[u]_{(m,n)} = u^m E u^n$. Thus, $u \in u^m E u^n$. Hence, E is (m,n)-regular. □

Theorem 3.11. *Let E be a semigroup and m, n be positive integers with $m \geqslant 2$ or $n \geqslant 2$. Then E is (m,n)-regular if and only if $L = L^2$ for each (m,n)-ideal L of E.*

Proof. Let E be (m,n)-regular semigroup and L be (m,n)-ideal of E. Then, by Theorem 3.9, $L = L^m E L^n$. Thus, $L = L^m E L^n = L^m E (L^m E L^n)^n = L^m E (L^m E L^n)(L^m E L^n) \cdots (L^m E L^n) \subseteq (L^m E L^n)(L^m E L^n) = LL$. Inverse inclusion is obvious because L is an (m,n)-ideal of E. Hence, $L = L^2$. Conversely, suppose that $L = L^2$ for each (m,n)-ideal L of E and $u \in E$. Since $[u]_{(m,n)}$ is an (m,n)-ideal of E we have

$$[u]_{(m,n)} = [u]_{(m,n)}[u]_{(m,n)}$$

$$
\begin{aligned}
&= [u]_{(m,n)}([u]_{(m,n)}[u]_{(m,n)}) \\
&\vdots \\
&= ([u]_{(m,n)})^{m+n+1} \\
&= ([u]_{(m,n)})^{m}[u]_{(m,n)}([u]_{(m,n)})^{n} \\
&\subseteq ([u]_{(m,n)})^{m}E([u]_{(m,n)})^{n} = u^m E u^n.
\end{aligned}
$$

Since $u \in [u]_{m,n}$, we have $u \in u^m E u^n$. Hence, E is (m, n)-regular. □

Theorem 3.12. *Let $\overline{\sigma}$ be an IVF (m, n)-ideal of E and m, n positive integers with $m \geqslant 2$ or $n \geqslant 2$. Then E is (m, n)-regular if and only if $\overline{\sigma} = \overline{\sigma} \circ \overline{\sigma}$.*

Proof. Let E be an (m, n)-regular semigroup, $\overline{\sigma}$ be IVF (m, n)-ideal of E and $n \geqslant 2$. Then by Theorem 3.10, $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$. Thus,

$$
\begin{aligned}
\overline{\sigma} &= \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \\
&= \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)^n \\
&= \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \circ (\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)^{n-3} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \\
&= \overline{\sigma}^m \circ (\overline{\mathcal{E}} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}}) \circ \overline{\sigma}^n \circ \overline{\sigma}^m \circ (\overline{\mathcal{E}} \circ \overline{\sigma}^n \circ (\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n)^{n-3} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}}) \circ \overline{\sigma}^n \\
&\sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^m \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n = \overline{\sigma} \circ \overline{\sigma}.
\end{aligned}
$$

Thus, $\overline{\sigma} \sqsubseteq \overline{\sigma} \circ \overline{\sigma}$. Inverse inclusion is obvious since $\overline{\sigma}$ is an IVF (m, n)-ideal of E. Hence, $\overline{\sigma} = \overline{\sigma} \circ \overline{\sigma}$. Conversely, suppose that $\overline{\sigma} = \overline{\sigma} \circ \overline{\sigma}$and let L be an (m, n)-ideal of E and $u \in L$. Then by hypothesis $\overline{\chi}_L(u) = (\overline{\chi}_L \circ \overline{\chi}_L)(u) = \overline{\chi}^2_L(u)$. Since $\overline{\chi}_L(u) = \overline{1}$ and $\overline{\chi}^2_L(u) = \overline{1}$ we have $u \in L^2$. Thus, $L \subseteq L^2$, inverse inclusion is obvious since L is an (m, n)-ideal of E. Thus, $L = L^2$. Hence by Theorem 3.11, E is (m, n)-regular. Similarly, Theorem 3.12 can be proved for the case $m \geqslant 2$. □

Lemma 3.13. *Let $\overline{\sigma}$ is an IVF (m, n)-ideal of a semigroup E and $\overline{\tau}$ is an IVF subsemigroup of E, such that $\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \sqsubseteq \overline{\tau} \sqsubseteq \overline{\sigma}$. Then $\overline{\tau}$ is an IVF (m, n)-ideal of E for any positive integers m, n.*

