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Exploring multi-index functions with significance relations and associated with general properties in fractional calculus

Rana Safdar Ali[a,*], Amna Liaqat[a], Khadiga Wadi Nahar Tajer[b], Rabab Moustafa Boshnak[c], Badria Almaz Ali Yousif[d]

[a]Department of Mathematics, University of Lahore, Lahore, Pakistan.

[b]Department of Mathematics, College of Science and Arts, Qassim University, Ar Rass 51452, Saudi Arabia.

[c]Department of Mathematics, College of Science and Arts, Qassim University, Al-Mithnab 51951, Saudi Arabia.

[d]Department of Mathematics, College of Science and Arts, Qassim University, Unayzah 51911, Saudi Arabia.

Abstract

The extensions and generalization of the special functions including in particular Pochhammer symbol, hypergeometric functions, Mittag-Leffler type functions, and Bessel-Maitland functions are the main core for the development of fractional operators by means of its kernel. We present a new extension of the multi-inde Mittag Leffler (MML) function and multi-index Bessel-Maitland (MBM) function by using the generalized Pochhammer symbol. Moreover, we establish some significance relations of such type of multi-index functions with other existing versions of Mittag-Leffler and Bessel-Maitland functions. Moreover, we analyze the behavior of some well-known fractional operators like as Saigo's fractional integral (SFI), Erdeyli fractional integral (EFI) and Riemann fractional integral (RFI) with the product of newly describe multi-index functions.

Keywords: Bessel-Maitland function, fractional operators, hypergeometric functions, Mittag-Leffler function, mathematical operators

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1. Introduction

The rapid developments of fractional calculus has increased the demand of integral operators and transformations. To resolve this issue many researches discuss and formulate new results [3, 4, 7, 14–16], which seen the theoretical achievements of fractional operators. Recently, many mathematicians have been developed a large number of integral formulas [2, 5, 6, 13, 32] involving a verity of special functions as its kernel. Several integral operator associate product of bessel functions have been developed and play consequential role in different physical problems of physics and engineering.

*Corresponding author

Email addresses: rsafdar0@gmail.com (Rana Safdar Ali), Amnaliaqat2511@gmail.com (Amna Liaqat), khadiganahar@gmail.com (Khadiga Wadi Nahar Tajer), R.Boshnaqe@qu.edu.sa (Rabab Moustafa Boshnak), B.Yousif@qu.edu.sa (Badria Almaz Ali Yousif)

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The gamma function [28] for $\Re(u) > 0$ is defined as

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy.$$

The beta function [28] is defined as follows

$$\beta(p, h) = \int_0^1 y^{p-1}(1-y)^{h-1} dy, \tag{1.1}$$

where $\Re(p) > 0, \Re(h) > 0$. The beta function in the form of gamma function is as follows

$$\beta(p, h) = \frac{\Gamma(p)\Gamma(h)}{\Gamma(p+h)}. \tag{1.2}$$

The Pochhammer's symbol defined [28] for $p \in \mathbb{C}$ and $q \in \mathbb{N}$, then

$$\frac{\Gamma(p+q)}{\Gamma(p)} = (p)_q = \begin{cases} 1, & \text{for } q = 0,\ p \neq 0, \\ p(p+1)(p+2)\cdots(p+q-1), & \text{for } q \geqslant 1. \end{cases} \tag{1.3}$$

The Gauss hypergeometric function [32] is defined for $\mu,\ \nu,\ \omega \in \mathbb{C}$, $\omega \neq 0, -1, -2 \cdots\ |z| < 1$, as follows

$${}_2F_1(\mu, \nu; \omega; z) = \sum_{q=0}^{\infty} \frac{(\mu)_q (\nu)_q}{(\omega)_q} \frac{z^q}{q!}, \tag{1.4}$$

where $(\mu)_q$, $(\nu)_q$, and $(\omega)_q$ are Pochhammer's symbols. When $z = 1$, then Gauss hypergeometric function in gamma form is as

$${}_2F_1(\mu, \nu; \omega; 1) = \frac{\Gamma(\omega)\Gamma(\omega-\mu-\nu)}{\Gamma(\omega-\mu)\Gamma(\omega-\nu)}. \tag{1.5}$$

The generalized hypergeometric function [32] is defined as

$${}_eF_h(c_1 \cdots c_e; d_1 \cdots d_h; z) = \sum_{q=0}^{\infty} \frac{(c_1)_q \cdots (c_e)_q}{(d_1)_q \cdots (d_h)_q} \frac{z^q}{q!},$$

where $c_i, d_j \in \mathbb{C}$, $d_j \neq 0, -1, \ldots$ $(i = 1, 2, \ldots, e;\ j = 1, 2, \ldots, h)$. The Saigo's generalized fractional integral operators [23] is defined for $y > 0$, $\Re(\mu) > 0$, and $\mu,\ \nu,\ \omega \in \mathbb{C}$, as follows

$$(I_{0,z}^{\mu,\nu,\omega} f)(z) = \frac{z^{-\mu-\nu}}{\Gamma(\mu)} \int_0^z (z-y)^{\mu-1} {}_2F_1(\mu+\nu, -\omega; \mu; 1-\frac{y}{z}) f(y) dy, \tag{1.6}$$

where ${}_2F_1$ is a Gauss hypergeometric function used as kernel.

