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Robust impulse nonlinear delayed multi-agent systems: an exponential synchronization

N. Kaewbanjak[a], A. Stephen[b], A. Srinidhi[c], R. Raja[d,e], K. Mukdasai[f], J. Thipcha[g], P. Singkibud[h,*]

[a]Faculty of Science at Sriracha, Kasetsart University, Sriracha Campus, Chon Buri, 20230, Thailand.

[b]Center for Computational Modeling, Chennai Institute of Technology, Chennai-600069, Tamil Nadu, India.

[c]Department of Mathematics, Alagappa University, Karaikudi-630 004, India.

[d]Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi-630 004, India.

[e]Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon.

[f]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand.

[g]Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai 50290, Thailand.

[h]Department of Applied Mathematics and Statistics, Faculty of Science and Liberal Arts, Rajamangala University of Technology Isan, Nakhon Ratchasima 30000, Thailand.

Abstract

This work explores the use of state-feedback pinning control in the presence of time-varying delay to solve the synchronization problem of uncertain nonlinear multi-agent systems (MASs). To begin, it is assumed that the agent's communication topology is a directed, static network. Second, the synchronization issue is transformed into the typical closed-loop system stability issue by employing Laplacian matrix inequality (LMI). The primary goal of this study is to construct a state-feedback pinning controller that yields a closed-loop system that is stable under all permissible uncertainty and impulsive cases. To achieve this goal, we develop a new set of delay-dependent synchronization criteria for the closed-loop system by constructing an appropriate Lyapunov functional and making use of Kronecker product features in conjunction with matrix inequality approaches. All that's needed to construct the optimal state-feedback controller is a set of constraints in the form of linear matrix inequalities, which can be solved with any number of powerful optimization methods. To further illustrate the practicality and efficiency of the suggested control design system, a numerical example and associated simulations are provided.

Keywords: Multi-agent systems, time delay, parameter uncertainty, linear matrix inequality, exponential synchronization.

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1. Introduction

Cooperative control has earned a lot of interest from scientists in the past few years for its wide range of real-life applications in various fields, namely circuits, vehicle formulation, and mobile networks [9, 11, 31]. Cooperative tracking (synchronization) and the cooperative regulator (consensus) two general types of coordinated control of MASs. Distributed controllers are developed for every agent (node) such that all agents subsequently converge to an unknown standard value. This value may be time-varying

*Corresponding author

Email address: peerapongpat.si@rmuti.ac.th (P. Singkibud)

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or a fixed constant and is usually a function of the initial states of the agents and the transmission network topology. Consider a leader agent that serves as a command generator for the tracking problem, generating the desired reference response while ignoring data from the follower agents.

In linear MASs, many necessary synchronization and consensus control results have been established. According to the relative neighbors' output signals and distributed observer-type protocols in [40], the consensus of general linear MASs has been achieved. In [39], the agreement of heterogeneous linear MASs under DoS attack subject to aperiodic sampled data had examined. The authors dealt in [13] network-based practical set consensus of MASs subject to input saturation. Besides, distributed consensus control for MASs with DoS has been investigated in [19].

Even though most of the real-world MASs contain nonlinearities innately and are more complex than linear dynamics, these consensus control methods are constrained only to the linear multi-agent scenario. As a result, synchronization control for nonlinear MASs has gained more attention from numerous academics [12, 18, 41] and has practical value. The authors in [36] have investigated fuzzy theory adaptive finite-time consensus of high-order multi-agent under the dead zone. Synchronization analysis of MASs with stochastic packet losses and external perturbations via a memory state-feedback controller have been discussed in [33]. In addition, the authors derived in [32] the quasi-synchronization of heterogeneous dynamical networks under event-triggered impulsive controls.

It is well acclaimed in many applications that the effects of time delay should be addressed. For the time delay, many noteworthy results arose in consensus control of linear MASs. For linear MASs, the existing consensus control approaches for linear delayed MASs can not be applied directly to the nonlinear case due to the mathematical analysis and complex dynamic involved under nonlinear MASs. Only some results were available in consensus control for nonlinear MASs with time delays [8, 17, 23, 30] so far. Synchronization of MASs with state time delays had yet to be addressed comprehensively, which offers significant challenges. Channel noises may destroy.

Innumerable good properties of practical control systems. Impulsive noise is familiar, even though channel disturbance may express itself in many forms. Throughout the information transmission process of MASs connected by communications topologies, it is reasonable to consider the effects of impulse perturbations.

The research on the dynamical properties of neural networks (NNs) has earned massive attention because of their potential real-life applications in numerous areas, such as signal, image processing, pattern recognition, and combinatorial optimization [34, 35, 37]. Uncertain behaviors, including bifurcation, attractors, and periodic oscillations, were exhibited by delayed NNs, as unique artificial dynamical systems. A delayed chaotic neural system is an infinite system with infinitely large positive Lyapunov exponents in general, which has a wide range of applications, including secure communication to increase security. Also, examining numerous interacting delayed NNs might help us understand synchronized oscillation events that occur in physics, nature, society, control systems, and other contexts see in [4, 28, 29].

Numerous coupled dynamic systems are modelable as chaotic nonlinear systems in several contexts, including physics, nature, and society [22, 25, 26]. Therefore, there is significant research value in studying chaotic systems and their synchronization. Additionally, in recent years, researchers in various domains, including biological systems, chemical reactions, and secure communication, have paid a great deal of attention to the master-slave synchronization of two chaotic systems [14–16, 24]. This shows that the findings in this research have practical applications in expanding secure communications and chaotic synchronization to MASs.

Because of its ease of implementation, the state-feedback pinning synchronization control problem has received a lot of attention [1, 5, 38]. State-feedback control allows one agent in MASs to have its state converge on a specific common form by leveraging its condition and the states of its neighbors. In practice, obtaining the complete state information of MASs takes time and effort. Instead, measurement-based control protocols like observer-based consensus control protocols have been developed. The feedback gain is included in the synchronization protocol's control input, implying that enough circumstances are deduced for the systems to achieve their goals. On the other hand, the feedback gained should be tailored to meet the required criteria.

In the controller design, further examination of MASs in the presence of the time-varying delays, under state-feedback pinning control is carried out. Derived from the aforementioned conversations, investigations into MASs exhibit the subsequent constraints.

- Due to certain limitations in the applicability of current research, it is evident that further investigation is required regarding the comparison example of MASs with state-feedback pinning control.
- The prior research has not exhaustively explored the correlation between varying time delays, MASs synchronization, and the design of state-feedback pinning control.

This work explores the use of state-feedback pinning control in the presence of time-varying delay to solve the synchronization problem of uncertain nonlinear MASs. To begin, it is assumed that the agent's communication topology is a directed, static network. Second, the synchronization issue is transformed into the typical closed-loop system stability issue by employing LMI. The primary goal of this study is to construct a state-feedback pinning controller that yields a closed-loop system that is stable under all permissible uncertainty and impulsive cases. To achieve this goal, we develop a new set of delay-dependent synchronization criteria for the closed-loop system by constructing an appropriate Lyapunov functional and making use of Kronecker product features in conjunction with matrix inequality approaches. All that's needed to construct the optimal state-feedback controller is a set of constraints in the form of linear matrix inequalities, which can be solved with any number of powerful optimization methods. To further illustrate the practicality and efficiency of the suggested control design system, a numerical example and associated simulations are provided.

This manuscript finds a class of impulsive MASs inspired by the prior discussion with time delay. The main contribution to this chapter relates to the following aspects.

