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Inverse nodal problem with fractional order conformable type derivative

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Abstract

The present paper is about the inverse nodal problem for Sturm-Liouville problem with eigenparameter in the boundary condition using the conformable derivative approach. We defined a function $f(\mu)$ generally in the boundary condition and we found the zeros of the eigenfunctions (nodal points) by nucleus function $K(x,t)$, which is a derived transformation operator. Then, we obtained the potential function by using the nodal parameters.

Keywords: Sturm Liouville problem, conformable derivative, nodal points, potential function, eigenparameter.

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1. Introduction

When we compare the usual fractional derivatives approaches with the conformable derivative, we see that conformable derivative has some advantages, for example, some properties related to calculus and some theorems which are commonly used in classical derivatives, do not lose their functionality in conformable derivatives. In addition, the conformable derivative can be more appropriately applied to many models in physics, chemistry and engineering compared to the usual fractional derivatives approaches. Thus, it is possible to numerically solve the differential equations with conformable derivative that we obtained from such models. This approach extended the derivative to the whole interval $[0,1]$ and details on these can be found as [1] introduced conformable derivatives. Further, valuable work has been done ion the area of fractional calculus including history, new results, new operators and applications in diverse fields of science and technology, for which we refer some literature as [5–8, 14, 17–19, 24, 25, 29, 32, 33, 35, 36].

In Sturm-Liouville theory, the solution of the inverse problem means finding the coefficients of the equation and the constants in the boundary conditions using spectral parameters. These spectral parameters are; eigenvalues, eigenfunctions, norming constants, and also nodal points can be considered. By

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changing these parameters, the problem can be named differently, for example, if the problem is treated with eigenvalues the inverse eigenvalue problem occurs and if its with the nodal points the inverse nodal problem arises. However there are many studies on fractional Sturm-Liouville Problem (SLP) and inverse SLP as discussed in [2, 4, 11, 16, 20, 23, 28, 30, 31, 34] and the inverse nodal problem can be found in [3, 9, 10, 12, 13, 15, 21, 22, 26, 27], it seems that these results are not enough as such many different researches are continuously being carried out in this field. Some authors introduced a formula for the potential function in inverse nodal problem as in [3]. The authors in [31] treated the inverse eigenvalue problem of the SLP using conformable derivative approach with eigenparameter in the boundary condition and here in this work we treated the same problem under the same derivative approach and developed the inverse nodal problem.

2. Some basic terms

The following definitions and theorems can be found in [1, 26].

Definition 2.1. The α order conformable derivative of any $g : [0, \infty) \to R$ is defined as

$$D_x^\alpha g(x) = \lim_{e \to 0} \frac{g(x + ex^{1-\alpha}) - g(x))}{e}$$

for all $x > 0$, $\alpha \in (0, 1]$.

Definition 2.2. For a function g, the conformable integral of g of order α is given as

$$I_\alpha g(x) = \int_0^x g(t) d_\alpha t = \int_0^x t^{\alpha-1} g(t) dt$$

for all $x > 0$.

Lemma 2.3. *If g is differentiable, then for $x > a$, $D_x^\alpha I_\alpha g(x) = g(x) - g(a)$.*

Theorem 2.4. *Let g, h are two differentiable functions. Then*

$$\int_a^b g(x) D_x^\alpha (h(x)) d_\alpha x = gh|_a^b - \int_a^b h(x) D_x^\alpha (g(x)) d_\alpha x.$$

Lemma 2.5. *The α-Leibniz integral rule gives*

$$D_x^\alpha \left[\int_{a(x)}^{b(x)} g(x,t) d_\alpha t \right] = D_x^\alpha b(x) g(x, b(x)) b(x)^{\alpha-1} - D_x^\alpha a(x) g(x, a(x)) a(x)^{\alpha-1} + \int_a^b D_x^\alpha (g(x,t)) d_\alpha t$$

for $a(x) \leqslant t \leqslant b(x)$ while $a(x)$ and $b(x)$ are both α-differentiable for $x_0 \leqslant x \leqslant x_1$.