Saigo's generalized fractional differential operators [35] are defined for $y > 0$ and $\mu,\ \nu,\ \omega \in \mathbb{C}$, $\Re(\mu) > 0$ as

$$(D_{0,z}^{\mu,\nu,\omega} f)(z) = (\frac{d}{dz})^p \frac{z^{\mu+\nu}}{\Gamma(p-\mu)} \int_0^z (z-y)^{p-\mu-1} {}_2F_1(-\mu-\nu, -\nu-\omega+p; p-\mu; 1-\frac{y}{z}) f(y) dy, \tag{1.7}$$

where p is the pth-derivative of the integral.

The Erdeyli-Kober fractional integral operator [39] is defined for $y > 0$, $\Re(\mu) > 0$, and $\mu.\nu \in \mathbb{C}$, then

$$(K_{o,z}^{\mu,\nu} f)(z) = \frac{z^{-\mu-\nu}}{\Gamma(\mu)} \int_0^z (z-y)^{\mu-1} y^{\nu} f(y) dy. \tag{1.8}$$

The Riemann-Liouville fractional integral operators [35] are defined for $y > 0$, $\Re(\mu) > 0$, and $\mu, \nu, \omega \in \mathbb{C}$,

$$(I^{\zeta}_{0,z} f)(z) = \frac{1}{\Gamma(\zeta)} \int_0^z (z-y)^{\zeta-1} f(y) dy. \tag{1.9}$$

Riemann Liouville fractional differential operators [35] are defined for $y > 0$, $\Re(\mu) > 0$, $p = [\Re(\zeta) + 1]$, and $\mu, \nu, \omega \in \mathbb{C}$ as

$$(D^{\zeta}_{0,z} f)(z) = (\frac{d}{dz})^p \frac{1}{\Gamma(p-\zeta)} \int_0^z (z-y)^{p-\zeta-1} f(y) dy. \tag{1.10}$$

The Pochhammer symbols $(z;p)_q$ [17] is defined as

$$(z;p)_q = \begin{cases} \frac{\Gamma_p(z+q)}{\Gamma(z)}, & \Re(p), p, q \in \mathbb{C}, \\ (p)_q, & p = 0,\ z, q \in \mathbb{C} \setminus 0. \end{cases}$$

The Pochhammer symbols $(z;p,\nu)_q$ [31] are defined as

$$(z;p,\nu)_q = \begin{cases} \frac{\Gamma_\nu(z+q;p)}{\Gamma(z)}, & \Re(p), \Re(\nu) > 0,\ z, q \in \mathbb{C}, \\ (p;\nu)_q, & \nu = 0,\ z, q \in \mathbb{C} \setminus 0. \end{cases}$$

The generalized hypergeometric function extension was presented by Srivastava et al. [37] as

$${}_uF_v[(\lambda_1;\rho,\nu)\cdots(\lambda_u);(\xi_1)\cdots(\xi_v);z] = \sum_{q=0}^{\infty} \frac{(\lambda_1;\rho,\nu)_q \cdots (\lambda_u)_q}{(\xi_1)_q \cdots (\xi_v)_q} \frac{z^q}{q!},$$

where $\lambda_j \in \mathbb{C}$ for $j = 1, \dots, u$, $\xi_k \in \mathbb{C}$, $k = 1, \dots, v$, and $\xi_k \neq 0, -1, \dots$ and where $(\lambda_1;\rho,\nu)_q$ is the extension of generalized Pochhammer symbol. Specially, the related extension Gauss hypergeometric function ${}_2F_1$ is given as

$${}_2F_1[(\lambda_1;r,s),\mu;\xi;z] = \sum_{q=0}^{\infty} \frac{(\lambda_1;r,s)_q (\mu)_q}{(\xi)_q} \frac{z^q}{q!}.$$

The confluent hypergeometric function ${}_1F_1$ is given as

$$\Phi[(\lambda_1;r,s);\mu;z] = {}_1F_1[(\lambda_1;\rho,\nu);\xi;z] = \sum_{q=0}^{\infty} \frac{(\lambda_1;\rho,\nu)_q}{(\xi)_q} \frac{z^q}{q!}. \tag{1.11}$$

For the interested researchers and readers may consult [9, 10, 12, 26, 27, 30, 39] for the various extensions of special function and applications in the different area. The Pochhammer's symbol in gamma form can be written as

$$(p)_q = \frac{\Gamma(p+q)}{\Gamma(p)}, \tag{1.12}$$

where $p \in \mathbb{C}$ and $q \in \mathbb{N}$.

2. Modified version of MML function and its special cases

In this section, we discuss the significance versions of Mittag-Leffler functions, and also describe the extension of multi-index Mittag-Leffler (MML) function by utilizing generalized Pochhammer symbols. Moreover, we establish immense relation of newly describe function with other existing versions of Mittag-Leffler functions in the form of special cases.

In 1903, Mittag Leffler function [19, 25] in a single parameter is defined as

$$E_{\alpha}(z) = \sum_{q=0}^{\infty} \frac{z^q}{\Gamma(\alpha q + 1)}, \tag{2.1}$$

where $z \in \mathbb{C}$ and $\alpha > 0$. The generalized Mittag Leffler function [1] in two parameters, which also introduced in entire complex plane, is given as

$$E_{\alpha,\beta}(z) = \sum_{q=0}^{\infty} \frac{z^q}{\Gamma(\alpha q + \beta)}, \tag{2.2}$$

where α, $\beta > 0$, and $z \in \mathbb{C}$. In [29], the generalization of Mittag-Leffler (ML) function (2.2) is defined as follows

$$E^{x}_{\alpha,\beta}(z) = \sum_{q=0}^{\infty} \frac{(x)_q}{\Gamma(\alpha q + \beta)} \frac{z^q}{q!}, \tag{2.3}$$