Proof. By assumption, $\overline{\tau}$ is an IVF subsemigroup of E. Then, by Theorem 3.4, $\overline{\tau}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n \sqsubseteq \overline{\tau}$. Thus $\overline{\tau}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n(u) \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n(u) \sqsubseteq \overline{\tau}(u)$. Hence, $\overline{\tau}$ is an IVF (m, n)-ideal of E. □

Lemma 3.14. *Let $\overline{\sigma}$ be an IVF (m, n)-ideal and $\overline{\tau}$ be an IVF subset of a semigroup E and m, n be positive integers. If $\overline{\sigma} \circ \overline{\tau} \sqsubseteq \overline{\tau}$ or $\overline{\tau} \circ \overline{\sigma} \sqsubseteq \overline{\sigma}$, then the following statements hold:*

(1) $\overline{\sigma} \circ \overline{\tau}$ *is an IVF* (m, n)*-ideal of* E;

(2) $\overline{\tau} \circ \overline{\sigma}$ *is an IVF* (m, n)*-ideal of* E.

Proof. Suppose that $\overline{\sigma} \circ \overline{\tau} \sqsubseteq \overline{\tau}$ or $\overline{\tau} \circ \overline{\sigma} \sqsubseteq \overline{\sigma}$.

(1) By assumption we have $(\overline{\sigma} \circ \overline{\tau}) \circ (\overline{\sigma} \circ \overline{\tau}) \sqsubseteq \overline{\sigma} \circ (\overline{\sigma} \circ \overline{\tau}) = \overline{\sigma} \circ \overline{\sigma} \circ \overline{\tau} \sqsubseteq \overline{\sigma} \circ \overline{\tau}$. Thus $\overline{\sigma} \circ \overline{\tau}$ is an IVF subsemigroup of E. So, we have

$$
\begin{aligned}
(\overline{\sigma} \circ \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \circ \overline{\tau})^n &= (\overline{\sigma} \circ \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \circ \overline{\tau})^{n-1} \circ (\overline{\sigma} \circ \overline{\tau}) \\
&\sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^{n-1} \circ (\overline{\sigma} \circ \overline{\tau}) = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \circ \overline{\tau} \sqsubseteq \overline{\sigma} \circ \overline{\tau}.
\end{aligned}
$$

Thus, $\overline{\sigma} \circ \overline{\tau}$ is an IVF (m, n)-ideal of E.

(2) Similar to (1) we can show that $\overline{\tau} \circ \overline{\sigma}$ is an IVF (m, n)-ideal of E. □

4. $(m,0)$-regular and $(0,n)$-regulars

In this section, we defined $(m,0)$-regular and $(0,n)$-regulars in semigroup and integrated properties of its.

Definition 4.1. An IVF subsemigroup of a semigroup E is said to be an *IVF $(m,0)$-ideal* of E if $\overline{\sigma}(u_1u_2 \cdots u_m z) \succeq \overline{\sigma}(u_1) \curlywedge \overline{\sigma}(u_2) \cdots \curlywedge \overline{\sigma}(u_m)$ for all $u_1, u_2, \ldots, u_m, z \in E$. And we said to be an *IVF $(0,n)$-ideal* of E if $\overline{\sigma}(zv_1v_2 \cdots v_n) \succeq \overline{\sigma}(v_1) \curlywedge \overline{\sigma}(v_2) \cdots \overline{\sigma}(v_n)$ for all $u_1, u_2, \ldots, u_m, z, v_1, v_2, \ldots, v_n \in E$ and m, n are positive integers.

Theorem 4.2. *Let E be a semigroup and m, n be positive integers. Then L is an $(m,0)$-ideal ($(0,n)$-ideal) of E if and only if the interval valued characteristic function $\overline{\chi}_L$ is an IVF an $(m,0)$-ideal ($(0,n)$-ideal) of E.*

Proof. Suppose that L is an $(m,0)$-ideal of E and let $u_1, u_2, \ldots, u_m, z \in E$. Then we have following cases.

Case 1. If $u_i \notin L$ for some $i \in \{1,2,\ldots,m\}$, then $\overline{\chi}_L(u_i) = \overline{0}$ for some $i \in \{1,2,\ldots,m\}$. Thus $\overline{\chi}_L(u_1u_2 \cdots u_m z) \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_L(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m)$.