Let us consider the conformable SLP $L_\alpha(q, f)$ from [31],

$$-D_x^\alpha D_x^\alpha y + q(x) y = \mu y \tag{2.1}$$

with

$$D_x^\alpha y(0) = 0, \; D_x^\alpha y(\pi) + f(\mu) y(\pi) = 0, \tag{2.2}$$

where D_x^α defines the conformable derivative of order α, $0 < \alpha \leqslant 1$, q is a real valued continuous function and

$$f(\mu) = a_1 \sqrt{\mu} + a_2 \sqrt{\mu}^2 + \cdots + a_r \sqrt{\mu}^r, \quad a_i \in R, a_r \neq 0, r \in Z^+.$$

Let $S(x,\mu)$ be solution of (2.1) under the condition $y(0)=1$. Then, $S(x,\mu)$ can be expressed as

$$S(x,\mu)=\cos\left(\frac{\sqrt{\mu}}{\alpha}x^{\alpha}\right)+\int_0^x K(x,t)\cos\left(\frac{\sqrt{\mu}}{\alpha}t^{\alpha}\right)d_{\alpha}t. \tag{2.3}$$

The kernel $K(x,t)$ is solution of

$$D_x^{\alpha}(D_x^{\alpha}K(x,t))-q(x)K(x,t)=D_t^{\alpha}(D_t^{\alpha}K(x,t)), \tag{2.4}$$
$$K(x,x)=\int_0^x\frac{1}{1+h^{\alpha-1}}q(h)d_{\alpha}h.$$

Conversely, if $K(x,t)$ is a solution of (2.4), then (2.3) satisfies (2.1) under the condition $y(0)=1$. This is known as transformation operator in the Sturm-Liouville theory. It should be noted that the classical case of this problem was considered by the authors of [11].

It is well known that μ is an eigenvalue of the problem (2.1) under (2.2) if and only if

$$W_{\alpha}(\mu)=D_x^{\alpha}S(\pi,\mu)+f(\mu)S(\pi,\mu)=0. \tag{2.5}$$

Then from [31] and by (2.5), $W_{\alpha}(\mu)$ can be written as

$$\begin{aligned}W_{\alpha}(\mu)=&-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)+K(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)\\&+\int_0^{\pi}D_x^{\alpha}K(\pi,t)\cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)d_{\alpha}t+f(\mu)\cos\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)\\&+\frac{f(\mu)}{\sqrt{\mu}}K(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\alpha}\pi^{\alpha}\right)-\frac{f(\mu)}{\sqrt{\mu}}\int_0^{\pi}D_t^{\alpha}K(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\alpha}t^{\alpha}\right)d_{\alpha}t=0.\end{aligned}$$

3. The main results

In this part, we will give the nodal points, nodal lengths and the reconstruction of the potential with proof for each. The following lemma gives us the asymptotic form of the eigenvalues for the problem (2.1) in addition eigenvalues are roots of the equation $W_{\alpha}(\mu)=0$. It is clear that this function just depends on μ and α not x.

Let $\{\mu_n\}_{n\geqslant 0}$ be the spectrum of (2.1) under (2.2) and $y(x,\mu_n)$ be the eigenfunctions subject to the eigenvalues. Obviously from Oscillation theorem, $y(x,\mu_n)$ have exactly n roots which lie in $(0,\pi)$. Further, define

$$Y_{\alpha}=\left\{y_j^n=\frac{(x_j^n)^{\alpha}}{\alpha},x_j^n\in X_{\alpha}\right\},n>0,$$