where $\alpha, \beta, x \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0$. Moreover, ML function generalization introduced by [17] is as follows

$$E^{x}_{\alpha,\beta;p}(z) = \sum_{q=0}^{\infty} \frac{(x;p)_q}{\Gamma(\alpha q + \beta)} \frac{z^q}{q!}, \tag{2.4}$$

where $q \geqslant 0, \alpha, \beta, x \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0$. Rahman et al. [31], defined the following extension of Mittag Leffler functin by

$$E^{x}_{\alpha,\beta;p,\nu}(z) = \sum_{q=0}^{\infty} \frac{(x;p,\nu)_q}{\Gamma(\alpha q + \beta)} \frac{z^q}{q!}, \tag{2.5}$$

where $q \geqslant 0, \alpha, \beta, x \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \nu \geqslant 0$. There are many applications for these functions which are found in the papers [21, 33, 34, 36, 40]. Multi-index Mittag Leffler function (mML) as the extension of generalized Mittag Leffler function, has introduced and studied by Kiryakova [24]. Multi-index Mittag Leffler function [24] is defined for $m \geqslant 1$ and $\Re(\rho_1) \cdots \Re(\rho_m) > 0$, then

$$E_{\frac{1}{\rho_j},\mu_j}(z) = \sum_{q=0}^{\infty} \frac{z^q}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{q}{\rho_j})}, \tag{2.6}$$

where $\mu_1, \ldots, \mu_m$ are arbitrary parameters. If we replace $m = 1$, $\rho_1 = \frac{1}{\alpha}$, and $\mu_1 = \beta$, then it gives the classical result as given in equation (2.2). The modified version of MML function is defied as follows

$$E^{\lambda;r,s}_{\frac{1}{\rho_j},\mu_j}(z) = \sum_{q=0}^{\infty} \frac{(\lambda;r,s)_q}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{q}{\rho_j})} \frac{z^q}{q!}, \tag{2.7}$$

where $\Re(r) > 0, r \geqslant 0, \lambda \in \mathbb{C}, s \geqslant 0$, $m \geqslant 1$ and $\Re(\rho_1) \cdots \Re(\rho_m) > 0$, $\mu_1 \cdots \mu_m$ are arbitrary parameters.

Special cases:

- If we put $j = 1 = m$, $\frac{1}{\rho_1} = \rho_1$ in (2.7), then we get the extension of ML function as (2.5).
- If we put $r = 0, s = 0$, $j = 1 = m$, $\frac{1}{\rho_1} = \rho_1$ in (2.7), then we get the generalization of ML function as (2.3).

- If we replace $s = 0$, $j = 1 = m$, $\frac{1}{\rho_1} = \rho_1$ in (2.7), then we obtained the generalized ML function as defined in (2.4).
- After replacing $r = s = 0$ and $\lambda = 1$ in (2.7), then we have another version of multi-index ML function, which defined in (2.6).
- If we insert $r = s = 0$, $j = 1 = m$, and $\lambda = 1$, $\frac{1}{\rho_1} = \rho_1, \mu_1 = 1$ in (2.7), then we get the ML function as (2.1).
- We put $r = s = 0$, $j = 1 = m$, and $\lambda = 1$, $\frac{1}{\rho_1} = \rho_1$ in (2.7), then we get the ML function as (2.2).
- We replace $j = 1 = m$, $\rho_1 = 1$ in (2.7), then we have the confluent hypergeometric function defined in (1.11).

3. Generalized versions of Bessel-Maitland functions and its special cases

In this sections, we discuss the different type of Bessel-Maitland functions, and also introduced the modified version of Bessel-Maitland function by utilizing the Pochhammer symbols, and establish some relation with existing version of Bessel-Maitland function.

Bessel functions [11, 45] have great importance in many significance area of mathematics, especially in applied mathematics. Bernoulli was whose introduced the concept of Bessel function in 1732. The Bessel function of first kind of order n is defined as follows

$$J_n(z) = \sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+n}}{\Gamma(n+q+1)q!}. \tag{3.1}$$

Edward Maitland Wright [46] established the generalized version of Bessel function, named Bessel-Maitland function. In [41, 42], Suthar et al. discussed certain properties of Bessel-Maitland function and further some extensions of Bessel-Maitland function also discussed in [8, 38] for $z, \eta \in \mathbb{C}$, $\alpha > 0$ as

$$J_\eta^\beta(z) = \sum_{q=0}^{\infty} \frac{(-z)^q}{q!\Gamma(\beta q + \eta + 1)}. \tag{3.2}$$

Waseem et al. [22] introduced the generalized Bessel-Maitland function and also discussed the integrals formulas of generalized Bessel-Maitland function for $z \in \mathbb{C}/(-\infty, 0]$; $\beta, \eta, \gamma \in \mathbb{C}$, $\Re(\beta) \geqslant 0$, $\Re(\eta) \geqslant -1$, $\Re(\gamma) \geqslant 0$, $k \in (0,1) \cup \mathbb{N}$ and ongoing extension and derivations of Bessel-Maitland function in [8, 44] without breakup as

$$J_{\eta,k}^{\beta,\gamma}(z) = \sum_{q=0}^{\infty} \frac{(\gamma)_{kq}(-z)^q}{q!\Gamma(\beta q + \eta + 1)}. \tag{3.3}$$