- First, exponential synchronization conditions of MASs with time delays and impulse effects are investigated.
- By employing linear matrix inequality (LMI) and LF, it has been possible to study the exponential synchronization for MASs under time delays and impulse proposed. Compared to previous publications [20], the MASs explored in this manuscript are more advanced.
- A pinning controller with satisfactory performance for achieving exponential synchronization has been developed.
- The synchronization control scheme, which is ideal for MASs with unknown parameters, is constructed through a state-feedback pinning control strategy and the achievement of the rule.
- Finally, to avoid computational complexity in this study, we construct the LF based on Lyapunov theory and graph theory, which is connected to the topology of the MASs and the LF.

Throughout the entire paper, we use the notations in Table 1.

Table 1: Notations.

Notation	Description
R^n	n-dimensional Euclidean space
$R^{m\times n}$	Set of all $m \times n$ real matrices
$\lambda_{min}(\mathbb{T})(\lambda_{max}(\mathbb{T}))$	The minimal(maximal) eigenvalue of matrix $\mathbb{T}$
$\|.\|$	The Euclidean norm
$*$	Symmetric matrix

2. Network model and preliminaries

Let $G = (\sigma, \varepsilon, \mathcal{W})$ be an interconnection graph to describe the connection topology of N agents, each of which is represented by a node in $\sigma = \{\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_N\}$ and is connected to some other nodes by at least one edge $\varepsilon \subseteq \sigma \times \sigma$. If the edge $(\mathfrak{z}_i, \mathfrak{z}_j) \in \varepsilon$ connecting node $\mathfrak{z}_i$ and node $\mathfrak{z}_j$ exists, $\mathfrak{z}_j$ is a neighbor of $\mathfrak{z}_i$. $\mathcal{W} = [a_{ij}]_{N\times N}$ is the adjacent weighted matrix, where the existence of the edge $(\mathfrak{z}_i, \mathfrak{z}_j)$ implying a_{ij} must be zero. N_i refers the group of neighbors of agent i corresponding to node $\mathfrak{z}_1$. A path from node $\mathfrak{z}_j$ towards node $\mathfrak{z}_i$ is of a sequence of edges $(\mathfrak{z}_i, \mathfrak{z}_{l_1}), (\mathfrak{z}_{l_1}, \mathfrak{z}_{l_2}), \ldots, (\mathfrak{z}_{l_m}, \mathfrak{z}_j)$. When a directed path exists from one arbitrary node to the other nodes, the directed topology is called linked. $L = (l_{ij})_{N\times N}$ is called the Laplacian matrix and it is represented by $l_{ij} = -a_{ij}$, $i \neq j$; and $l_{ii} = \sum_{j=1, j\neq i}^{N} l_{ij}$ $(i, j = 1, 2, 3, \ldots, N)$.

Let us assume the follower system is described as follows:

$$\begin{cases} \dot{z}_i(t) = -Az_i(t) + Bz_i(t - r(t)) + C\hat{g}(z_i(t)) + D\hat{g}(z_i(t - r(t))) + \mathcal{K}_i(t), & t \neq t_{\mathfrak{K}}, \\ \Delta z_i(t_{\mathfrak{K}}) = \mathcal{J} z_i(t_{\mathfrak{K}}^-), \quad t = t_{\mathfrak{K}}, \quad \mathfrak{K} \in Z_+, \end{cases}$$

here, $i = 1, 2, 3, \ldots, N$; $z_i(t) \in R^n$ refers to state of i^{th} agent; $\hat{g} : R^n \to R^n$ refers to nonlinear agent function. Let $A = \text{diag}\{a_1, \ldots, a_n\}, B, C$, and D are known matrices of $R^{n\times n}$. $r(t)$ refers time delay satisfies $0 \leqslant r_1 \leqslant r(t) \leqslant r_2$, $\dot{r}(t) \leqslant \iota$, and $r = \max\{r_1, r_2\}$, where r_1, r_2, and ι are constant. $\mathcal{K}_i(t)$ denotes control input, and $\mathcal{J}$ is a impulsive gain.

Now, we assume the leader system is represented by

$$\begin{cases} \dot{z}_0(t) = -Az_0(t) + Bz_0(t - r(t)) + C\hat{g}(z_0(t)) + D\hat{g}(z_0(t - r(t))), & t \neq t_{\mathfrak{K}}, \\ \Delta z_0(t_{\mathfrak{K}}) = \mathcal{J} z_0(t_{\mathfrak{K}}^-), \quad t = t_{\mathfrak{K}}, \ \mathfrak{K} \in Z_+, \end{cases}$$

the leader state denote $z_0(t) \in R^n$; $\hat{g} : R^n \to R^n$ refers to nonlinear agent function. Let $\varpi_i(t) = z_i(t) - z_0(t)$, we assume the error as:

$$\begin{cases} \dot{\varpi}_i(t) = -A\varpi_i(t) + B\varpi_i(t - r(t)) + Cf(\varpi_i(t)) + Df(\varpi_i(t - r(t))) + \mathcal{K}_i(t), & t \neq t_{\mathfrak{K}}, \\ \Delta \varpi_i(t_{\mathfrak{K}}) = \mathcal{J} \varpi_i(t_{\mathfrak{K}}^-), \quad t = t_{\mathfrak{K}}, \quad \mathfrak{K} \in Z_+, \end{cases} \tag{2.1}$$

here, $f(\varpi_i(t)) = \hat{g}(z_i(t)) - \hat{g}(z_0(t))$, and $\mathcal{J}$ is a impulsive gain. The state-feedback pinning control design is

$$\begin{aligned} \mathcal{K}_i(t) &= m_i\left\{ \sum_{j=1}^{N} a_{ij}[z_j(t) - z_i(t)] - \psi_i[z_i(t) - z_0(t)] \right\} \\ &= m_i\left\{ -\sum_{j=1}^{N} l_{ij}(\varpi_j(t)) - \psi_i(\varpi_j(t)) \right\}, \ i = 1, 2, \ldots, N, \ \text{ and } \ j = 1, 2, \ldots, N, \end{aligned} \tag{2.2}$$

the compact form is,

$$\mathcal{K}(t) = -(L \otimes M)\varpi(t) - (\mathfrak{R} \otimes M)\varpi(t),$$

where, $M = \{m_1, m_2, \ldots, m_N\}$ is the control gain and $\mathfrak{R} = \{\psi_1, \psi_2, \ldots, \psi_N\}$ is pinning matrix. The compact representation of system (2.1) is

$$\begin{cases} \dot{\varpi}(t) = -(\mathcal{I} \otimes A)\varpi(t) + (\mathcal{I} \otimes B)\varpi(t - r(t)) + (\mathcal{I} \otimes C)f(\varpi(t)) + (\mathcal{I} \otimes D)f(\varpi(t - r(t))) + \mathcal{K}(t), \ t \neq t_{\mathfrak{K}}, \\ \Delta \varpi(t_{\mathfrak{K}}) = (\mathcal{I} \otimes \mathcal{J})\varpi(t_{\mathfrak{K}}^-), \ t = t_{\mathfrak{K}}, \ \mathfrak{K} \in Z_+, \end{cases} \tag{2.3}$$

where $\varpi(t) = (\varpi_1, \varpi_2, \ldots, \varpi_N)^T$ refers the error and $\mathcal{K}(t) = (\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_N)^T$ is the designed controller.