Case 2. If $u_i \notin L$ for each $i \in \{1,2,\ldots,m\}$, then $\overline{\chi}_L(u_i) = \overline{0}$ for each $i \in \{1,2,\ldots,m\}$. Thus $\overline{\chi}_L(u_1u_2 \cdots u_m z) \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_L(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m)$.

Therefore, $\overline{\chi}_L$ is an IVF $(m,0)$-ideal of E. Conversely, suppose that $\overline{\chi}_L$ is an IVF $(m,0)$-ideal of E. Then $\overline{\chi}_L(u_1u_2 \cdots u_m z) = \overline{1} \succeq \overline{\chi}_L(u_1) \curlywedge \overline{\chi}_L(u_2) \curlywedge \cdots \curlywedge \overline{\chi}_L(u_m)$. Thus, $u_1u_2 \cdots u_m z \in I$. Hence, $L^m E L^n \subseteq L$. Therefore, L is an $(m,0)$-ideal of E. □

Theorem 4.3. *Let $\overline{\sigma}$ an IVF subsemigroup of E and m, n positive integers. Then, $\overline{\sigma}$ is an IVF $(m,0)$-ideal ($(0,n)$-ideal) of E if and only if $\overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}$ ($\overline{\mathcal{E}} \circ \overline{\sigma}^n \sqsubseteq \overline{\sigma}$).*

Proof. Straightforward. □

Lemma 4.4. *Let E be a semigroup and m, n be positive integers. Then every IVF right (left) ideal of E is an IVF $(m,0)$-ideal (IVF $(0,n)$-ideal) of E.*

Proof. Let $\overline{\sigma}$ be an IVF right ideal and $u_1, u_2, \ldots, u_m, z \in E$. Then

$$\overline{\sigma}(u_1u_2 \cdots u_m z) \succeq \overline{\sigma}(u_1u_2 \cdots u_m) \succeq \overline{\sigma}(u_1) \curlywedge \overline{\sigma}(u_2) \cdots \curlywedge \overline{\sigma}(u_m).$$

Thus, $\overline{\sigma}$ is an IVF $(m,0)$-ideal of E. □

Definition 4.5. Let E be a semigroup and m, n be positive integers. Then E is said to be $(m,0)$-*regular*($(0,n)$-regular) if for each $u \in E$ there exists $x \in E$ such that $u = u^m x$ ($u = yu^n$).

Theorem 4.6. *Let E be a semigroup and m, n be positive integers. If E is $(m,0)$-regular ((0, n)-regular), then IVF $(m,0)$-ideals ((0, n)-ideals) and IVF right (left) ideals coincide.*

Proof. Let E be an $(m,0)$-regular, $\overline{\sigma}$ be an IVF $(m,0)$-ideal of E, and $u, v \in E$. Then there exists $x \in E$ such that $uv = u^m xv$. Thus,

$$\overline{\sigma}(uv) = \overline{\sigma}(u^m xv) = \overline{\sigma}(u^m(xv)) \succeq \overline{\sigma}(u^m) \succeq \overline{\sigma}(u).$$

Hence, $\overline{\sigma}$ is an IVF right ideal of E. □

Definition 4.7. An IVF subset $\overline{\sigma}$ of a semigroup E is called *idempotent* if $\overline{\sigma} \circ \overline{\sigma} = \overline{\sigma}$.

Theorem 4.8. *Let E be an (m,n)-regular semigroup. Then the IVF $(m,0)$-ideals ($(0,n)$-ideals) of E are idempotent.*

Proof. Suppose that $\overline{\sigma}$ is an $(m,0)$-ideal of E. Then, $\overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}$. By assumption and Theorem 3.8, we have

$$\begin{aligned}\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^{n-1} \circ \overline{\sigma} &\sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^{n-1} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \\ &\sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\mathcal{E}} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma} \circ \overline{\sigma}.\end{aligned}$$

Thus, $\overline{\sigma} \sqsubseteq \overline{\sigma} \circ \overline{\sigma}$. Clearly $\overline{\sigma} \circ \overline{\sigma} \sqsubseteq \overline{\sigma}$. Hence, $\overline{\sigma} \circ \overline{\sigma} = \overline{\sigma}$. Therefore, $\overline{\sigma}$ is an idempotent. □