Generalized multi-index Bessel function discussed by Suthar et al. in [43] with fractional integrals acting Guess hypergeometric function kernel in context of fractional calculus for $z, \beta_i, \eta_i, \gamma \in \mathbb{C}$, $(i = 1, 2 \cdots m)$, $\Re(\beta_i) > 0$, $\Re(\eta_i) > -1$, $\sum_{i=1}^{m} \Re(\beta)_i > \max\{0; \Re(k) - 1\}$, $k > 0$ is as

$$J_{\gamma,k}^{(\beta_i,\eta_i)_m}(z) = \sum_{q=0}^{\infty} \frac{(\gamma)_{kq}(-z)^q}{q! \prod_{i=1}^{m} \Gamma(\beta_i q + \eta_i + 1)}. \tag{3.4}$$

The Extended multi-index Bessel function (emB) is defined as ([20])

$$\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}(z) = \sum_{q=0}^{\infty} \frac{c^q(\lambda)_{hq}(-z)^q}{(\delta)_q \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})},$$

where $\beta_i, \eta_i, \lambda, p, c \in \mathbb{C}$ $(i = 1, 2, 3, \ldots, w)$ be such that $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(x) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, and $\Re(\delta) > 0$.

We describe the new modified version of multi-index Bessel-Maitland function by using generalized Pochhammer symbols as follows

$$\mathbb{J}^{(\beta_i)_w,\lambda;r,s}_{(\eta_i)_w,x,\delta,u}(z) = \sum_{q=0}^{\infty} \frac{(x)_q (\lambda; r, s)_{uq}}{(\delta)_q \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + 1)} \frac{(-z)^q}{q!}, \tag{3.5}$$

where $\delta, x, \lambda, \beta_i, \eta_i \in \mathbb{C}$, $i = 1, \ldots, w$, $\Re(\beta_i) \geqslant -1$, $\Re(\eta_i) > 0$, $\Re(r) > 0$, $r \geqslant 0$, $s \geqslant 0$.

Special cases:

- If we replace $\lambda = u = x = \delta = 1$, $r = s = 0$, $i = 1 = w$, $\eta_1 = 0$, and replace $z = -z$ in (3.5), then we get the function is defined in (2.1).
- If we put $\lambda = u = x = \delta = 1$, $r = s = 0$, $i = 1 = w$, $\eta_1 = \eta - 1$, and replace $z = -z$ in (3.5), then we obtain (2.2).
- We adjust the parameters $\lambda = u = \delta = 1$, $r = s = 0$, $i = 1 = w$, $\eta_1 = \eta - 1$, and replace $z = -z$ in (3.5), then we get the equation (2.3).
- If we put $x = \delta = u = 1$, $s = 0$, $i = 1 = w$, $\eta_1 = \eta - 1$, and replace $z = -z$ in (3.5), then we get (2.4).
- After replacing the parameters $x = \delta = u = 1$, $i = 1 = w$, $\eta_1 = \eta - 1$, and replacing $z = -z$ in (3.5), then we have (2.5).
- If we insert $x = \delta = u = \lambda = 1$, $\eta_i = \eta_i - 1$, $\beta_i = \frac{1}{\beta_i}$ and replace $z = -z$ in (3.5), then we get the ML function (2.6).
- If we replace $x = \delta = u = 1$, $\eta_i = \eta_i - 1$, $\beta_i = \frac{1}{\beta_i}$ and replace $z = -z$ in (3.5), then we get the MML function in (2.7).
- We replace $\delta = x = \lambda = 1$, $r = s = u = 0$, $i = i = w$, $\beta_1 = 1$ and replace $z = \frac{z^2}{4}$ and multiplying by z^{η_1} in (3.5), then we have the Bessel function of first kind (3.1) if of the form $\frac{1}{z^{\eta_1}} J_{\eta_1}(z)$.
- If we put $x = \delta = 1$, $u = r = s = 0$, $i = 1 = w$ in (3.5), then we obtained the function in (3.2).
- We insert $r = s = 0$, $x = \delta = 1$, $i = 1 = w$ in (3.5), then we get another version of BM function (3.3).
- We put $r = s = 0$, $x = \delta = 1$, in (3.5), then we get the BM function defined in (3.4).
- After putting the values of $x = 1$, $r = s = 0$, $\eta_i = \eta_i + \frac{p-1}{2}$, and replacing $z = cz$ in (3.5), then we get the extended multi-index Bessel function defined in (1.3).

4. Differential properties of multi-index type series functions

In this section, we discuss the various dynamical differential properties of multi-index Mittag-Leffler and multi-index Bessel-Maitland functions.

Theorem 4.1. *Let $\Re(r) > 0, r \geqslant 0, \lambda \in \mathbb{C}, s \geqslant 0$, $m \geqslant 1$ and $\Re(\rho_1) \cdots \Re(\rho_m) > 0$, $\mu_1 \cdots \mu_m$ be arbitrary parameters, then the following relation holds*

$$E^{\lambda;r,s}_{\frac{1}{\rho_j},\mu_j}(z) = \prod_{j=1}^{m} (\mu_j) E^{\lambda;r,s}_{\frac{1}{\rho_j},\mu_j+1}(z) + \prod_{j=1}^{m} \left(\frac{z}{\rho_j}\right) \frac{d}{dz} E^{\lambda;r,s}_{\frac{1}{\rho_j},\mu_j+1}(z) \tag{4.1}$$

and also

$$E^{\lambda;r}_{\frac{1}{\rho_j},\mu_j}(z)=\prod_{j=1}^{m}(\mu_j)E^{\lambda;r}_{\frac{1}{\rho_j},\mu_j+1}(z)+\prod_{j=1}^{m}\left(\frac{z}{\rho_j}\right)\frac{d}{dz}E^{\lambda;r}_{\frac{1}{\rho_j},\mu_j+1}(z), \tag{4.2}$$

$$E^{\lambda}_{\frac{1}{\rho_j},\mu_j}(z)=\prod_{j=1}^{m}(\mu_j)E^{\lambda}_{\frac{1}{\rho_j},\mu_j+1}(z)+\prod_{j=1}^{m}\left(\frac{z}{\rho_j}\right)\frac{d}{dz}E^{\lambda}_{\frac{1}{\rho_j},\mu_j+1}(z). \tag{4.3}$$