and $j=1,2,\dots,n$. Besides, $X_{\alpha}=\left\{x_j^n:n\in\mathbb{N}\right\}$ denotes a set which is consisting of the nodal set of (2.1) under (2.2). That is , $y(x_j^n,\mu_n)=0$. In addition, representing j^{th} nodal domain of the n^{th} element by $I_{\alpha j}^n=\left[y_j^n,y_{j+1}^n\right]$. Also, if $l_j^n=y_{j+1}^n-y_j^n$ be the nodal length of the j^{th} domain. Another mapping is needed need to describe which will play main role in solving the inverse nodal problem $j_n(y)$ for being the largest index j, such that $0\leqslant y_j^n\leqslant y$. Then, $j=j_n(y)$ if and only if $y\in\left[y_j^n,y_{j+1}^n\right)$.

The authors of [11] obtained the classical asymptotic form of the eigenvalues of this problem and below are the asymptotic form of the eigenvalues using the conformable derivative.

Lemma 3.1. *For $r=1$, then the eigenvalues $\{\mu_n\}$ has the asymptotic form as $n\to\infty$,*

$$\sqrt{\mu_n}=\frac{\alpha n}{\pi^{\alpha-1}}+\frac{\alpha\arctan a_1}{\pi^{\alpha}}+\frac{\alpha}{\pi^{\alpha}n}\int_0^{\pi}\frac{q(h)}{(1+h^{\alpha-1})}d_{\alpha}h+O\left(\frac{1}{n^2}\right),$$

for $r = 2$,

$$\sqrt{\mu_n} = \frac{\alpha n}{\pi^{\alpha-1}} + \frac{\alpha}{2\pi^{\alpha-1}} + \frac{\alpha}{\pi^{\alpha}(n+\frac{1}{2})}\left[\int_0^{\pi} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h - \frac{1}{a_2}\right] + O\left(\frac{1}{n^2}\right),$$

for $r \geqslant 3$,

$$\sqrt{\mu_n} = \frac{\alpha n}{\pi^{\alpha-1}} + \frac{\alpha}{2\pi^{\alpha-1}} + \frac{\alpha}{\pi^{\alpha}(n+\frac{1}{2})}\left[\int_0^{\pi} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n^2}\right).$$

Theorem 3.2. *The nodal points of the problem* (2.1) *under* (2.2) *provide the asymptotic form as* $n \to \infty$, *for* $r = 1$

$$y_j^n = \left(j - \frac{1}{2}\right)\frac{\pi^{\alpha}}{\alpha n} + \left(j - \frac{1}{2}\right)\frac{\pi^{\alpha-1}\arctan a_1}{\alpha n^2} + \frac{\pi^{2\alpha-2}}{\alpha^2 n^2} K(y_j^n, y_j^n) + O\left(\frac{1}{n^3}\right),$$

and for $r \geqslant 2$

$$y_j^n = \left(j - \frac{1}{2}\right)\left(1 + \frac{1}{2n}\right)\frac{\pi^{\alpha}}{\alpha n} + \left(1 + \frac{1}{2n}\right)^2 \frac{\pi^{2\alpha-2}}{\alpha^2 n^2} K(y_j^n, y_j^n) + O\left(\frac{1}{n^3}\right).$$

Proof. Considering the solution of (2.1) under (2.2),

$$S(x,\mu) = \cos\left(s\frac{x^{\alpha}}{\alpha}\right) + \int_0^x K(x,t)\cos\left(s\frac{t^{\alpha}}{\alpha}\right) d_\alpha t$$

or

$$S(x,\mu) = \cos\left(s\frac{x^{\alpha}}{\alpha}\right) + \frac{K(x,x)}{s}\sin\left(s\frac{t^{\alpha}}{\alpha}\right) + O\left(\frac{1}{s}\right),$$

