J. Math. Computer Sci., 34 (2024), 152–161

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Analytical study to solving the inhomogeneous pantograph delay equation: the exact solution

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Abstract

This paper solves the pantograph delay equation which includes inhomogeneous term. The inhomogeneous term is in the form of a class of exponential functions. An efficient transformation is introduced to reduce the inhomogeneous pantograph delay differential equation (IPDDE) to the homogeneous pantograph delay differential equation (HPDDE), which is a homogeneous model. It is found that the solution of the IPDDE depends mainly on the solution of HPDDE. It is well-known in the literature that the analytical solution of the HPDDE is already obtained in a closed series form. Such ready solution of the HPDDE is invested in this paper and accordingly, the solution of the IPDDE under consideration is established. Also, several exact solutions of the present model are determined at specific conditions and values of the parameters. The solutions of some examples in the literature are obtained in exact forms as special cases of the current results. Moreover, the properties of the obtained solutions are theoretically and graphically addressed.

Keywords: Pantograph, delay, inhomogeneous, exact solution, series solution.

2020 MSC: 34A99, 34A30.

1. Introduction

In this paper, we consider the inhomogeneous pantograph delay differential equation (IPDDE):

$$y'(t) = ay(t) + by(ct) + \alpha e^{\beta t}, \quad y(0) = \lambda, \quad t \geqslant 0, \tag{1.1}$$

where a, b, β, c, and λ are real constants. In the absence of the inhomogeneous term, i.e., at $\alpha = 0$, the IPDDE (1.1) reduces to the homogeneous pantograph delay differential equation (HPDDE):

$$y'(t) = ay(t) + by(ct), \quad y(0) = \lambda, \quad t \geqslant 0. \tag{1.2}$$

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doi: 10.22436/jmcs.034.02.05

Received: 2023-10-30 *Revised:* 2023-11-22 *Accepted:* 2024-01-31

The case $b = 0$ transforms the delay model (1.1) to an initial value problem (IVP) of linear ordinary differential equation (ODE), i.e., $y'(t) - ay(t) = \alpha e^{\beta t}$, which can be easily solved exactly. In addition, the case $c = 1$ leads to the ODE $y'(t) - (a + b)y(t) = \alpha e^{\beta t}$, where the exact solution is also available in this case. Beside, the case $c = 0$ is not of interest and accordingly the cases $b = 0$, $c = 0$, and $c = 1$ reduce Eqs. (1.1) to three different linear first-order ODEs which will be excluded when solving the IPDDE (1.1).

The HPDDE (1.2) has been solved by many authors in the literature [19, 20, 26–29] using different numerical techniques. In addition, multi and generalized forms of the pantograph model have been addressed by the authors [17, 18]. Moreover, considerable efforts have been observed to deduce different analytical solutions for the HPDDE (1.2) such as Al-Enazy et al. [3] using the Adomian decomposition method (ADM), Albidah et al. [2] via the Homotopy Perturbation Method (HPM), El-Zahar and Ebaid [16] utilizing a direct ansatz approach, and Alrebdi and Al-Jeaid [6] through applying the Laplace Transform (LT). A popular model known as Ambartsumian equation is an interesting special case of the HPDDE (1.2) which has been exactly solved by Bakodah and Ebaid [8] and approximately by Ebaid et al. [11]. Although the current model is linear and can be dealt with by the LT approach, it requires comprehensive work as can be observed when applying the LT to solve different models in physics, science and engineering, see Refs. [4, 5, 7, 9, 10, 12, 14, 15, 21, 23–25].

Thus, it is main objective of this work to introduce a simple but effective method to solve the IPDDE (1.1). In this context, it will be indicated that the IPDDE (1.1) can be transformed to the HPDDE: $z'(t) = az(t) + bz(ct)$ via a suitable transformation which contains a certain relation between the exponent β and the constants a and c. By the aide of the ready solutions of the HPDDE in Refs. [2, 3, 6, 16], the solution of the present model will be established in different analytical forms. Furthermore, the exact solutions at specific conditions and values of the parameters are to be derived. Additionally, the behaviors of the present solutions and their properties will be analyzed.