Lemma 2.1 ([10]). *Linear matrix inequality (LMI) is given as*

$$\begin{pmatrix} \mathbb{T}_{11} & \mathbb{T}_{12} \\ \mathbb{T}_{12}^T & \mathbb{T}_{22} \end{pmatrix} < 0,$$

where $\mathbb{T}_{11} = \mathbb{T}_{11}^T, \mathbb{T}_{22} = \mathbb{T}_{22}^T$, *is equivalent to the following conditions:*

1. $\mathbb{T}_{11} < 0, \mathbb{T}_{22} - \mathbb{T}_{12}^T \mathbb{T}_{11}^{-1} \mathbb{T}_{12} < 0$;
2. $\mathbb{T}_{22} < 0, \mathbb{T}_{11} - \mathbb{T}_{12} \mathbb{T}_{22}^{-1} \mathbb{T}_{12}^T < 0$.

Lemma 2.2 ([20]). *Consider the non-negative function* $\mathcal{V}(t)$, $t \in [-r, \infty)$, *satisfies*

$$\dot{\mathcal{V}}(t) \leqslant -\mu \mathcal{V}(t) + \eta \tilde{\mathcal{V}}(t), \ t \geqslant t_0,$$

where $0 \leqslant \eta < \mu$, $\tilde{\mathcal{V}}(t) = \sup_{s \in [t-r,t]} \mathcal{V}(s)$, *then,* $\mathcal{V}(t) = \tilde{\mathcal{V}}(t_0) e^{-\omega(t-t_0)}$, $t \geqslant t_0$, *where,* $\omega > 0$ *is the unique solution of the equation* $\omega = \mu - \eta e^{\omega r}$.

Lemma 2.3 ([21]). *For any vectors* $x, y \in R^n$, *scalar* $\epsilon > 0$, *and positive definite matrix* $Q \in R^{n \times n}$, *the following inequality holds:*

$$2x^T y \leqslant \epsilon x^T Q x + \epsilon^{-1} y^T Q^{-1} y.$$

Definition 2.4. The MASs (2.1) is cooperative synchronization [7] if

$$\lim_{t\to\infty} \|\varpi_i(t)\| = \lim_{t\to\infty} \|z_i(t) - z_0(t)\| = 0, \quad \text{for all} \ \ i = 0, 1, \ldots, N.$$

Assumption 2.5. Assume that $\exists$ constant $k > 0$ such that $\forall x, y \in R^n$, satisfying

$$\|\hat{g}(x) - \hat{g}(y)\| \leqslant k \|x - y\|.$$

3. Main results

We will investigate exponential synchronization criteria for leader-follower problems for specific and uncertain MASs with delay using Halanay's inequality.

Theorem 3.1. *Under Assumption 2.5, there exist positive constants* $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4$, *and* $\bar{\gamma}_5$, *some constant* $0 \leqslant \eta < \mu$ *and positive matrix* $\mathbb{T}$ *in order to attain following conditions:*

$$\bar{\Theta} = \begin{bmatrix} \bar{\psi} & (\mathcal{I} \otimes \mathbb{T}B) & (\mathcal{I} \otimes \mathbb{T}C) & (\mathcal{I} \otimes \mathbb{T}D) & (\mathcal{I} \otimes \mathfrak{H}_1) \\ * & -\bar{\gamma}_1 & 0 & 0 & 0 \\ * & * & -\bar{\gamma}_2 & 0 & 0 \\ * & * & * & -\bar{\gamma}_3 & 0 \\ * & * & * & * & -\bar{\gamma}_4 \end{bmatrix} < 0, \tag{3.1}$$

$$\bar{\Omega} = \begin{bmatrix} \bar{\chi} & (\mathcal{I} \otimes \mathfrak{H}_1) \\ * & -\bar{\gamma}_5 \end{bmatrix} < 0, \tag{3.2}$$

where, $\bar{\psi} = -2(\mathcal{I} \otimes \mathbb{T}A) - 2(L \otimes \mathbb{T}M) - 2(\mathfrak{R} \otimes \mathbb{T}M) + \mu' + \bar{\gamma}_2 k^2 - \bar{\gamma}_5 - 2(\mathcal{I} \otimes \mathfrak{H}_1)$, $\bar{\chi} = \bar{\gamma}_1 + \bar{\gamma}_3 k^2 + \bar{\gamma}_4 + 2(\mathcal{I} \otimes \mathfrak{H}_1) - \eta'$. *The error of MASs (2.3) can achieve exponential under time constant* $\frac{\omega}{2}$, *where* $\omega \in (0, \ \mu - \eta]$ *is the solution for the equation below*

$$\omega = \mu - \eta e^{\omega r}.$$

Proof. Design the Lyapunov function as:

$$\mathcal{V}(t) = \varpi^T(t)(\mathcal{I}\otimes \mathbb{T})\varpi(t). \tag{3.3}$$

Depending on (3.3), we get

$$\begin{aligned}
\dot{\mathcal{V}}(t) &= 2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T})\dot{\varpi}(t),\\
\dot{\mathcal{V}}(t) &= 2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T})[-(\mathcal{I}\otimes A)\varpi(t) + (\mathcal{I}\otimes B)\varpi(t-r(t)) + (\mathcal{I}\otimes C)f(\varpi(t))\\
&\quad + (\mathcal{I}\otimes D)f(\varpi(t-r(t))) - (L\otimes M)\varpi(t) - (\mathfrak{R}\otimes M)\varpi(t)]\\
&= -2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}A)\varpi(t) + 2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}B)\varpi(t-r(t)) + 2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}C)f(\varpi(t))\\
&\quad + 2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}D)f(\varpi(t-r(t))) - 2\varpi^T(t)(L\otimes \mathbb{T}M)\varpi(t) - 2\varpi^T(t)(\mathfrak{R}\otimes \mathbb{T}M)\varpi(t).
\end{aligned} \tag{3.4}$$

In view of Lemma 2.3, we have

$$\begin{aligned}
2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}B)\varpi(t-r(t)) &\leqslant \frac{1}{\bar{\gamma}_1}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}BB^T\mathbb{T})\varpi(t) + \bar{\gamma}_1\varpi^T(t-r(t))\varpi(t-r(t)),\\
2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}C)f(\varpi(t)) &\leqslant \frac{1}{\bar{\gamma}_2}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}CC^T\mathbb{T})\varpi(t) + \bar{\gamma}_2 f^T(\varpi(t))f(\varpi(t))\\
&\leqslant \frac{1}{\bar{\gamma}_2}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}CC^T\mathbb{T})\varpi(t) + \bar{\gamma}_2 k_2\varpi^T(t)\varpi(t)),\\
2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}D)f(\varpi(t-r(t))) &\leqslant \frac{1}{\bar{\gamma}_3}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}DD^T\mathbb{T})\varpi(t) + \bar{\gamma}_3 f^T(\varpi(t-r(t)))f(\varpi(t-r(t)))\\
&\leqslant \frac{1}{\bar{\gamma}_3}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}DD^T\mathbb{T})\varpi(t) + \bar{\gamma}_3 k_3\varpi^T(t-r(t))\varpi(t-r(t))).
\end{aligned}$$