where $s = \sqrt{\mu}$, then by definition of nodal points, it yields

$$\begin{aligned} S(x,\mu) = \cos\left(s\frac{x^{\alpha}}{\alpha}\right) + \frac{K(x,x)}{s}\sin\left(s\frac{t^{\alpha}}{\alpha}\right) + O\left(\frac{1}{s}\right) &= 0, \\ \cot\left(s\frac{x^{\alpha}}{\alpha}\right) &= -\frac{K(x,x)}{s} + O\left(\frac{1}{s}\right), \\ \left(s\frac{x^{\alpha}}{\alpha}\right) &= \operatorname{arccot}\left[-\frac{K(x,x)}{s} + O\left(\frac{1}{s}\right)\right], \end{aligned}$$

and using Taylor's expansion of $\operatorname{arccot}(x)$ near to zero we have

$$y_j^n = \frac{(j-\frac{1}{2})\pi}{s_n} + \frac{K(y_j^n, y_j^n)}{s_n^2} + O\left(\frac{1}{s_n^2}\right). \tag{3.1}$$

If $r = 1$,

$$s_n = \frac{\alpha n}{\pi^{\alpha-1}} + \frac{\alpha \arctan a_1}{\pi^{\alpha}} + O\left(\frac{1}{n^2}\right), \quad \frac{1}{s_n} = \frac{\pi^{\alpha-1}}{\alpha n} + \frac{\arctan a_1}{\alpha n^2}\pi^{\alpha-2} + O\left(\frac{1}{n^3}\right), \tag{3.2}$$

and also

$$\frac{1}{s_n^2} = \frac{\pi^{2\alpha-2}}{\alpha^2 n^2} + O(\frac{1}{n^3}). \tag{3.3}$$

Then inserting (3.2) and (3.3) in (3.1), gives the first case. For $r \geqslant 2$,

$$\begin{aligned} s_n &= \frac{\alpha n}{\pi^{\alpha-1}} + \frac{\alpha}{2\pi^{\alpha-1}} + O\left(\frac{1}{n}\right), \\ \frac{1}{s_n} &= \frac{\pi^{\alpha-1}}{\alpha n} + \frac{\pi^{\alpha-1}}{2\alpha n^2} + O\left(\frac{1}{n^3}\right) = \frac{\pi^{\alpha-1}}{\alpha n}\left(1 + \frac{1}{2n}\right) + O\left(\frac{1}{n^3}\right). \end{aligned} \tag{3.4}$$

This implies that

$$\frac{1}{s_n^2} = \frac{\pi^{2\alpha-2}}{\alpha^2 n^2}\left(1+\frac{1}{2n}\right)^2 + O\left(\frac{1}{n^3}\right). \tag{3.5}$$

Then using (3.4) and (3.5) in (3.1), we have

$$y_j^n = (j-\frac{1}{2})\left(1+\frac{1}{2n}\right)\frac{\pi^\alpha}{\alpha n} + \left(1+\frac{1}{2n}\right)^2 \frac{\pi^{2\alpha-2}}{\alpha^2 n^2} K(y_j^n, y_j^n) + O\left(\frac{1}{n^3}\right),$$

which completes the proofs. □

Theorem 3.3. *The nodal lengths of the problem* (2.1) *under* (2.2) *are*

$$l_j^n = \frac{\pi^\alpha}{n\alpha} + \frac{\pi^{\alpha-1}\arctan a_1}{\alpha n^2} + \frac{\pi^{2\alpha-2}}{\alpha^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n^3}\right)$$

for $r = 1$, *and for* $r \geqslant 2$,

$$l_j^n = \left(1+\frac{1}{2n}\right)\frac{\pi^\alpha}{n\alpha} + \left(1+\frac{1}{2n}\right)^2 \frac{\pi^{2\alpha-2}}{\alpha^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n^2}\right).$$

Proof. By the definition of the nodal lengths, $l_j^n = y_{j+1}^n - y_j^n$, the proofs for $r = 1$ and $r \geqslant 2$ can be easily obtained, respectively. □

3.1. *Reconstruction of the potential*

Besides calculating the spectral parameters in two cases so far, we will give the potential function for the two cases. Formulas for potential function q with respect to two different cases of eigenvalues can be given in two cases, $r = 1$, and $r \geqslant 2$, as follows.