2. Theoretical results

Theorem 2.1. *For $b \in \mathbb{R} - \{0\}$, the solution of the IPDDE (1.1) under the constrain $\beta = ac$ is*

$$y(t) = z(t) - \frac{\alpha}{b} e^{at}, \tag{2.1}$$

where $z(t)$ is a solution of the HPDDE:

$$z'(t) = az(t) + bz(ct),$$

subject to the IC:

$$z(0) = \lambda + \frac{\alpha}{b}.$$

Proof. Let us assume a transformation in the form:

$$y(t) = z(t) + \rho e^{\omega t}, \tag{2.2}$$

where ρ and ω are to determined. Substituting Eq. (2.2) into Eqs. (1.1) gives

$$z'(t) + \rho\omega e^{\omega t} = a\left(z(t) + \rho e^{\omega t}\right) + b\left(z(ct) + \rho e^{c\omega t}\right) + \alpha e^{\beta t},$$

which can be written as

$$z'(t) = az(t) + bz(ct) + \left[\rho(a - \omega)e^{\omega t} + \rho b e^{c\omega t} + \alpha e^{\beta t}\right]. \tag{2.3}$$

Setting $\omega = a$, then Eq. (2.3) becomes

$$z'(t) = az(r) + bz(ct) + \left[\rho b e^{c\omega t} + \alpha e^{\beta t}\right].$$

The expression in the bracket vanishes under the constraints:

$$\alpha = -\rho b, \quad \beta = \omega c. \tag{2.4}$$

Since $\omega = a$, from above, and substituting into (2.4) gives

$$\rho = -\frac{\alpha}{b}, \quad \beta = ac,$$

which completes the proof. □

Remark 2.2. Based on Theorem 2.1, the solution of the IPDDE:

$$y'(t) = ay(t) + by(ct) + \alpha e^{act}, \quad y(0) = \lambda, \quad t \geqslant 0, \tag{2.5}$$

is given by

$$y(t) = -\frac{\alpha}{b}e^{at} + z(t), \quad b \neq 0,$$

where $z(t)$ is a solution of the IVP:

$$z'(t) = az(t) + bz(ct), \quad z(0) = \lambda + \frac{\alpha}{b}. \tag{2.6}$$

Lemma 2.3. *The power series solution (PSS) of the IVP* (2.6) *is*

$$z(t) = \left(\lambda + \frac{\alpha}{b}\right) \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \left(a + bc^{k-1}\right)\right) \frac{t^i}{i!}, \quad b \neq 0.$$

Proof. Following Ref. [1] or Ref. [22], one can get

$$z(t) = \left(\lambda + \frac{\alpha}{b}\right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \left(a + bc^{k-1}\right)\right) \frac{t^i}{i!}\right],$$

i.e.,

$$z(t) = \left(\lambda + \frac{\alpha}{b}\right) \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \left(a + bc^{k-1}\right)\right) \frac{t^i}{i!}, \tag{2.7}$$

where $\prod_{k=1}^{i} \left(a + bc^{k-1}\right) = 1$ at $i = 0$. □

Lemma 2.4. *For* $b \in \mathbb{R} - \{0\}$, *the series* (2.7) *has infinite radius of convergence if* $|c| < 1$.

Proof. The proof follows immediately by applying the ratio test for convergence, see [22]. □

Lemma 2.5. *In view of Lemma* 2.3, *the solution of the IPDDE:* $y'(t) = ay(t) + by(ct) + \alpha e^{act}, y(0) = \lambda$, *is*

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right) \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \left(a + bc^{k-1}\right)\right) \frac{t^i}{i!}, \quad b \neq 0. \tag{2.8}$$

Proof. The proof follows by inserting the result of Lemma 2.3 into the transformation (2.1). □

3. Exact solution at special cases

Before launching to the main results of the present section, it may be reasonable to express the solution in the form:

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right) \sum_{i=0}^{\infty} g_i(c) \frac{t^i}{i!}, \tag{3.1}$$

where $g_i(c)$ is defined by

$$g_i(c) = \prod_{k=1}^{i} \left(a + bc^{k-1}\right). \tag{3.2}$$

It is shown in this section that $y(t)$ can be obtained in exact forms under particular constraints/values of

a, b, and c.