Theorem 4.9. *Let E be a semigroup and m, n be positive integers. Then E is $(m, 0)$-regular ($(0, n)$-regular) if and only if $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}}$ ($\overline{\sigma} \sqsubseteq \overline{\mathcal{E}} \circ \overline{\sigma}^n$) for each IVF subset $\overline{\sigma}$ of E.*

Proof. Let E be an $(m, 0)$-regular semigroup and $u \in E$. Then, there exists $x \in E$ such that $u = u^m x$. Thus,

$$(\overline{\sigma}^m \circ \overline{\mathcal{E}})(u) = \curlyvee_{(p,q)\in F_u} \overline{\sigma}^m(p) \curlywedge \overline{\mathcal{E}}(q) = \curlyvee_{(p,q)\in F_{u^m x}} \overline{\sigma}^m(p) \curlywedge \overline{\mathcal{E}}(q)$$
$$\succeq \overline{\sigma}^m(u^m) \curlywedge \overline{\mathcal{E}}(x) = \overline{\sigma}^m(u^m) \curlywedge \overline{1} = \overline{\sigma}^m(u^m) \succeq \overline{\sigma}(u).$$

Hence, $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}}$. Conversely, suppose that $u \in E$. Since $\overline{\chi}$ is the interval valued characteristic function of E. By hypothesis and Theorem 2.12, we have that $\overline{\chi}_u \sqsubseteq \overline{\chi}_u^m \circ \overline{\mathcal{E}} = \overline{\chi}_{u^m E}$. Thus, $u \in u^m E$. Hence, E is $(m, 0)$-regular. □

Theorem 4.10 ([11]). *Let E be a semigroup, $\mathcal{R}_{(m,0)}$ be the set of all $(m, 0)$-ideals of E, and $\mathcal{L}_{(0,n)}$ be the set of all $(0, n)$-ideals of E. Then E is $(m, 0)$-regular ($(0, n)$-regular) if and only if $R = R^m \circ E$ ($L = E \circ L^n$), $\forall R \in \mathcal{R}_{(m,0)}$ ($\forall L \in \mathcal{L}_{(m,0)}$), where m, n are positive integer.*

Theorem 4.11. *Let E be a semigroup and m, n be positive integers. Then E is $(m, 0)$-regular ($(0, n)$-regular) if and only if $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}}$ ($\overline{\sigma} = \overline{\mathcal{E}} \circ \overline{\sigma}^n$) for each IVF $(m, 0)$-ideal ($(0, n)$-ideal) $\overline{\sigma}$ of E.*

Proof. Suppose that E is an $(m, 0)$-regular semigroup and $\overline{\sigma}$ is an IVF $(m, 0)$-ideal of E. Then, by Theorems 4.9 and 4.3, we have $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}}$ and $\overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}$. Thus, $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}}$.

Conversely, suppose that $\overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}}$ and $\overline{\sigma}$ is an IVF $(m, 0)$-ideal of E. Let R be $(m, 0)$-ideal of E. Then, $R^2 \subseteq R$ and $R^m E \subseteq R$. By Lemma 2.12 (3), $\overline{\chi}_R^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\chi}_R$. Thus $\overline{\chi}_R$ is an IVF subsemigroup of E. By Lemma 2.12 (2) we have $\overline{\chi}_R^m \circ \overline{\mathcal{E}} = \overline{\chi}_{R^m E} \sqsubseteq \overline{\chi}_R$ and by Theorem 4.3, $\overline{\chi}_R$ is an IVF $(m, 0)$-ideal of E. By assumption, $\overline{\chi}_R^m \circ \overline{\mathcal{E}} = \overline{\chi}_{R^m E} = \overline{\chi}_R$. Thus, $R^m E = R$. Hence by Theorem 4.10, E is $(m, 0)$-regular. □

Theorem 4.12 ([13]). *Let E be a semigroup. Then, E is (m, n)-regular if and only if $R \cap L = R^m L^n$ for every $(m, 0)$-ideal R of E and for every $(0, n)$-ideal L of E.*

Theorem 4.13. *Let $\overline{\sigma}$ be an IVF $(m, 0)$-ideal and $\overline{\tau}$ be an IVF $(0, n)$-ideal of a semigroup E and m, n be positive integers. Then E is (m, n)-regular of E if and only if $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{\tau}^n$.*