Then, for any matrix $\mathfrak{M}_1$, it is straightforward that

$$0 = 2\xi^T(t)(\mathcal{I}\otimes \mathfrak{M}_1)[\varpi(t-r(t)) - \varpi(t) + \int_{t-r(t)}^{t}\dot{\varpi}(s)ds], \tag{3.5}$$

where, $\xi^T(t) = \begin{bmatrix}\varpi^T(t) & \varpi^T(t-r(t))\end{bmatrix}$. From Lemma 2.3, one obtains

$$2\varpi^T(t)(\mathcal{I}\otimes \mathfrak{M}_1)\varpi(t-r(t)) \leqslant \frac{1}{\bar{\gamma}_4}\varpi^T(t)(\mathcal{I}\otimes \mathfrak{M}_1\mathfrak{M}_1^T)\varpi(t) + \bar{\gamma}_4\varpi^T(t-r(t))\varpi(t-r(t)), \tag{3.6}$$

$$2\varpi^T(t-r(t))(\mathcal{I}\otimes \mathfrak{M}_1)\varpi(t) \leqslant \frac{1}{\bar{\gamma}_5}\varpi^T(t-r(t))(\mathcal{I}\otimes \mathfrak{M}_1\mathfrak{M}_1^T)\varpi(t-r(t)) + \bar{\gamma}_5\varpi^T(t)\varpi(t). \tag{3.7}$$

Combining equation (3.1), (3.2), and (3.4)-(3.7), we obtain

$$\begin{aligned}
\dot{\mathcal{V}}(t) &\leqslant -2\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}A)\varpi(t) + \frac{1}{\bar{\gamma}_1}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}BB^T\mathbb{T})\varpi(t) + \bar{\gamma}_1\varpi^T(t-r(t))\varpi(t-r(t))\\
&\quad + \frac{1}{\bar{\gamma}_2}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}CC^T\mathbb{T})\varpi(t) + \bar{\gamma}_2 k_2\varpi^T(t)\varpi(t)) + \frac{1}{\bar{\gamma}_3}\varpi^T(t)(\mathcal{I}\otimes \mathbb{T}DD^T\mathbb{T})\varpi(t)\\
&\quad + \bar{\gamma}_3 k_3\varpi^T(t-r(t))\varpi(t-r(t))) + \frac{1}{\bar{\gamma}_4}\varpi^T(t)(\mathcal{I}\otimes \mathfrak{M}_1\mathfrak{M}_1^T)\varpi(t) + \bar{\gamma}_4\varpi^T(t-r(t))\varpi(t-r(t))\\
&\quad - \frac{1}{\bar{\gamma}_5}\varpi^T(t-r(t))(\mathcal{I}\otimes \mathfrak{M}_1\mathfrak{M}_1^T)\varpi(t-r(t)) - \bar{\gamma}_5\varpi^T(t)\varpi(t) - 2\varpi^T(t)(I\otimes \mathfrak{M}_1)\varpi(t)\\
&\quad + 2\varpi^T(t-r(t))(\mathcal{I}\otimes \mathfrak{M}_1)\varpi(t-r(t)) - 2\varpi^T(t)(L\otimes \mathbb{T}M)\varpi(t) - 2\varpi^T(t)(\mathfrak{R}\otimes \mathbb{T}M)\varpi(t),\\
\dot{\mathcal{V}}(t) &\leqslant \varpi^T(t)[-2(\mathcal{I}\otimes \mathbb{T}A) - 2(L\otimes \mathbb{T}M) + \bar{\gamma}_1^{-1}(\mathcal{I}\otimes \mathbb{T}BB^T\mathbb{T}) + \bar{\gamma}_2^{-1}(\mathcal{I}\otimes \mathbb{T}CC^T\mathbb{T}) + \bar{\gamma}_3^{-1}(\mathcal{I}\otimes \mathbb{T}DD^T\mathbb{T})
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\gamma}_2 k_2+\bar{\gamma}_4^{-1}(\mathcal{J}\otimes\mathfrak{M}_1\mathfrak{M}_1^T)-\bar{\gamma}_5-2(\mathcal{J}\otimes\mathfrak{M}_1)+\mu'-2(\mathfrak{R}\otimes\mathbb{T}M)]\varpi(t)\\
&+\varpi^T(t-\tau(t))[\bar{\gamma}_1+\bar{\gamma}_3 k_3+\bar{\gamma}_4-\bar{\gamma}_5^{-1}(\mathcal{J}\otimes\mathfrak{M}_1\mathfrak{M}_1^T)+2(\mathcal{J}\otimes\mathfrak{M}_1)-\eta']\varpi(t-\tau(t))\\
&-\mu'\varpi^T(t)\varpi(t)+\eta'\varpi^T(t-r(t))\varpi(t-r(t)),\\
\dot{\mathcal{V}}(t)&\leqslant\varpi^T(t)\bar{\Theta}\varpi(t)+\varpi^T(t-r(t))\bar{\Omega}\varpi(t-r(t))-\mu'\varpi^T(t)\varpi(t)+\eta'\varpi^T(t-r(t))\varpi(t-r(t)),\\
\dot{\mathcal{V}}(t)&\leqslant-\mu'\varpi^T(t)\varpi(t)+\eta'\varpi^T(t-r(t))\varpi(t-r(t))\\
&\leqslant-\frac{\mu'}{\lambda_{\max}(\mathbb{T})}\varpi^T(t)\varpi(t)+\frac{\eta'}{\lambda_{\min}(\mathbb{T})}\varpi^T(t-r(t))\varpi(t-r(t))\\
&\leqslant-\mu\varpi^T(t)\varpi(t)+\eta\varpi^T(t-r(t))\varpi(t-r(t))\leqslant-\mu\mathcal{V}(t)+\eta\mathcal{V}(t-r(t))\leqslant-\mu\mathcal{V}(t)+\eta\widetilde{\mathcal{V}}(t),
\end{aligned}
$$

where, $\mu'=\lambda_{\max}(\mathbb{T})\mu$, $\eta'=\lambda_{\min}(\mathbb{T})\eta$, $\widetilde{\mathcal{V}}(t)=\sup_{s\in[t-r,t]}\mathcal{V}(s)$, we get from Lemma 2.2 that $\mathcal{V}(t)\leqslant\widetilde{\mathcal{V}}(t_0)e^{-\omega(t-t_0)}$, where $\omega\in(0,\ \mu-\eta]$ is the solution of $\omega=\mu-\eta e^{\omega r}$. From $\mathcal{V}(t)=\varpi^T(I\otimes\mathbb{T})\varpi$, we can get

$$
\lambda_{\min}(\mathbb{T})\|\varpi(t)\|^2=\lambda_{\min}(\mathbb{T})\varpi^T\varpi\leqslant\mathcal{V}(t),\quad \lambda_{\min}(\mathbb{T})\|\varpi(t)\|^2\leqslant\mathcal{V}(t),\ \ \lambda_{\min}(\mathbb{T})\|\varpi(t)\|^2\leqslant\widetilde{\mathcal{V}}(t_0)e^{-\omega(t-t_0)}
$$

$$
\|\varpi(t)\|^2\leqslant\frac{\widetilde{\mathcal{V}}(t_0)}{\lambda_{\min}(\mathbb{T})}e^{-\omega(t-t_0)},\qquad \|\varpi(t)\|\leqslant\frac{\sqrt{\widetilde{\mathcal{V}}(t_0)}}{\lambda_{\min}(\mathbb{T})}e^{\frac{-\omega}{2}(t-t_0)},
$$