3.1. $a + b = 0$

This is an interesting case which gives an exact solution in explicit form in terms of exponential function. The below theorem addresses this point.

Theorem 3.1. *If $a + b = 0$, then the IPDDE (2.5) yields*

$$y'(t) = ay(t) - ay(ct) + \alpha e^{act}, \quad y(0) = \lambda, \quad t \geqslant 0, \tag{3.3}$$

which has the exact solution:

$$y(t) = \lambda + \frac{\alpha}{a}\left(1 - e^{at}\right), \quad a \neq 0. \tag{3.4}$$

Proof. From (3.1), we have

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)\left[g_0(c) + g_1(c)t + g_2(c)\frac{t^2}{2!} + \cdots\right].$$

From Eq. (3.2), we deduce that

$$g_0(c) = 1, \quad g_1(c) = a + b = 0, \quad g_2(c) = g_1(c)(a + bc) = 0, \quad g_3(c) = g_2(c)(a + bc^2) = 0,$$

and hence, $g_i(c) = 0, \forall\, i \geqslant 1$. Thus

$$y(t) = -\frac{\alpha}{b}e^{at} + \lambda + \frac{\alpha}{b},$$

which is equivalent to (3.4) when replacing b with $-a$. Moreover, Eq. (2.5) takes the form (3.3), which completes the proof. □

Remark 3.2. It should be noted that the exact solution (3.4) is independent of c. This case means that the exact solution of the IPDDE (3.3) is not affected by the value of c, i.e., the solution (3.4) remains the same whatever the value of c.

3.2. $a + bc = 0$

The exact solution of this case combines the exponential function with a polynomial of first degree in t and discussed in the next theorem.

Theorem 3.3. *If $a + bc = 0$, then the exact solution of the IVP:*

$$y'(t) = ay(t) + by\left(-\frac{a}{b}t\right) + \alpha e^{-\frac{a^2}{b}t}, \quad y(0) = \lambda, \quad t \geqslant 0, \tag{3.5}$$

is given by

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)[1 + (a + b)t].$$

Proof. Following the analysis in the previous theorem, we have from Eq. (3.2), when $a + bc = 0$, that

$$g_0(c) = 1, \quad g_1(c) = a + b, \quad g_2(c) = (a + b)(a + bc) = 0, \quad g_3(c) = g_2(c)(a + bc^2) = 0,$$

and so $g_i(c) = 0, \forall\, i \geqslant 2$. Therefore

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)[1 + (a + b)t].$$

This case implies $c = -\frac{a}{b}$ and hence Eq. (2.5) takes the form (3.5), which completes the proof. □

3.3. $a + bc^n = 0$, $n \in \mathbb{N}^+$

This case generalizes the previous case, where the solution is given as a sum of the exponential function term $\left(-\frac{\alpha}{b}e^{at}\right)$ and a polynomial of degree n in t.