Proof. Suppose that E is an (m, n)-regular semigroup of E. By Theorem 3.8, $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq (\overline{\sigma} \curlywedge \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \curlywedge \overline{\tau})^n$. Since $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\sigma}$ and $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\tau}$, we have

$$(\overline{\sigma} \curlywedge \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \curlywedge \tau)^n \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n.$$

By Theorem 4.3, $\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n = \overline{\sigma}^m \circ \overline{\tau}$ and by Lemma 4.8, we get $\overline{\tau} = \overline{\tau}^n$. Thus, $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\sigma}^m \circ \overline{\tau}^n$. Since $\overline{\sigma}$ and $\overline{\tau}$ are IVF $(m, 0)$-ideal and IVF $(0, n)$-ideal of E, we have $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{\tau}^n$.

Conversely, suppose that $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{\tau}^n$ and R, L be an $(m, 0)$-ideal and a $(0, n)$-ideal of E, respectively. By Lemma 4.2, $\overline{\chi}_R$ and $\overline{\chi}_L$ are IVF $(m, 0)$-ideal and IVF $(0, n)$-ideal of E. By hypothesis we have $\overline{\chi}_{R\cap L} = \overline{\chi}_R \curlywedge \overline{\chi}_L = \overline{\chi}_R^m \circ \overline{\chi}_L^n = \overline{\chi}_{R^m L^n}$. Thus, $R \cap L = R^m L^n$. Hence by Theorem 4.12, E is (m, n)-regular. □

The following result is an immediate consequence of Lemma 4.8 and Theorem 4.13.

Corollary 4.14. *Let $\overline{\sigma}$ be an IVF $(m, 0)$-ideal and $\overline{\tau}$ be an IVF $(0, n)$-ideal of a semigroup E and m, n be positive integers. Then E is (m, n)-regular of E if and only if $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma} \circ \overline{\tau}$.*

Lemma 4.15 ([11]). *Let $\mathcal{R}_{(m,0)}$ and $\mathcal{L}_{(0,n)}$ be the set of all $(m, 0)$-ideal and $(0, n)$-ideal of a semigroup E, respectively, and m, n be positive integers. Then E is (m, n)-regular if and only if $R \cap L = R^m L \cap R L^n$ for each $R \in \mathcal{R}_{(m,0)}$ and $L \in \mathcal{L}_{(0,n)}$.*

Theorem 4.16. *Let $\overline{\sigma}$ be an IVF $(m, 0)$-ideal and $\overline{g}$ be an IVF $(0, n)$-ideal of a semigroup E and m, n be positive integers. Then E is (m, n)-regular of E if and only if $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{\tau} \curlywedge \overline{\sigma} \circ \overline{\tau}^n$.*

Proof. Suppoe that E is an (m,n)-regular semigroup of E. Then,

$$\overline{\sigma} \curlyvee \overline{\tau} \sqsubseteq (\overline{\sigma} \curlywedge \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \curlywedge \overline{\tau})^n \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n \sqsubseteq \overline{\sigma}^m \circ \overline{\tau}.$$

Thus, $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\sigma}^m \circ \overline{\tau}$. Similarly $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\sigma} \circ \overline{\tau}^n$. Thus, $\overline{\sigma} \curlywedge \overline{\tau} \sqsubseteq \overline{\sigma}^m \circ \overline{\tau} \curlywedge \overline{\sigma} \circ \overline{\tau}^n$. By assumption, $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{\tau} \curlywedge \overline{\sigma} \circ \overline{\tau}^n$.

Conversely, suppose that $\overline{\sigma} \curlywedge \overline{\tau} = \overline{\sigma}^m \circ \overline{g} \curlywedge \overline{\sigma} \circ \overline{\tau}^n$ and let R, L be an $(m,0)$-ideal and a $(0,n)$-ideal of E, respectively. Then by Lemma 4.2, $\overline{\chi}_R$ and $\overline{\chi}_L$ are IVF $(m,0)$-ideal and IVF $(0,n)$-ideal of E. By hypothesis we have $\overline{\chi}_{R\cap L} = \overline{\chi}_R \curlywedge \overline{\chi}_L = \overline{\chi}_R^m \curlywedge \overline{\chi}_L^n = \overline{\chi}_{R^mL^n}$. Thus, $R \cap L = R^mL^n$. Hence by Lemma 4.15, E is (m,n)-regular. □