when $t=t_{\mathfrak{K}}$,

$$
\begin{aligned}
\mathcal{V}(\varpi(t_{\mathfrak{K}}))-\mathcal{V}(\varpi(t_{\mathfrak{K}}^-))&=\varpi^T(t_{\mathfrak{K}})(I\otimes\mathbb{T})\varpi(t_{\mathfrak{K}})-\varpi^T(t_{\mathfrak{K}}^-)(I\otimes\mathbb{T})\varpi(t_{\mathfrak{K}}^-)\\
&=\varpi^T(t_{\mathfrak{K}}^-)(I\otimes\hat{\mathcal{J}}^T\mathbb{T}\hat{\mathcal{J}})\varpi(t_{\mathfrak{K}}^-)-\varpi T(t_{\mathfrak{K}}^-)(I\otimes\mathbb{T})\varpi(t_{\mathfrak{K}}^-)\\
&=\varpi^T(t_{\mathfrak{K}}^-)((I\otimes\hat{\mathcal{J}}^T\mathbb{T}\hat{\mathcal{J}})-(I\otimes\mathbb{T}))\varpi(t_{\mathfrak{K}}^-)\leqslant 0\\
\mathcal{V}(\varpi(t_{\mathfrak{K}}))-\mathcal{V}(\varpi(t_{\mathfrak{K}}^-))&\leqslant 0,\\
\mathcal{V}(\varpi(t_{\mathfrak{K}}))&\leqslant\mathcal{V}(\varpi(t_{\mathfrak{K}}^-)),\ \ \mathfrak{K}\in Z_+.
\end{aligned}
$$

As a consequence, the impulse effect of synchronization error is exponential. □

Theorem **??** is used to acquire the control gain matrix for the pinning control scheme (2.2) to exponentially synchronize the considered synchronization error system (2.3) with the help of Theorem 3.1.

Theorem 3.2. *Under Assumption 2.5, there exist positive constant $\bar{\gamma}_1,\bar{\gamma}_2,\bar{\gamma}_3,\bar{\gamma}_4$, and $\bar{\gamma}_5$, some constant $0\leqslant\eta<\mu$ and positive matrix $\mathbb{T}$ in order to attain the conditions:*

$$
\hat{\Theta}=\begin{bmatrix}\hat{\psi} & (\mathcal{J}\otimes\hat{\mathbb{T}}B) & (\mathcal{J}\otimes\hat{\mathbb{T}}C) & (\mathcal{J}\otimes\hat{\mathbb{T}}D) & (\mathcal{J}\otimes\hat{\mathfrak{H}}_1)\\ * & -\bar{\gamma}_1 & 0 & 0 & 0\\ * & * & -\bar{\gamma}_2 & 0 & 0\\ * & * & * & -\bar{\gamma}_3 & 0\\ * & * & * & * & -\bar{\gamma}_4\end{bmatrix}<0, \tag{3.8}
$$

$$
\hat{\Omega}=\begin{bmatrix}\hat{\chi} & (\mathcal{J}\otimes\hat{\mathfrak{H}}_1)\\ * & -\bar{\gamma}_5\end{bmatrix}<0, \tag{3.9}
$$

where, $\hat{\psi}=-2(\mathcal{J}\otimes\hat{\mathbb{T}}A)-2(L\otimes\hat{\mathbb{T}}M)-2(\mathfrak{R}\otimes\hat{\mathbb{T}}M)+\mu'+\bar{\gamma}_2k_2-\bar{\gamma}_5-2(\mathcal{J}\otimes\hat{\mathfrak{H}}_1)$, $\hat{\chi}=\bar{\gamma}_1+\bar{\gamma}_3k_2+\bar{\gamma}_4+2(\mathcal{J}\otimes\hat{\mathfrak{H}}_1)-\eta'$. The error of MASs (2.3) can achieve exponential under time constant $\frac{\omega}{2}$, where $\omega\in(0,\ \mu-\eta]$ is the solution for the equation below

$$
\omega=\mu-\eta e^{\omega r},
$$

and the gain matrix M is given by $M=SQ^{-1}$.

Proof. Let $\mathbb{T}^{-1}=R$, $MQ=S$, $\hat{\mathbb{T}}_j=Q\mathbb{T}Q^T$. Then, pre and post multiplying $\text{diag}\{\underbrace{(I\otimes R),(I\otimes R)}_{2\text{ times}}\}$ with the LMIs (3.1) and (3.2), we can obtain the LMIs (3.8) and (3.9). □

3.1. Robust synchronization criteria

Let us assume the synchronization of uncertain MASs (UMASs) as:

$$\begin{cases} \dot{\varpi}_i(t) = -(A+\Delta A(t))\varpi_i(t) + (B+\Delta B(t))\varpi_i(t-r(t)) + (C+\Delta C(t))f(\varpi_i(t)) \\ \qquad +(D+\Delta D(t))f(\varpi_i(t-r(t))) + \mathcal{K}_i(t), \quad t \neq t_{\mathfrak{K}}, \\ \Delta\varpi_i(t_{\mathfrak{K}}) = \hat{\mathcal{J}}_i\varpi_i(t_{\mathfrak{K}}^-), \quad t = t_{\mathfrak{K}}, \quad \mathfrak{K} \in Z_+. \end{cases} \tag{3.10}$$

Compact form of UMASs is

$$\begin{cases} \dot{\varpi}(t) = -(I\otimes(A+\Delta A(t)))\varpi(t) + (I\otimes(B+\Delta B(t)))\varpi(t-r(t)) + (I\otimes(C+\Delta C(t)))f(\varpi(t)) \\ \qquad +(I\otimes(D+\Delta D(t)))f(\varpi(t-r(t))) + \mathcal{K}(t), \quad t \neq t_{\mathfrak{K}}, \\ \Delta\varpi(t_{\mathfrak{K}}) = (I\otimes\hat{\mathcal{J}})\varpi(t_{\mathfrak{K}}^-), \quad t = t_{\mathfrak{K}}, \quad \mathfrak{K} \in Z_+, \end{cases} \tag{3.11}$$

where $\varpi(t) = (\varpi_1, \varpi_2, \ldots, \varpi_N)^T$ denotes the error vector.

Theorem 3.3. *Under Assumption 2.5, there exist positive constant $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3$, $\bar{\gamma}_4$, and $\bar{\gamma}_5$, some constant $0 \leqslant \eta < \mu$ and positive matrix $\mathbb{T}$ in order to attain the conditions:*

$$\bar{\Theta} = \begin{bmatrix} \bar{\psi} & (\mathcal{J}\otimes\mathbb{T}B) & (\mathcal{J}\otimes\mathbb{T}C) & (\mathcal{J}\otimes\mathbb{T}D) & (\mathcal{J}\otimes\mathfrak{H}_1) \\ * & -\bar{\gamma}_1 & 0 & 0 & 0 \\ * & * & -\bar{\gamma}_2 & 0 & 0 \\ * & * & * & -\bar{\gamma}_3 & 0 \\ * & * & * & * & -\bar{\gamma}_4 \end{bmatrix} < 0, \tag{3.12}$$

$$\bar{\Omega} = \begin{bmatrix} \bar{\chi} & (\mathcal{J}\otimes\mathfrak{H})_1 \\ * & -\bar{\gamma}_5 \end{bmatrix} < 0, \tag{3.13}$$

where $\bar{\psi} = -2(\mathcal{J}\otimes\hat{\mathbb{T}}A) - 2(L\otimes\hat{\mathbb{T}}M) - 2(\mathfrak{R}\otimes\hat{\mathbb{T}}M) + (\mathcal{J}\otimes W_1^TW_1) + (\mathcal{J}\otimes W_3^TW_3) + \mu' + \bar{\gamma}_2k_2 - \bar{\gamma}_5 - 2(\mathcal{J}\otimes\hat{\mathfrak{H}}_1) + 4(\mathcal{J}\otimes\hat{\mathbb{T}}RR^T\hat{\mathbb{T}})$; $\bar{\chi} = \bar{\gamma}_1 + \bar{\gamma}_3k_2 + (\mathcal{J}\otimes W_2^TW_2) + k_2(\mathcal{J}\otimes W_4^TW_4) + \bar{\gamma}_4 + 2(\mathcal{J}\otimes\hat{\mathfrak{H}}_1) - \eta'$. *The error of UMASs (3.11) can achieve exponential under time constant $\frac{\omega}{2}$, where $\omega \in (0,\ \mu-\eta]$ is the solution for the equation*

$$\omega = \mu - \eta e^{\omega r}.$$

Proof. Design the Lyapunov function

$$\mathcal{V}(t) = \varpi^T(t)(\mathcal{J}\otimes\mathbb{T})\varpi(t). \tag{3.14}$$