Theorem 3.4. *If $a + bc^n = 0$, $n \in \mathbb{N}^+$, then the exact solution of the IVP:*

$$y'(t) = ay(t) + by\left(\left(-\frac{a}{b}\right)^{1/n} t\right) + \alpha e^{a\left(-\frac{a}{b}\right)^{1/n} t}, \quad y(0) = \lambda, \quad t \geqslant 0, \tag{3.6}$$

is

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right) \sum_{i=0}^{n} g_i \frac{t^i}{i!}, \tag{3.7}$$

where g_i is given in terms of a and b by

$$g_i = \prod_{k=1}^{i} \left(a + b\left(-\frac{a}{b}\right)^{(k-1)/n}\right). \tag{3.8}$$

Proof. In view of the above analysis, one can prove that $g_i(c) = 0$, $\forall\, i \geqslant n+1$ if $a + bc^n = 0$ for any finite number $n \in \mathbb{N}^+$. Thus, the solution in this case is obtained from Eq. (3.1) by the truncated series:

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)\left[g_0(c) + g_1(c)t + g_2(c)\frac{t^2}{2!} + \cdots + g_n(c)\frac{t^n}{n!}\right],$$

which is equivalent to (3.7). In addition, Eq. (2.5) transforms to (3.6) and g_i takes the form (3.8) through the substitution $c = \left(-\frac{a}{b}\right)^{1/n}$, which completes the proof. □

4. Solution of the inhomogeneous Ambartsumian equation

At $a = -1$ and $b = c = 1/q$, the IPDDE (2.5) takes the form:

$$y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right) + \alpha e^{-t/q}, \quad y(0) = \lambda, \quad q > 1, \tag{4.1}$$

which is considered as the inhomogeneous Ambartsumian equation. The standard Ambartsumian equation is homogeneous in nature and it is a special case of the model (4.1), when $\alpha = 0$. The solution of the current inhomogeneous version of the Ambartsumian equation can be obtained by inserting the quantities $a = -1$ and $b = c = 1/q$ into Eq. (2.8), this yields

$$y(t) = -\alpha q e^{-t/q} + (\lambda + \alpha q) \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \left(q^{-k} - 1\right)\right) \frac{t^i}{i!}.$$

This solution reduces, at $\alpha = 0$, to

$$y(t) = \lambda \sum_{i=0}^{\infty} \left(\prod_{k=1}^{i} \left(q^{-k} - 1\right)\right) \frac{t^i}{i!},$$

which agrees with previous results in Refs. [1, 8, 11, 22]. In addition, the inhomogeneous model of the Ambartsumian equation (4.1) has the exact solution:

$$y(t) = -\alpha q e^{-t/q},$$

if the condition $\lambda + \alpha q = 0$. In this case, the IC $y(0) = \lambda$ must be related to coefficient α by $\lambda = -\alpha q$. This means that the IVP:

$$y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right) + \alpha e^{-t/q}, \quad y(0) = -\alpha q, \quad q > 1,$$

has the exact solution (4).

5. Exact solution of inhomogeneous differential-difference equation at $c = -1$

At $c = -1$, the IPDDE (2.5) takes the form of the following inhomogeneous differential-difference equation:

$$y'(t) = ay(t) + by(-t) + \alpha e^{-at}, \quad y(0) = \lambda. \tag{5.1}$$

Based on Remark 2.2, the solution of this model is

$$y(t) = -\frac{\alpha}{b}e^{at} + z(t), \quad b \neq 0,$$

where $z(t)$ is the solution of the homogeneous differential-difference equation:

$$z'(t) = az(t) + bz(-t), \quad z(0) = \lambda + \frac{\alpha}{b}. \tag{5.2}$$

Following Ebaid and Al-Jeaid [13], one can obtain the exact solution of Eqs. (5.2) in the periodic form:

$$z(t) = \left(\lambda + \frac{\alpha}{b}\right)\left[\cos\left(\sqrt{b^2 - a^2}\, t\right) + \sqrt{\frac{b+a}{b-a}}\sin\left(\sqrt{b^2 - a^2}\, t\right)\right], \quad \text{if} \quad b > a.$$

Also, the solution of Eqs. (5.2) in terms hyperbolic functions can be obtained as

$$z(t) = \left(\lambda + \frac{\alpha}{b}\right)\left[\cosh\left(\sqrt{a^2 - b^2}\, t\right) + \sqrt{\frac{a+b}{a-b}}\sinh\left(\sqrt{a^2 - b^2}\, t\right)\right], \quad \text{if} \quad a > b.$$