Proof. Consider the multi-index relation

$$\begin{aligned}
&\prod_{j=1}^{m}(\mu_j)E^{\lambda;r}_{\frac{1}{\rho_j},\mu_j+1}(z)+\prod_{j=1}^{m}\left(\frac{z}{\rho_j}\right)\frac{d}{dz}E^{\lambda;r}_{\frac{1}{\rho_j},\mu_j+1}(z)\\
&=\prod_{j=1}^{m}(\mu_j)\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}+\prod_{j=1}^{m}\left(\frac{z}{\rho_j}\right)\frac{d}{dz}\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}\\
&=\prod_{j=1}^{m}(\mu_j)\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}+\sum_{q=0}^{\infty}\frac{\prod_{j=1}^{m}(\frac{z}{\rho_j})(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{qz^{q-1}}{q!}\\
&=\prod_{j=1}^{m}(\mu_j)\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}+\sum_{q=0}^{\infty}\frac{\prod_{j=1}^{m}(\frac{q}{\rho_j})(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}\\
&=\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j}+1)}\frac{z^q}{q!}\prod_{j=1}^{m}(\mu_j+\frac{q}{\rho_j})\\
&=\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\Gamma(\mu_1+\frac{q}{\rho_1}+1)\cdots\Gamma(\mu_m+\frac{q}{\rho_m}+1)}\frac{z^q}{q!}[(\mu_1+\frac{q}{\rho_1})\cdots(\mu_m+\frac{q}{\rho_m})]\\
&=\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q[(\mu_1+\frac{q}{\rho_1})\cdots(\mu_m+\frac{q}{\rho_m})]}{(\mu_1+\frac{q}{\rho_1})\Gamma(\mu_1+\frac{q}{\rho_1})\cdots(\mu_m+\frac{q}{\rho_m})\Gamma(\mu_m+\frac{q}{\rho_m})}\frac{z^q}{q!}\\
&=\sum_{q=0}^{\infty}\frac{(\lambda;r,s)_q}{\prod_{j=1}^{m}\Gamma(\mu_j+\frac{q}{\rho_j})}\frac{z^q}{q!}=E^{\lambda;r,s}_{\frac{1}{\rho_j},\mu_j}(z)\ \ (\text{using } \Gamma(z+1)=z\Gamma(z)).
\end{aligned}$$

The result (4.2) can be obtained if we set $s=0$ in (4.1). The result (4.3) can be obtained if we set $r=0$ in (4.2). □

Theorem 4.2. *Let $\delta,x,\lambda,\beta_i,\eta_i\in\mathbb{C}$, $i=1,\dots,w$, $\Re(\beta_i)\geqslant-1,\Re(\eta_i)>0$, $\Re(r)>0,r\geqslant0,s\geqslant0$, then the following relation holds*

$$\mathbb{J}^{(\beta_i)_w,\lambda;r,s}_{(\eta_i)_w,x,\delta,u}(z)=\prod_{i=1}^{w}(\eta_i+1)\mathbb{J}^{(\beta_i)_w,\lambda;r,s}_{(\eta_i)_w+1,x,\delta,u}(z)+\prod_{i=1}^{w}\left(z\beta_i\right)\frac{d}{dz}\mathbb{J}^{(\beta_i)_w,\lambda;r,s}_{(\eta_i)_w+1,x,\delta,u}(z)$$

and also

$$\mathbb{J}^{(\beta_i)_w,\lambda;r}_{(\eta_i)_w,x,\delta,u}(z)=\prod_{i=1}^{w}(\eta_i+1)\mathbb{J}^{(\beta_i)_w,\lambda;r}_{(\eta_i)_w+1,x,\delta,u}(z)+\prod_{i=1}^{w}\left(z\beta_i\right)\frac{d}{dz}\mathbb{J}^{(\beta_i)_w,\lambda;r}_{(\eta_i)_w+1,x,\delta,u}(z),$$

$$\mathbb{J}^{(\beta_i)_w,\lambda}_{(\eta_i)_w,x,\delta,u}(z)=\prod_{i=1}^{w}(\eta_i+1)\mathbb{J}^{(\beta_i)_w,\lambda}_{(\eta_i)_w+1,x,\delta,u}(z)+\prod_{i=1}^{w}\left(z\beta_i\right)\frac{d}{dz}\mathbb{J}^{(\beta_i)_w,\lambda}_{(\eta_i)_w+1,x,\delta,u}(z).$$

Proof. To obtain the required result, we repeat the same procedure as in Theorem 4.1. □

5. Analyzing of fractional behavior with product of generalized multi-index functions

In this section, we discuss the fractional differential and integrals behavior with the product of extended multi-index Bessel function and multi-index Mittag-Leffler function.