Depending on (3.14), we get

$$\begin{aligned} \dot{\mathcal{V}}(t) &= 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T})\dot{\varpi}(t), \\ \dot{\mathcal{V}}(t) &= 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T})[-(\mathcal{J}\otimes(A+\Delta A(t)))\varpi(t) + (\mathcal{J}\otimes(B+\Delta B(t)))\varpi(t-r(t)) + (\mathcal{J}\otimes(C \\ &\quad +\Delta C(t)))f(\varpi(t)) + (\mathcal{J}\otimes(D+\Delta D(t)))f(\varpi(t-r(t))) - (L\otimes M)\varpi(t) - (\mathfrak{R}\otimes M)\varpi(t)]. \end{aligned} \tag{3.15}$$

Replace $\Delta A(t), \Delta B(t), \Delta C(t), \Delta D(t)$ with $RS(t)W_1, RS(t)W_2, RS(t)W_3, RS(t)W_4$, respectively. Then, we obtain

$$\begin{aligned} 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}\Delta A(t))\varpi(t) &= 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}RS(t)W_1)\varpi(t) \leqslant \varpi^T(t)(\mathcal{J}\otimes(\mathbb{T}RR^T\mathbb{T} + W_1^TW_1))\varpi(t), \\ 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}\Delta B(t))\varpi(t-r(t)) &= 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}RS(t)W_2)\varpi(t-r(t)) \\ &\leqslant \varpi^T(t)(\mathcal{J}\otimes\mathbb{T}RR^T\mathbb{T})\varpi(t) + \varpi^T(t-r(t))(\mathcal{J}\otimes W_2^TW_2)\varpi(t-r(t)), \\ 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}\Delta C(t))f(\varpi(t)) &= 2\varpi^T(t)(\mathcal{J}\otimes\mathbb{T}RS(t)W_3)f(\varpi(t)) \\ &\leqslant \varpi^T(t)(\mathcal{J}\otimes\mathbb{T}RR^T\mathbb{T})\varpi(t) + f^T(\varpi(t))(\mathcal{J}\otimes W_3^TW_3)f(\varpi(t)) \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\leqslant \varpi^T(t)(\mathcal{J}\otimes \mathbb{T}RR^T\mathbb{T})\varpi(t) + k_2\varpi^T(t)(\mathcal{J}\otimes W_3^TW_3)\varpi(t),\\ 2\varpi^T(t)(\mathcal{J}\otimes \mathbb{T}\Delta D(t))f(\varpi(t-r(t))) &= 2\varpi^T(t)(\mathcal{J}\otimes \mathbb{T}RS(t)W_4)f(\varpi(t-r(t)))\\ &\leqslant \varpi^T(t)(\mathcal{J}\otimes \mathbb{T}RR^T\mathbb{T})\varpi(t) + k_2\varpi^T(t-r(t))(\mathcal{J}\otimes W_4^TW_4)\varpi(t-r(t)). \end{aligned}$$

As a consequence, the impulse effect of UMASs is exponentially synchronization by applying (3.15)-(3.16) in (3.12) and (3.13). □

Remark 3.4. The algorithm presented below outlines the process of constructing the control gain matrix using the LMI condition as defined in Theorem 3.2.

Algorithm 1: Grid search algorithm
1: **function** LYAPUNOV FUNCTION $(A, B, C, D, r_1, r_2)\triangleright$
where A, B, D, and D are system parameters with positive constants $r_1,\ r_2,\ r\ \iota,\ \mu\ \eta$.
2: **if** the matrix $\mathbb{T}$, $\bar{\gamma}_j$, $(j = 1, \dots, 6) > 0$, any matrices $\mathcal{M}_1$, $\mathfrak{H}_1$ exists
and for a given upper bound, LMIs (3.8) and (3.9) true **then**
3: Compute, the control gain matrix $M = SQ^{-1}$
4: **else**
5: **repeat** with a superior upper limit
6: **end if**
7: $N = 0$ to End time
8: **Input:** In Examples, the system values are shown
9: **for** $i = 1$ to N **do**
10: Derive the system by R-K fourth order method
11: **end for**
12: **end function**

Remark 3.5. All of the findings mentioned above rely on the Lyapunov stability theory, in which the number of variables in the resulting LMIs is critical to consider. In all likelihood, the computing complexity increases directly to the number of variables. On the other hand, double integral terms are introduced in the constructed LF (3.3) to reduce conservatism. However, the addition of these terms will, in general, increase the computational complexity. Furthermore, this work is distinct in that it proposes novel integral inequalities and LKF, which provide less conservatism on outcomes than earlier results. Based on these findings, the protocol established in this work has more extensive applicability in MAS control design.

Remark 3.6. The pinning controller plays an essential role in achieving the goal of global exponential synchronization. The controller mechanism will reduce the time delay compared to other controllers, and the proposed controller will be more effective. The follower agent fractional receives the details from the leader agent.

4. Numerical examples

The numerical results are implemented in this section to demonstrate the efficacy of the determined conditions for assumed MASs.

Example 4.1. Assume the system (2.1) containing of leader and followers indexed by 0 and $1, \dots, 4$, respectively. Consider the error system as:

$$\dot{\varpi}_i(t) = -\mathfrak{A}\varpi_i(t) + \mathfrak{B}\varpi_i(t-r(t)) + \mathfrak{C}\delta(\varpi_i(t)) + \mathfrak{D}\delta(\varpi_i(t-r(t))) + \mathcal{K}_i(t).$$

Consider the function is $\delta(\varpi_i(t)) = 0.01\sinh(\varpi_i(t))$. The discrete delay is $r(t) = 0.73\sin(t) + 0.67$, $r_1 = 0.67$, $r_2 = 1.4$, $\iota = 0.9$, and $r = \max\{0.67, 1.4\} = 1.4$, $\mu = 2.8545$, and $\eta = 1.4367$. The system matrices are

$$\mathfrak{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \mathfrak{B} = \begin{bmatrix} 0.013 & 0.01 \\ 0.02 & 0.12 \end{bmatrix},\quad \mathfrak{C} = \begin{bmatrix} 0.004 & 0.01 \\ 0.03 & 0.006 \end{bmatrix},\quad \mathfrak{D} = \begin{bmatrix} 0.014 & 0.02 \\ 0.3 & 0.12 \end{bmatrix}.$$

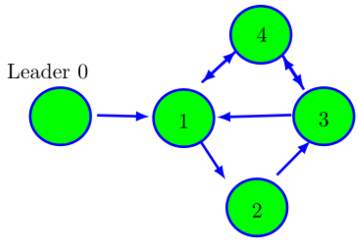

Figure 1: Topology structure of MASs.

Table 2: Comparisons of time delays.