Accordingly, the differential-difference model (5.1) has the following solution in terms of exponential and trigonometric functions:

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)\left[\cos\left(\sqrt{b^2 - a^2}\, t\right) + \sqrt{\frac{b+a}{b-a}}\sin\left(\sqrt{b^2 - a^2}\, t\right)\right], \quad b > a. \tag{5.3}$$

Also, the solution in terms of exponential and hyperbolic functions reads

$$y(t) = -\frac{\alpha}{b}e^{at} + \left(\lambda + \frac{\alpha}{b}\right)\left[\cosh\left(\sqrt{a^2 - b^2}\, t\right) + \sqrt{\frac{a+b}{a-b}}\sinh\left(\sqrt{a^2 - b^2}\, t\right)\right], \quad a > b, \tag{5.4}$$

provided that $b \neq 0$.

6. Results

In the previous sections, we showed that the exact solutions of several classes can be obtained under certain constraints of the parameters contained in each of these classes.

6.1. *Exact solution of some examples in the literature*

The authors in [19] applied a numerical method to solve the IVP:

$$y'(t) = -y(t) + \frac{1}{2}y(ct) - \frac{1}{2}e^{-ct}, \quad y(0) = 1. \tag{6.1}$$

They have compared the numerical solution with the exact one, $y(t) = e^{-t}$. However, this exact solution can be obtained in a direct manner via using our results as follows.

In view of Eqs. (6.1) and Eqs. (2.5), we have $\lambda = 1$, $a = -1$, $b = \frac{1}{2}$, and $\alpha = -\frac{1}{2}$. These values lead to that $\lambda + \frac{\alpha}{b} = 0$, hence, the result in Lemma 2.5 (Eq. (2.8)) leads directly to the exact solution $y(t) = e^{-t}$.

In addition, the IVP:

$$y'(t) = -y(t) + \frac{c}{2}y(ct) - \frac{c}{2}e^{-ct}, \quad y(0) = 1,$$

has been analyzed in Ref. [27] by means of Taylor method. The authors [27] pointed out to that their method enjoys better accuracy in contrast to the collocation method. Their conclusion was based on comparing the two approaches with the available exact solution $y(t) = e^{-t}$. This result can be derived directly by observing that $\lambda = 1$, $a = -1$, $b = \frac{c}{2}$, and $\alpha = -\frac{c}{2}$, hence, $\lambda + \frac{\alpha}{b} = 0$. Employing these values in Eq. (2.8) gives $y(t) = e^{-t}$, which agrees with the above mentioned exact solution.

6.2. *Behavior of the current exact solutions*

This section is devoted to study the behavior of some of the obtained exact solutions in previous sections. In Eq. (3.7), it was shown that the exact solution for the IVP $y'(t) = ay(t) + by\left(\left(-\frac{a}{b}\right)^{1/n} t\right) + \alpha e^{a\left(-\frac{a}{b}\right)^{1/n} t}$, $y(0) = \lambda$ depends mainly on the truncated number n of the standard series solution (3.1). In this context, Fig. 1 shows the plots of the exact solution (3.7) at different values of $n \in \mathbb{N}^+$. The behavior of all solutions in this figure indicates that they are increasing functions and the rate of such increase is influenced by the increase in the value of n. Another situation is displayed in Fig. 2 for the exact solution