Theorem 3.4. *The reconstruction of the potential of the problem* (2.1) *under* (2.2) *for* $q \in C[0, \pi]$ *is, for* $r = 1$,

$$q(x) = (1+x^{\alpha-1}) \lim_{n\to\infty} s_n^2 \left(\frac{s_n l_j^n}{\pi} - 1\right),$$

and for $r \geqslant 2$,

$$q(x) = (1+x^{\alpha-1}) \lim_{n\to\infty} \frac{s_n^3}{\pi}\left(1+\frac{1}{2n}\right).$$

Proof. However we study conformable problem, the proof will be similar with the classical case. By definition of nodal lengths, we have

$$\left(l_j^n - \pi\left(\frac{\pi^{\alpha-1}}{n\alpha} + \frac{\pi^{\alpha-2}\arctan a_1}{\alpha n^2}\right)\right)\frac{\alpha^2 n^2}{\pi^{2\alpha-2}} = \int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h + O\left(\frac{1}{n^3}\right).$$

Then, from (3.2) and (3.3) we have

$$\left(l_j^n - \frac{\pi}{s_n}\right) s_n^2 = \int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h + O\left(\frac{1}{n^3}\right),$$

that is

$$\left(\frac{s_n l_j^n}{\pi} - 1\right) = \frac{1}{s_n \pi}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1+h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n}\right).$$

By mean value theorem for the integrals there is a $z \in (y_j^n, y_{j+1}^n)$ such that

$$\left(\frac{s_n l_j^n}{\pi} - 1\right) = \frac{l_j^n}{s_n \pi} \frac{q(z)}{(1 + z^{\alpha-1})} + O\left(\frac{1}{n}\right).$$

From the fact that, as $n \to \infty$, $l_j^n \to \frac{\pi}{s_n}$, we obtain

$$q(x) = (1 + x^{\alpha-1}) \lim_{n\to\infty} s_n^2 \left(\frac{s_n l_j^n}{\pi} - 1\right),$$

and for the proof for $r \geqslant 2$, from (3.1), (3.4), and (3.5), we obtain

$$\left(1 + \frac{1}{2n}\right)^2 l_j^n = \pi\left(1 + \frac{1}{2n}\right)\frac{1}{s_n} + \frac{1}{s_n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1 + h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n^5}\right),$$

which becomes

$$\left(1 + \frac{1}{2n}\right) l_j^n \frac{s_n^3}{\pi} = \left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(h)}{(1 + h^{\alpha-1})} d_\alpha h\right] + O\left(\frac{1}{n^5}\right),$$

also, by mean value theorem as in the above case, we have

$$\left(1 + \frac{1}{2n}\right) l_j^n \frac{s_n^3}{\pi} = l_j^n \frac{q(z)}{(1 + z^{\alpha-1})} + O\left(\frac{1}{n^5}\right),$$

and by taking limit as $n \to \infty$ we have

$$q(x) = (1 + x^{\alpha-1}) \lim_{n\to\infty} \frac{s_n^3}{\pi}\left(1 + \frac{1}{2n}\right).$$

□

The following function consists of fractional integral of q and nodal lengths. Note that this function with classical derivative was given in [3]. Let us consider

$$F_n(x) = \frac{2n^2\alpha^2}{\pi^{2\alpha-2}}\left[\frac{n\alpha l_{\alpha j}^n}{\pi} + \frac{l_{\alpha j}^n}{2n^2\alpha\pi^2}\int_0^\pi q(t) d_\alpha t - 1\right]. \tag{3.6}$$

Following lemmas are to complete the proof of Theorem 3.4 and because of similarity with the classical case, the proof is omitted.