Lemma 4.17. *Let $\overline{\sigma}$ be an IVF $(m,0)$-ideal (fuzzy $(0,n)$-ideal) of a semigroup E and m,n be positive integers. Then $\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}$ ($\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n$).*

Proof. The proof is obvious. □

Lemma 4.18. *Let E is (m,n)-regular of a semigroup E and m,n be positive integers. Then for each IVF (m,n)-ideal $\overline{\sigma}$ of E, there exist an IVF $(m,0)$-ideal $\overline{\tau}$ and an IVF $(0,n)$-ideal $\overline{\sigma}$ of E such that $\overline{\sigma} = \overline{\tau} \circ \overline{\sigma}$.*

Proof. Suppose that $\overline{\sigma}$ is an IVF (m,n)-ideal of E. Then, $\overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n \sqsubseteq \overline{\sigma}$. Since E is (m,n)-regular we have $\overline{\sigma} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n$. Thus, $\overline{\sigma} = \overline{\sigma}^m \overline{\mathcal{E}} \circ \circ \overline{\sigma}^n$. Let $\overline{\tau} = \overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}$ and $\overline{\sigma} = \overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n$. Then by Lemma 4.17, $\overline{\tau}$ and $\overline{h}$ are IVF $(m,0)$-ideal and IVF $(0,n)$-ideal of E, respectively. Since E is (m,n)-regular we have $\overline{\tau} = \overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}} = \overline{\sigma}^m \circ \overline{\mathcal{E}}, \overline{h} = \overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n = \overline{\mathcal{E}} \circ \overline{\sigma}^n$ and $\overline{\mathcal{E}} \sqsubseteq \overline{\mathcal{E}}^m \overline{\mathcal{E}}\, \overline{\mathcal{E}}^n = \overline{\mathcal{E}}^{m+n+1} \sqsubseteq \overline{\mathcal{E}}^2 \sqsubseteq \overline{\mathcal{E}}$. This implies that $\overline{\mathcal{E}} = \overline{\mathcal{E}}^2$. Thus, $\overline{\tau} \circ \overline{\sigma} = \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\mathcal{E}} \circ \overline{\sigma}^n = \overline{\sigma}^m \circ \overline{\mathcal{E}}^2 \circ \overline{\sigma}^n = \overline{\sigma}$. □

Lemma 4.19. *Let E is (m,n)-regular of a semigroup E and m,n be positive integers. Then $\overline{\sigma} \circ \overline{g}$ is an IVF (m,n)-ideal of E, for each $\overline{\sigma}$ and $\overline{\tau}$ are IVF (m,n)-ideal and IVF subset of E, respectively.*

Proof. Suppose that $\overline{\sigma}$ and $\overline{\tau}$ are IVF (m,n)-ideal and IVF subset of E. Then,

$$\begin{aligned}(\overline{\sigma} \circ \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \circ \overline{\tau})^n &= \underbrace{(\overline{\sigma} \circ \overline{\tau}) \circ (\overline{\sigma} \circ \overline{\tau}) \circ \cdots \circ (\overline{\sigma} \circ \overline{\tau})}_{m\text{-times}} \circ \overline{\mathcal{E}} \circ \underbrace{(\overline{\sigma} \circ \overline{\tau}) \circ (\overline{\sigma} \circ \overline{\tau}) \circ \cdots \circ (\overline{\sigma} \circ \overline{\tau})}_{n\text{-times}} \\ &= (\overline{\sigma} \circ \overline{\tau}) \circ \underbrace{(\overline{\sigma} \circ \overline{\tau}) \circ (\overline{\sigma} \circ \overline{\tau}) \circ \cdots \circ (\overline{\sigma} \circ \overline{\tau})}_{m-1\text{-times}} \circ \overline{\mathcal{E}} \circ \underbrace{(\overline{\sigma} \circ \overline{\tau}) \circ \cdots \circ (\overline{\sigma} \circ \overline{\tau})}_{n--1\text{-times}} \circ (\overline{\sigma} \circ \overline{\tau}) \\ &\sqsubseteq (\overline{\sigma} \circ \overline{\tau}) \circ \overline{\mathcal{E}} \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \circ \overline{\tau}) \sqsubseteq \overline{\sigma} \circ \overline{\mathcal{E}} \circ \overline{\tau} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \circ \overline{\tau} \sqsubseteq \overline{\sigma} \circ \overline{\mathcal{E}} \circ \overline{\tau}.\end{aligned}$$

Thus, $\overline{\sigma} \circ \overline{\tau}$ is an IVF (m,n)-ideal of E. □

The following result is an immediate consequence of Lemma 4.18 and 4.19.