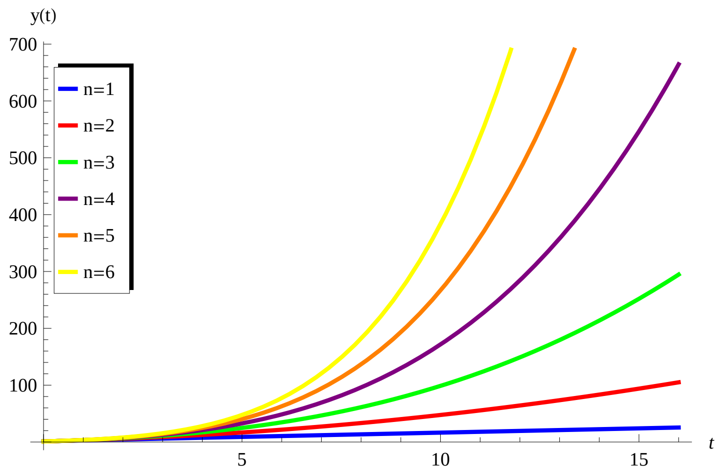

Figure 1: Plots of the exact solution (3.7) for the IVP $y'(t) = ay(t) + by\left(\left(-\frac{a}{b}\right)^{1/n} t\right) + \alpha e^{a\left(-\frac{a}{b}\right)^{1/n} t}$, $y(0) = \lambda$, when $a = -1$, $b = 2$, $\lambda = 1$, and $\alpha = 1$ at different values of $n \in \mathbb{N}^+$.

(5.3) of the inhomogeneous differential-difference IVP $y'(t) = ay(t) + by\,(-t) + \alpha e^{-at}$, $y(0) = \lambda$. This figure reveals that the exact solution (5.3) acquires the oscillatory property at different negative values of $a = -1.25, -1, -0.75, -0.5$. However, the nature of the solution (5.3) is changed to the exponential behavior by reversing the sign of the above values of a, as can be seen in Fig. 3. Also, Fig. 4 displays the the exponential behavior of solution (5.3) at different negative values of b. Finally, Fig. 5 presents the exponential growth of the exact solution (3.4) for the IPDDE $y'(t) = ay(t) - ay\,(ct) + \alpha e^{act}$, $y(0) = \lambda$, at different negative values of a, where c is arbitrary in this case. It can be seen from this figure that as $t \to \infty$, the exact solution (3.4) approaches a certain value given by $\lambda + \alpha/a$.

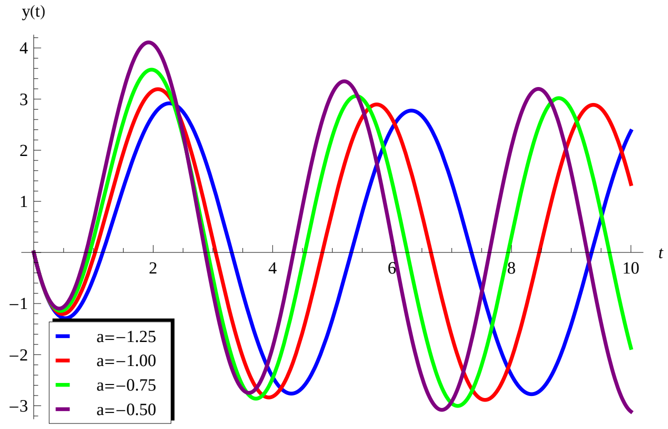

Figure 2: Plots of the exact solution (5.3) for the inhomogeneous differential-difference IVP $y'(t) = ay(t) + by(-t) + \alpha e^{-at}$, $y(0) = \lambda$, when $b = 2$, $\lambda = 0$, and $\alpha = -5$ at different negative values of $a = -1.25, -1, -0.75, -0.5$, oscillatory behavior.