Lemma 3.5. *Suppose that the sequence* $g_k \in C[0,\pi]$ *converges to any function* g *in the space* L_1^α. *Then for any* $\varepsilon > 0$, *with* $j = j_n(y)$,

$$\left\|\frac{\mu_n}{\alpha\pi}\int_{y_j^n}^{y_{j+1}^n} (g_k(t) - g(t))\, d_\alpha t\right\|_1 < \varepsilon.$$

Lemma 3.6. *Assume that* $q \in L_1^\alpha(0,\pi)$ *and* $r = 1$. *Then, as* $n \to \infty$,

$$\left\|\frac{\mu_n}{\alpha\pi}\int_{y_j^n}^{y_{j+1}^n} q(t) d_\alpha t - q(y)\right\|_1 \to 0, \quad j = j_n(y).$$

Theorem 3.7. $F_n(x)$ *converges to the potential function* $q(x)$ *in* L_1^α.

Proof. Proof of the theorem is omitted. □

4. Conclusion

We have shown that generally, the inverse nodal problem is solvable for a SLP depending on conformable derivative and also containing the spectral parameter in boundary conditions. We also reconstructed the potential function by using the spectral parameters. This study is significant in two respects. First, the function we consider in the boundary condition is an n^{th} degree polynomial. Thus, special cases can be easily obtained. Secondly, the conformable derivative we have discussed here is more general than the classical derivative. Eventually, it is possible to obtain some different results of spectral theory by conformable derivative.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66. 1, 2
[2] İ. Adalar, A. S. Ozkan, *Inverse problems for a conformable fractional Sturm-Liouville operator*, J. Inverse Ill-Posed Probl., **28** (2020), 775–782. 1
[3] W. F. Alfwzan, H. Khan, J. Alzabut, *Stability analysis for a fractional coupled Hybrid pantograph system with p-Laplacian operator*, Results Control Optim., **14** (2024), 1–11. 1, 3.1
[4] B. P. Allahverdiev, H. Tuna, Y. Yalçinkaya, *Conformable fractional Sturm-Liouville equation*, Math. Methods Appl. Sci., **42** (2019), 3508–3526. 1
[5] M. Al-Refai, T. Abdeljawad, *Fundamental results of conformable Sturm-Liouville eigenvalue problems*, Complexity, **2017** (2017), 7 pages. 1
[6] A. Atangana, D. Baleanu, A. Alsaedi, *New properties of conformable derivative*, Open Math., **13** (2015), 889–898.
[7] D. Baleanu, Z. B. Güvenç, J. A. T. Machado, *New trends in nanotechnology and fractional calculus applications*, Springer, New York, (2010).
[8] D. Baleanu, J. Alzabut, J. M. Jonnalagadda, Y. Adjabi, M. M. Matar, *A coupled system of generalized Sturm-Liouville problems and Langevin fractional differential equations in the framework of nonlocal and nonsingular derivatives*, Adv. Difference Equ., **2020** (2020), 30 pages. 1
[9] Y. Çakmak, *Inverse nodal problem for a conformable fractional diffusion operator*, Inverse Probl. Sci. Eng., **29** (2021), 1308–1322. 1
[10] X. Chen, Y. H. Cheng, C. K. Law, *Reconstructing potentials from zeros of one eigenfunction*, Trans. Amer. Math. Soc., **363** (2011), 4831–4851. 1
[11] Y. Chuanfu, Y. Xiaoping, *Ambarzumyan's theorem with eigenparameter in the boundary conditions*, Acta Math. Sci. Ser. B (Engl. Ed.), **31** (2011), 1561–1568. 1, 2, 3
[12] T. Gulsen, E. Yilmaz, S. Akbarpoor, *Numerical investigation of the inverse nodal problem by Chebisyhev interpolation method*, Therm. Sci., **22** (2018), 123–136. 1
[13] Y. Guo, G. Wei, *Inverse problems: dense nodal subset on an interior subinterval*, J. Differential Equations, **255** (2013), 2002–2017. 1
[14] S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, Y.-M. Chu, *On multi-step methods for singular fractional q-integro-differential equations*, Open Math., **19** (2021), 1378–1405. 1
[15] Y.-T. Hu, N. P. Bondarenko, C.-F. Yang, *Traces and inverse nodal problem for Sturm-Liouville operators with frozen argument*, Appl. Math. Lett., **102** (2020), 7 pages. 1
[16] B. Keskin, *Inverse problems for one dimensional conformable fractional Dirac type integro differential system*, Inverse Problems, **36** (2020), 10 pages. 1
[17] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70. 1
[18] Z. A. Khan, K. Shah, B. Abdalla, T. Abdeljawad, *A numerical study of complex dynamics of a chemostat model under fractal-fractional derivative*, Fractals, **31** (2023), 24 pages.
[19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, (2006). 1
[20] M. Klimek, O. P. Agrawal, *Fractional Sturm-Liouville problem*, Comput. Math. Appl., **66** (2013), 795–812. 1
[21] H. Koyunbakan, K. Shah, T. Abdeljawad, *Well-posedness of inverse Sturm-Liouville problem with fractional derivative*, Qual. Theory Dyn. Syst., **22** (2023), 15 pages. 1
[22] C. K. Law, J. Tsay, *On the well-posedness of the inverse nodal problem*, Inverse problems, **17** (2001), 1493–1512. 1
[23] S. Mosazadeh, *A new approach to uniqueness for inverse Sturm-Liouville problems on finite intervals*, Turkish J. Math., **41** (2017), 1224–1234. 1