Theorem 4.20. *Let E be a semigroup and m,n be positive integers. Then E is (m,n)-regular if and onlf if for each IVF (m,n)-ideal $\overline{\sigma}$ of E, there exist an IVF $(m,0)$-ideal $\overline{\tau}$ and an IVF $(0,n)$-ideal $\overline{\sigma}$ of E such that $\overline{\sigma} = \overline{\tau} \circ \overline{\sigma}$.*

Theorem 4.21. *Let f be an IVF $(m,0)$-ideal and $\overline{g}$ be an IVF $(0,n)$-ideal of a semigroup E such that $\overline{\sigma} \circ \overline{\tau} = \overline{\tau} \circ \overline{\sigma}$. Then the product $\overline{\sigma} \circ \overline{\tau}$ is an IVF (m,n)-ideal of E.*

Proof. By assumption, we have that

$$(\overline{\sigma} \circ \overline{\tau}) \circ (\overline{\sigma} \circ \overline{\tau}) = (\overline{\sigma} \circ \overline{\sigma}) \circ (\overline{\tau} \circ \overline{\tau}) \sqsubseteq \overline{\sigma} \circ \overline{\tau}.$$

Thus, $\overline{\sigma} \circ \overline{\tau}$ is an IVF subsemigroup of E. Also, we have

$$(\overline{\sigma} \circ \overline{\tau})^m \circ \overline{\mathcal{E}} \circ (\overline{\sigma} \circ \overline{\tau})^n = \overline{\sigma}^m \circ \overline{\tau}^m \circ (\overline{\mathcal{E}} \circ \overline{\tau}^n)\overline{\sigma}^n \sqsubseteq \overline{\sigma}^m \circ \overline{\tau}^n \circ \overline{\tau} \circ \overline{\sigma}^n = \overline{\sigma}^{m+n} \circ \overline{\tau}^{m+1} \sqsubseteq \overline{\sigma} \circ \overline{\tau}.$$

Hence, $\overline{\sigma} \circ \overline{\tau}$ is an IVF (m,n)-ideal of E. □

Definition 4.22. An IVF (m,n)-ideal $\overline{\sigma}$ of a semigroup E is said to be *minimal* if for each IVF (m,n)-ideal $\overline{\sigma}'$ of E, $\overline{\sigma}' \sqsubseteq \overline{\sigma}$ implies $\overline{\sigma}' = \overline{\sigma}$.

Theorem 4.23. *Let E be an (m,n)-regular semigroup and m,n be positive integers. Then an IVF subset $\overline{\sigma}$ of E is a minimal IVF (m,n)-ideal of E if and only if there exist a minimal IVF $(m,0)$-ideal $\overline{\tau}$ and a minimal IVF $(0,n)$-ideal $\overline{h}$ of E such that $\overline{\sigma} = \overline{g} \circ \overline{h}$.*