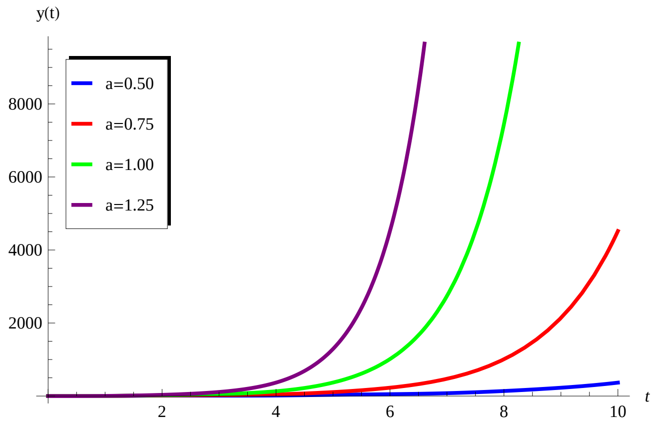

Figure 3: Plots of the exact solution (5.3) for the inhomogeneous differential-difference IVP $y'(t) = ay(t) + by(-t) + \alpha e^{-at}$, $y(0) = \lambda$, when $b = 2$, $\lambda = 0$, and $\alpha = -5$ at different positive values of $a = 0.5, 0.75, 1, 1.25$, exponential behavior.

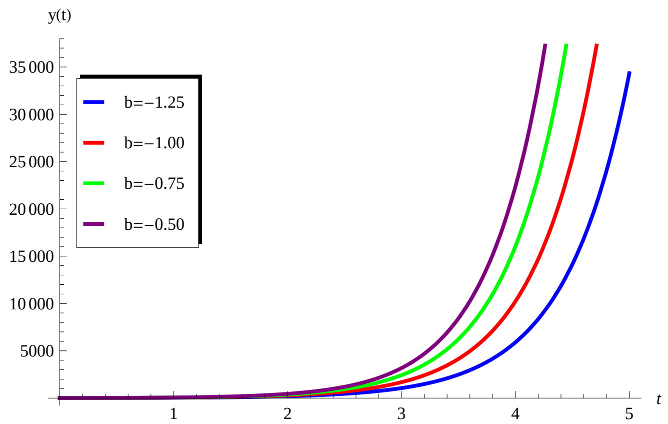

Figure 4: Plots of the exact solution (5.4) for the inhomogeneous differential-difference IVP $y'(t) = ay(t) + by(-t) + \alpha e^{-at}$, $y(0) = \lambda$, when $a = 2$, $\lambda = 10$, and $\alpha = 1$ at different negative values of $b = -1.25, -1, -0.75, -0.5$.

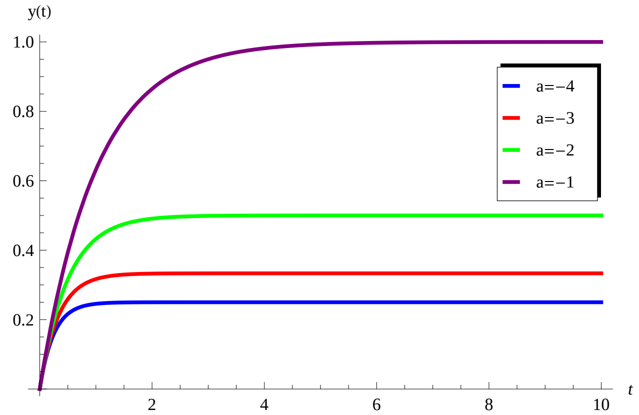

Figure 5: Plots of the exact solution (3.4) for the IPDDE $y'(t) = ay(t) - ay(ct) + \alpha e^{act}$, $y(0) = \lambda$, when $\lambda = 0$ and $\alpha = -1$, where c is arbitrary, at different negative values of $a = -4, -3, -2, -1$.

7. Conclusions

In this paper, a class of pantograph delay differential equations with inhomogeneous term was investigated. The inhomogeneous term was considered in the form of exponential function. An effective transformation was introduced and accordingly, the class of inhomogeneous pantograph delay differential equations was successfully reduced to the homogeneous version. The solution of the present class was established in several exact forms. Also, various exact solutions are determined under specific conditions and values of the model parameters. Existing solutions in the literature were obtained as special cases of the current ones. Further, the behaviors/properties of the obtained solutions were discussed and explained. Moreover, one of the main advantages of this study is the ability to obtain the exact solutions of some examples in the literature without resorting to any of the numerical approaches.

Acknowledgment

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2023/1/25302).

Funding

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2023/1/25302).

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