[24] J. T. Machado, V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 1140–1153. 1

[25] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, New York, (1993). 1

[26] H. Mortazaasl, A. J. Akbarfam, *Trace formula and inverse nodal problem for a conformable fractional Sturm-Liouville problem*, Inverse Probl. Sci. Eng., **28** (2020), 524–555. 1, 2

[27] J. R. McLaughlin, *Inverse spectral theory using nodal points as data—a uniqueness result*, J. Differential Equations, **73** (1988), 354–362. 1

[28] J. P. Pinasco, C. Scarola, *A nodal inverse problem for second order Sturm-Liouville operators with indefinite weights*, Appl. Math. Comput., **256** (2015), 819–830. 1

[29] M. Rivero, J. Trujillo, M. Velasco, *A fractional approach to the Sturm-Liouville problem*, Open Phys., **11** (2013), 1246–1254. 1

[30] M. K. Sadabad, A. J. Akbarfam, B. Shiri, *A numerical study of eigenvalues and eigenfunctions of fractional Sturm-Liouville problems via Laplace transform*, Indian J. Pure Appl. Math., **51** (2020), 857–868. 1

[31] A. Sa'idu, H. Koyunbakan, *Inverse fractional SLP with eigenparameter in the boundary conditions*, Mathematical Methods in the Applied Sciences, **45** (2022), 11003–11012. 1, 2, 2

[32] A. Sa'idu, H. Koyunbakan, *Transmutation of conformable Sturm-Liouville operator with exactly solvable potential*, Filomat, **37** (2023), 3383–390. 1

[33] A. Sa'idu, H. Koyunbakan, *A conformable inverse problem with constant delay*, J. Adv. App. Comput., **10** (2023), 26–38. 1

[34] Y. P. Wang, *Inverse problems for discontinuous Sturm-Liouville operators with mixed spectral data*, Inverse Probl. Sci. Eng., **23** (2015), 1180–1198. 1

[35] Y. Wang, J. Zhou, Y. Li, *Fractional Sobolev's spaces on time scales via conformable fractional calculus and their application to a fractional differential equation on time scales*, Adv. Math. Phys., **2016** (2016), 21 pages. 1

[36] G. Yang, B. Shiri, H. Kong, G.-C. Wu, *Intermediate value problems for fractional differential equations*, Comput. Appl. Math., **40** (2021), 20 pages. 1