Theorem 5.1. *Let* $\mu, \nu, \omega, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C} (i = 1, 2, 3, \ldots, w)$, $\Re(\mu) > 0$, $\Re(\sigma) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$ *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\nu}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\sigma n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_4\psi_{2+w}\left[\begin{matrix} (\lambda, h)(\sigma n + 1, 1)(\sigma n - \nu + \omega + 1, 1)(1, 1) \\ (\delta, 1)(\sigma n - \nu + 1, 1)(\mu + \sigma n + \omega + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

Proof. Considering the left-sided generalized Sagio fractional integral operator (1.6) with the product of multi-index functions, defined in (2.6) and (1.3), we have

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\mu-\nu}}{\Gamma(\mu)} \int_0^z {}_2F_1(\mu + \nu, -\omega; \mu; (1 - \frac{y}{z})) \times (z - y)^{\mu-1} \sum_{n,q=0}^{\infty} \frac{y^{\sigma n} c^q (\lambda)_{hq} (-1)^q y^q}{(\delta)_q \prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j}) \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})} dy. \tag{5.1}$$

Using (1.4) in equation (5.1) we have

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\nu-1}}{\Gamma(\mu)} \mathscr{E}_q^n \int_0^z (1 - \frac{y}{z})^{\mu-1} \sum_{h=0}^{\infty} \frac{(\mu + \nu)_h (-\omega)_h}{(\mu)_h h!} (1 - \frac{y}{z})^h y^{\sigma n + q} dy, \tag{5.2}$$

where

$$\mathscr{E}_q^n = \sum_{n,q=0}^{\infty} \frac{c^q (\lambda)_{hq} (-1)^q}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j}) (\delta)_q \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})}, \tag{5.3}$$

putting these values $\frac{y}{z} = t \implies dy = z dt$, $y = z \implies t = 1$, and $y = 0 \implies t = 0$ in equation (5.2) we have

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\nu-1}}{\Gamma(\mu)} \mathscr{E}_q^n \sum_{h=0}^{\infty} \frac{(\mu + \nu)_h (-\omega)_h}{(\mu)_h h!} \int_0^1 (1 - t)^{\mu+h-1} (zt)^{\sigma n + q} z dt. \tag{5.4}$$

Using (1.1) and (1.2) in equation (5.4) we obtain

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\nu}}{\Gamma(\mu)} \mathscr{E}_q^n z^{\sigma n + q} \sum_{h=0}^{\infty} \frac{(\mu + \nu)_h (-\omega)_h}{(\mu)_h h!} \frac{\Gamma(\mu + h)\Gamma(\sigma n + q + 1)}{\Gamma(\mu + \sigma n + q + h + 1)}. \tag{5.5}$$

Using (1.5) in equation (5.5) we have

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = z^{-\nu} \mathscr{E}_q^n z^{\sigma n + q} \frac{\Gamma(\sigma n + q - \nu + \omega + 1)\Gamma(\sigma n + q + 1)}{\Gamma(\sigma n + q - \nu + 1)\Gamma(\mu + \sigma n + q + \omega + 1)}. \tag{5.6}$$

Using (5.3) in equation (5.6) and simplifying, we have

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{-\nu}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n,q=0}^{\infty} \frac{c^q (-1)^q z^{\sigma n + q}}{\Gamma(\sigma n + q - \nu + 1) \prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} \times \frac{\Gamma(\lambda + hq)\Gamma(\sigma n + q + 1)\Gamma(\sigma n + q - \nu + \omega + 1)}{\Gamma(\delta + q)\Gamma(\mu + \sigma n + q + \omega + 1) \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})},$$

then we get the desired result

$$I_{0,z}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= \frac{z^{-\nu}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\sigma n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_4\psi_{2+w} \left[\begin{array}{c} (\lambda, h)(\sigma n + 1, 1)(\sigma n - \nu + \omega + 1, 1)(1, 1) \\ (\delta, 1)(\sigma n - \nu + 1, 1)(\mu + \sigma n + \omega + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{array} \Bigg| -zc \right].$$

□

The following right sided fractional operator with multi-index series function can be obtained by using the same pattern, which is discussed in Theorem 5.1

Theorem 5.2. *Let* $\mu, \nu, \omega, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}(i = 1, 2, 3, \dots, w)$, $\Re(\mu) > 0$, $\Re(\sigma) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$ *and* $\Re(\delta) > 0$, $y > 0$, *then following relation holds*

$$I_{z,\infty}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j} y^{-\sigma} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= \frac{z^{-\nu}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{-\sigma n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_4\psi_{2+w} \left[\begin{array}{c} (\lambda, h)(\sigma n + \nu, -1)(\sigma n + \omega, -1)(1, 1) \\ (\delta, 1)(\mu + \sigma n + \nu + \omega, -1)(\sigma n, -1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{array} \Bigg| -zc \right].$$

Theorem 5.3. *Let* $\mu, \nu, \omega, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}(i = 1, 2, 3, \dots, w)$, $\Re(\mu) > 0$, $\Re(\mathcal{L}) > 0$, $\Re(\beta) > 0$, $\Re(\Im) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$ *and* $\Re(\delta) > 0$, $y > 0$, *then following relation holds*

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}} E_{\frac{1}{\rho_j},\mu_j} y^{\beta+\Im} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= \frac{z^{\nu+\mathcal{L}}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\Im n + \beta n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})}$$
$$\times {}_4\psi_{3+w} \left[\begin{array}{c} (\lambda, h)(\mathcal{L} + \beta n + \Im n + 1, 1)(\nu + \mu + \omega + \mathcal{L} + \beta n + \Im n + 1, 1)(1, 1) \\ (\delta, 1)(\nu + \mathcal{L} + \beta n + \Im n + 1, 1)(\omega + \mathcal{L} + \beta n + \Im n + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{array} \Bigg| -zc \right].$$

Proof. The generalized Sagio fractional differential operator (1.7) with product of multi-index functions, defined in (2.6) and (1.3), is as