Proof. Suppose that $\overline{\sigma}$ is a minimal IVF (m,n)-ideal of E. By Lemma 4.18, $\overline{\sigma} = (\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}) \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n)$. Next to show that $\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}$ is a minimal IVF $(m,0)$-ideal of E. Suppose that $\overline{\sigma}'$ is an IVF $(m,0)$-ideal of E such that $\overline{\sigma}' \sqsubseteq (\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}})$. By assumption and by Corollary 4.14, $(\overline{\sigma} \curlyvee \overline{\sigma}^m \curlyvee \overline{\mathcal{E}}) \curlywedge (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) = (\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}) \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n)$. Similarly $\overline{\sigma}' \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) = \overline{\sigma}' \curlywedge (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) \sqsubseteq (\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}) \curlywedge (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) = \overline{\sigma}$. By Lemma 4.19, $\overline{\sigma}' \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n)$ is an IVF (m,n)-ideal of S. Since $\overline{\sigma}' \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) \sqsubseteq \overline{\sigma}$ we have $\overline{\sigma}' \circ (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) = \overline{\sigma}$. Thus, $(\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}) \curlywedge (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n) = \overline{\sigma}' \curlywedge (\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n)$. Since $\overline{\sigma} \sqsubseteq (\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}) \curlywedge (\overline{\sigma} \curlywedge \overline{\mathcal{E}} \circ \overline{\sigma}^n)$, we have $\overline{\sigma} \sqsubseteq \overline{\sigma}'$. Thus, $\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}'$. Hence, $\overline{\sigma}' = \overline{\sigma}$. Therefore, $\overline{\sigma} \curlyvee \overline{\sigma}^m \circ \overline{\mathcal{E}}$ is a minimal IVF $(m,0)$-ideal of E. Similarly, we can prove that $\overline{\sigma} \curlyvee \overline{\mathcal{E}} \circ \overline{\sigma}^n$ is a minimal IVF $(0,n)$-ideal of E.

Conversely, suppose that $\overline{\sigma} = \overline{\tau} \circ \overline{h}$ for some minimal IVF $(m,0)$-ideal $\overline{\tau}$ and minimal IVF $(0,n)$-ideal $\overline{\sigma}$ of E. By Theorem 4.20, $\overline{\sigma}$ is an IVF (m,n)-ideal of E. Let $\overline{\Omega}$ be IVF (m,n)-ideal of E such that $\overline{\Omega} \sqsubseteq \overline{\sigma}$. Then, $\overline{\Omega}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\sigma}^m \circ \overline{\mathcal{E}} \sqsubseteq (\overline{\tau} \circ \overline{\sigma})^m \circ \overline{\mathcal{E}} = ((\overline{\tau} \circ \overline{\sigma}) \circ (\overline{\tau} \circ \overline{\sigma}) \circ \cdots \circ (\overline{\sigma} \circ \overline{\sigma}) \circ \overline{\mathcal{E}}) \sqsubseteq (\overline{\tau} \circ \overline{\sigma}) \circ (\overline{\tau} \circ \overline{h}) \circ \cdots \circ (\overline{g} \circ \overline{\sigma}) \circ \overline{\mathcal{E}} \sqsubseteq \overline{\tau} \circ \overline{\mathcal{E}} \sqsubseteq (\overline{g}^m \circ \overline{\mathcal{E}} \circ \overline{\tau}^n) \circ \overline{\mathcal{E}} \sqsubseteq \overline{\tau}^m \circ \overline{\mathcal{E}} \sqsubseteq \overline{\tau}$. Since $\overline{\Omega}^m \circ \overline{\mathcal{E}}$ is an IVF $(m,0)$-ideal of E and $\overline{\tau}$ is a minimal IVF $(m,0)$-ideal of E, we have $\Omega^m \circ \overline{\mathcal{E}} = \overline{\tau}$. Similarly, we can prove that $\overline{\mathcal{E}} \circ \overline{\Omega}^n = \overline{\sigma}$. Thus, $\overline{\sigma} = \overline{\tau} \circ \overline{\sigma} = (\overline{\Omega}^m \circ \overline{\mathcal{E}}) \circ (\overline{\mathcal{E}} \circ \overline{\Omega}^n) = \overline{\Omega}^m \circ \overline{\mathcal{E}} \circ \overline{\mathcal{E}} \circ \overline{\Omega}^n \sqsubseteq \overline{\Omega}^m \circ \overline{\mathcal{E}} \circ \overline{\Omega}^n \sqsubseteq \overline{\Omega}$. Hence, $\overline{\sigma}$ is a minimal IVF (m,n)-ideal of E. □

Corollary 4.24. *Let E be an (m,n)-regular semigroup and m,n be positive integers. Then E has at least one minimal IVF (m,n)-ideal if and only if E has at least one minimal IVF $(m,0)$-ideal and at least one minimal IVF $(0,n)$-ideal.*

5. Conclusion

In this paper, we give the concept of interval valued fuzzy (m,n)-ideals and study the properties of interval valued fuzzy (m,n)-ideals in semigroups. In the future we study neurotrophic (m,n)-ideals in semigroup or algebraic.

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