Methods	[20]	Theorem 3.2
Time Delay r	1	1.4

For the communication topology displayed in Fig. 1, we calculate the Laplacian matrix and leader adjacency matrix as

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad \mathfrak{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and corresponding gain matrix is

$$M = \begin{bmatrix} 1.8752 & -0.2469 \\ -0.5361 & 0.7381 \end{bmatrix}.$$

In this case, the agent is chosen to be $\sigma_0(0) = [0.2, -0.2]^T$, $\sigma_1(0) = [0.02, -0.1]^T$, $\sigma_2(0) = [-0.03, 0.2]^T$, $\sigma_3(0) = [0.1, -0.15]^T$, and $\sigma_4(0) = [-0.1, 0.1]^T$, and the communication of MASs is shown in Fig. 1. The evolution of the leader and four followers under control inputs is displayed in Fig. 3, showing that the proper leader-following synchronization is obtained, whereas the state evolution of leader and four followers is shown in Fig. 2, and it's obvious that the leader and follower agents exactly synchronize. Additionally, the comparison of time delay is depicted in Table 2.

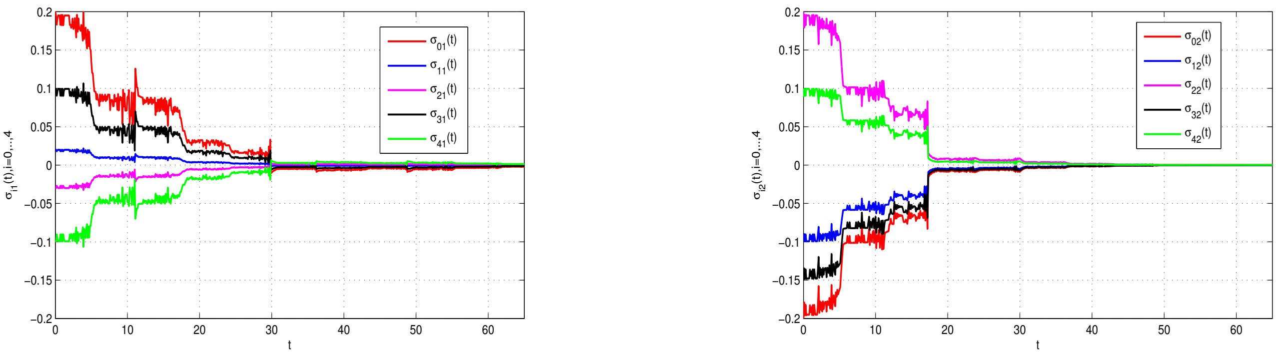

Figure 2: The plot of $\sigma_{i1}(t)$ and $\sigma_{i2}(t)$, $i = 0, 1, 2, 3, 4$ of the MASs guided by Theorem 3.2.

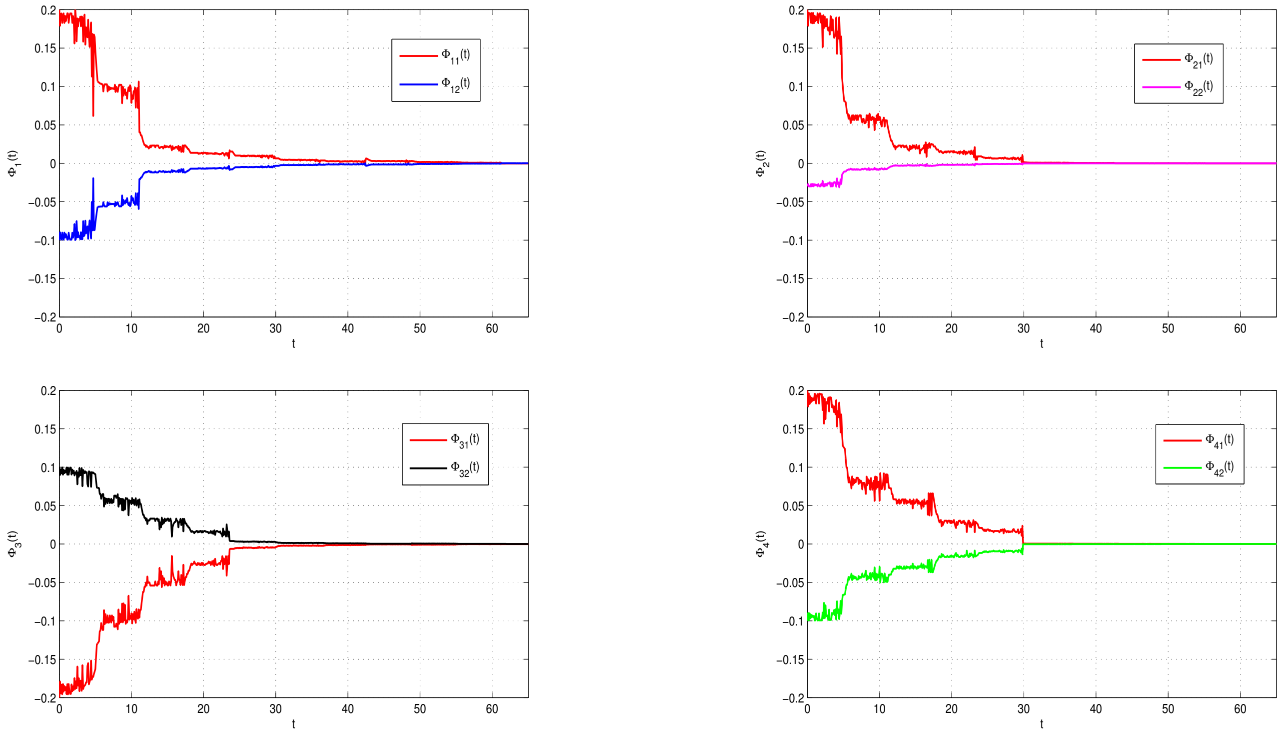

Figure 3: The plot of i^{th} follower agent's errors $\varpi_i(t)$, $i = 1, 2, 3, 4$ of the MASs.

Example 4.2. Consider the UMASs (3.10) indexed by 0 and 1, 2, 3, 4, respectively. Then consider the system

$$\dot{\varpi}_i(t) = -(\mathfrak{A} + \Delta\mathfrak{A}(t))\varpi_i(t) + (\mathfrak{B} + \Delta\mathfrak{B}(t))\varpi_i(t - r(t)) + (\mathfrak{C} + \Delta\mathfrak{C}(t))\delta(\varpi_i(t)) + (\mathfrak{D} + \Delta\mathfrak{D}(t))\delta(\varpi_i(t - r(t))) + \mathcal{K}_i(t).$$

Here considered non-linear function is $\delta(\varpi_i(t)) = 0.02 \sinh(\varpi_i(t))$. The discrete delay $r(t) = 0.62 \sin(t) + 0.58$, $r_1 = 0.58$, $r_2 = 1.2$, $\iota = 0.8$, and $r = \max\{0.58,\ 1.2\} = 1.4$ $\mu = 1.8050$, and $\eta = 1.0243$. The system matrices are

$$\mathfrak{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathfrak{B} = \begin{bmatrix} 1.013 & 0.21 \\ 0.02 & 0.12 \end{bmatrix}, \quad \mathfrak{C} = \begin{bmatrix} 3.004 & 1.01 \\ 0.43 & 0.6 \end{bmatrix}, \quad \mathfrak{D} = \begin{bmatrix} 2.014 & 0.92 \\ 0.3 & 0.21 \end{bmatrix}.$$

Uncertain parameters are taken below

$$W_1 = \begin{bmatrix} 0.03 & 0.04 \\ 0.02 & 0.05 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.03 & 0.02 \\ 0.004 & 0.001 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.06 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0.06 & 0.01 \\ 0.02 & 0.03 \end{bmatrix}, \quad R = \begin{bmatrix} 0.02 & 0.04 \\ 0.05 & 0.01 \end{bmatrix}.$$

The Laplacian matrix L and pinning matrix $\mathfrak{R}$ are

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad \mathfrak{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and control gain matrix M is

$$M = \begin{bmatrix} 2.0143 & 0.5412 \\ 0.5443 & 1.2782 \end{bmatrix}.$$

Table 3: Comparisons of time delays.