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}} E_{\frac{1}{\rho_j},\mu_j} y^{\beta+\Im} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = (\frac{d}{dz})^p \mathscr{E}_q^n \frac{z^{\mu+\nu}}{\Gamma(p-\mu)} \int_0^z (z-y)^{p-\mu-1} y^{\mathcal{L}+\beta n+\Im n+q}$$
$$\times {}_2F_1(-\mu-\nu, -\mu-\omega+p; p-\mu; 1-\frac{y}{z}) dy. \tag{5.7}$$

Using (1.4) in equation (5.7) we have

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}} E_{\frac{1}{\rho_j},\mu_j} y^{\beta+\Im} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= (\frac{d}{dz})^p \mathscr{E}_q^n \frac{z^{p+\nu-1}}{\Gamma(p-\mu)} \int_0^z (1-\frac{y}{z})^{p-\mu-1} y^{\mathcal{L}+\beta n+\Im n+q} \sum_{h=0}^{\infty} \frac{(-\mu-\nu)_h(-\mu-\omega+p)_h}{(p-\mu)_h h!} (1-\frac{y}{z})^h dy \tag{5.8}$$
$$= (\frac{d}{dz})^p \mathscr{E}_q^n \frac{z^{p+\nu-1}}{\Gamma(p-\mu)} \sum_{h=0}^{\infty} \frac{(-\mu-\nu)_h(-\mu-\omega+p)_h}{(p-\mu)_h h!} \int_0^z (1-\frac{y}{z})^{p+h-\mu-1} y^{\mathcal{L}+\beta n+\Im n+q} dy.$$

Now putting values $\frac{y}{z} = t \implies dy = zdt$, $y = z \implies t = 1$, and $y = 0 \implies t = 0$ in equation (5.8), we have

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}} E_{\frac{1}{\rho_j},\mu_j} y^{\beta+\Im} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= (\frac{d}{dz})^p \mathscr{E}_q^n \frac{z^{p+\nu-1}}{\Gamma(p-\mu)} \sum_{h=0}^{\infty} \frac{(-\mu-\nu)_h(-\mu-\omega+p)_h}{(p-\mu)_h h!} \int_0^1 (1-t)^{p+h-\mu-1} (zt)^{\mathcal{L}+\beta n+\Im n+q} z dt \tag{5.9}$$
$$= (\frac{d}{dz})^p \mathscr{E}_q^n \frac{z^{p+\nu+\mathcal{L}+\beta n+\Im n+q}}{\Gamma(p-\mu)} \sum_{h=0}^{\infty} \frac{(-\mu-\nu)_h(-\mu-\omega+p)_h}{(p-\mu)_h h!} \int_0^1 (1-t)^{p+h-\mu-1} t^{\mathcal{L}+\beta n+\Im n+q} dt.$$

Using (1.1) and (1.2) in equation (5.9) we get

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}}E_{\frac{1}{\rho_j},\mu_j}y^{\beta+\Im}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=(\frac{d}{dz})^p\mathscr{E}_q^n\frac{z^{p+\nu+\mathcal{L}+\beta n+\Im n+q}}{\Gamma(p-\mu)}\sum_{h=0}^{\infty}\frac{(-\mu-\nu)_h(-\mu-\omega+p)_h}{(p-\mu)_h h!}\frac{\Gamma(p+h-\mu)\Gamma(\mathcal{L}+\beta n+\Im n+q+1)}{\Gamma(p+h-\mu+\mathcal{L}+\beta n+\Im n+q+1)}. \tag{5.10}$$

Using (1.12) and (1.5) in equation (5.10) we obtain

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}}E_{\frac{1}{\rho_j},\mu_j}y^{\beta+\Im}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=\mathscr{E}_q^n(\frac{d}{dz})^p z^{p+\nu+\mathcal{L}+\beta n+\Im n+q}\frac{\Gamma(\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\nu+\mu+\omega+\mathcal{L}+\beta n+\Im n+q+1)}{\Gamma(p+\nu+\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\omega+\mathcal{L}+\beta n+\Im n+q+1)}. \tag{5.11}$$

Taking pth-derivative of equation (5.11), then

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}}E_{\frac{1}{\rho_j},\mu_j}y^{\beta+\Im}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=\mathscr{E}_q^n z^{\nu+\mathcal{L}+q+\beta n+\Im n}\frac{\Gamma(\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\nu+\mu+\omega+\mathcal{L}+\beta n+\Im n+q+1)}{\Gamma(\nu+\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\omega+\mathcal{L}+\beta n+\Im n+q+1)}. \tag{5.12}$$

Using (5.3) and (1.12) in equation (5.12), we have

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}}E_{\frac{1}{\rho_j},\mu_j}y^{\beta+\Im}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=\frac{z^{\nu+\mathcal{L}}\Gamma(\delta)}{\Gamma(\lambda)}\sum_{n,q=0}^{\infty}\frac{(-rz)^q z^{\beta n+\Im n}}{\Gamma(\delta+q)\prod_{j=1}^m\Gamma(\mu_j+\frac{n}{\rho_j})}$$
$$\times\frac{\Gamma(\lambda+hq)\Gamma(\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\nu+\mu+\omega+\mathcal{L}+\beta n+\Im n+q+1)}{\Gamma(\nu+\mathcal{L}+\beta n+\Im n+q+1)\Gamma(\omega+\mathcal{L}+\beta n+\Im n+q+1)\prod_{i=1}^w\Gamma(\beta_i q+\eta_i+\frac{1+p}{2})},$$