Methods	[20]	Theorem 3.3
Time Delay r	1	1.2

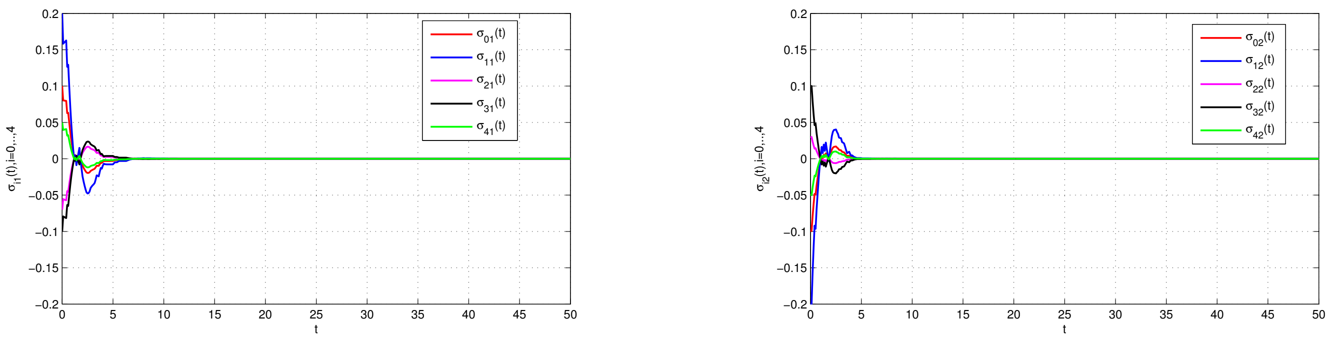

Figure 4: The plot of $\sigma_{i1}(t)$ and $\sigma_{i2}(t)$, $i = 0, 1, 2, 3, 4$ of the MASs guided by Theorem 3.3.

In this case, the agent is selected to be $\sigma_0(0) = [0.1, -0.1]^T$, $\sigma_1(0) = [0.2, -0.62]^T$, $\sigma_2(0) = [-0.07, 0.03]^T$, $\sigma_3(0) = [-0.1, 0.1]^T$, and $\sigma_4(0) = [0.05, -0.05]^T$, and the communication of MASs is shown in Fig. 1. The evolution of the leader and four followers under control inputs is displayed in Fig. 5, showing that the proper leader-following synchronization is obtained, whereas the state evolution of leader and four followers is shown in Fig. 4, and it's obvious that the leader and follower agents exactly synchronize. Additionally, the comparison of time delay is depicted in Table 3.

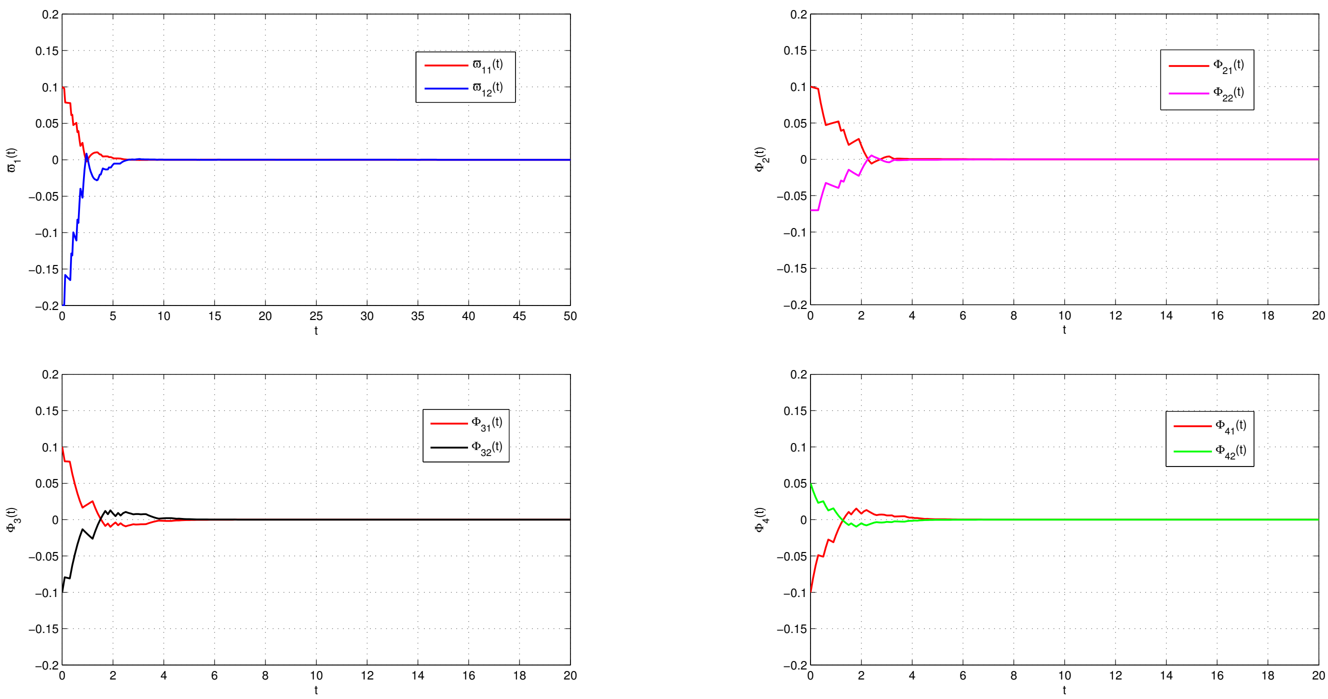

Figure 5: The plot of i^{th} follower agent's errors $\varpi_i(t)$, $i = 1, 2, 3, 4$ of the MASs.

5. Conclusion

In this manuscript, we analyzed the exponential synchronization problem of non-linear UMass with time delay and impulse effects. Here the exponential synchronization analysis for solved proper Lyapunov by utilizing LMI condition in MASs with a linear-feedback pinning controller scheme was proposed. Also, a leader-follower problem has been solved by constructing an appropriate Lyapunov function, and by using the LMI condition in the MASs, it achieved exponential synchronization. At last, two numerical validations under simulations have been presented to show and support the main proofs derived in the theoretical section.

In [3, 6, 27], the authors engaged in discussion regarding the synchronization for fractional-order systems with sampled-data control (SDC) approach and impulsive effect under cyber attack. Subsequently, in [2], the authors provided an explanation of T-S fuzzy systems accompanied by sliding-mode control (SMC) techniques. Inspiration from the insights presented in [2, 3, 6, 27], the proposed control methodology will offer applicability to T-S fuzzy fractional of MASs using an innovative approach. The proposed method in this research will be used to examine the H_∞ SMC for T-S fuzzy MASs with cyber-attack in future work.

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