then the desired result is

$$D_{0,z}^{\mu,\nu,\omega}[y^{\mathcal{L}}E_{\frac{1}{\rho_j},\mu_j}y^{\beta+\Im}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=\frac{z^{\nu+\mathcal{L}}\Gamma(\delta)}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{z^{\Im n+\beta n}}{\prod_{j=1}^m\Gamma(\mu_j+\frac{n}{\rho_j})}$$
$$\times{}_4\psi_{3+w}\left[\begin{matrix}(\lambda,h)(\mathcal{L}+\beta n+\Im n+1,1)(\nu+\mu+\omega+\mathcal{L}+\beta n+\Im n+1,1)(1,1)\\(\delta,1)(\nu+\mathcal{L}+\beta n+\Im n+1,1)(\omega+\mathcal{L}+\beta n+\Im n+1,1)(\eta_i+\frac{1+p}{2},\beta_i)|_{i=1}^w\end{matrix}\Bigg|-zc\right].$$

□

Similarly, we can obtain the behavior of right sided fractional differential operator with multi-index function by using the same pattern, which is discussed in Theorem (5.3).

Theorem 5.4. *Let* $\mu,\nu,\omega,\beta_i,\eta_i,\lambda,\delta,h,c\in\mathbb{C}\,(i=1,2,3,\ldots,w)$, $\Re(\mu)>0$, $\Re(\mathcal{L})>0$ *be such that* $\sum_{i=1}^w\Re(\beta_i)>\max\{0;\Re(h)-1\}$; $h>0$, $\Re(\eta_i)>0$, $\Re(\lambda)>0$, *and* $\Re(\delta)>0$, $y>0$, *then the following relation holds*

$$D_{z,\infty}^{\mu,\nu,\omega}[E_{\frac{1}{\rho_j},\mu_j}y^{-\mathcal{L}}\mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c}y](z)$$
$$=z^{\nu}\frac{\Gamma(\delta)}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{z^{-\mathcal{L}n}}{\prod_{j=1}^m\Gamma(\mu_j+\frac{n}{\rho_j})}{}_4\psi_{3+w}\left[\begin{matrix}(\lambda,h)(\mathcal{L}n-\nu,-1)(\mathcal{L}n+\mu+\omega,-1)(1,1)\\(\delta,1)(\mathcal{L}n,-1)(\mathcal{L}n-\nu+\omega,-1)(\eta_i+\frac{1+p}{2},\beta_i)|_{i=1}^w\end{matrix}\Bigg|-zc\right].$$

Theorem 5.5. *Let* $\mu, \nu, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$ $(i = 1,2,3,\ldots,w)$, $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\alpha) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$\begin{aligned} &K_{0,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) \\ &= \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\alpha n + \nu + 1, 1)(1,1) \\ (\delta, 1)(\alpha n + \nu + \mu + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right]. \end{aligned}$$

Proof. Consider the Erdeyli Kober fractional integral operator (1.8) with the product of functions given in (2.6) and (1.3),

$$\begin{aligned} K_{0,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= \frac{z^{-\mu-\nu}}{\Gamma(\mu)} \mathscr{E}_q^n \int_0^z (z-y)^{\mu-1} y^{\nu} y^{\alpha n + q} dy \\ &= \frac{z^{-\nu-1}}{\Gamma(\mu)} \mathscr{E}_q^n \int_0^z (1 - \frac{y}{z})^{\mu-1} y^{\alpha n + q + \nu} dy, \end{aligned} \tag{5.13}$$

putting values $\frac{y}{z} = t \implies dy = zdt$, $y = z \implies t = 1$, and $y = 0 \implies t = 0$ in equation (5.13) we have

$$\begin{aligned} K_{o,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= \frac{z^{-\nu-1}}{\Gamma(\mu)} \mathscr{E}_q^n \int_0^1 (1-t)^{\mu-1} (zt)^{\alpha n + q + \nu} z dt \\ &= \frac{z^{-\nu}}{\Gamma(\mu)} \mathscr{E}_q^n z^{\alpha n + q + \nu} \int_0^1 (1-t)^{\mu-1} t^{\alpha n + q + \nu} dt. \end{aligned} \tag{5.14}$$

Using (1.1) and (1.2) in equation (5.14), then

$$K_{o,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \mathscr{E}_q^n z^{\alpha n + q} \frac{\Gamma(\alpha n + q + \nu + 1)}{\Gamma(\alpha n + q + \nu + \mu + 1)}. \tag{5.15}$$

Using (5.3) and (1.12) in equation (5.15), we obtain

$$\begin{aligned} &K_{0,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) \\ &= \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} \sum_{q=0}^{\infty} \frac{\Gamma(\lambda + hq)(-rz)^q}{\Gamma(\delta + q) \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})} \frac{\Gamma(\alpha n + q + \nu + 1)}{\Gamma(\alpha n + q + \nu + \mu + 1)}, \end{aligned}$$

then we get the required result

$$\begin{aligned} &K_{0,z}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) \\ &= \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\alpha n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\alpha n + \nu + 1, 1)(1,1) \\ (\delta, 1)(\alpha n + \nu + \mu + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right]. \end{aligned}$$

□

Theorem 5.6. *Let* $\mu, \nu, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$ $(i = 1,2,3,\ldots,w)$, $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\alpha) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$\begin{aligned} &K_{z,\infty}^{\mu,\nu}[E_{\frac{1}{\rho_j},\mu_j} y^{-\alpha} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) \\ &= \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{-\alpha n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\nu + \alpha n, -1)(1,1) \\ (\delta, 1)(\mu + \nu + \alpha n, -1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right]. \end{aligned}$$

Theorem 5.7. *Let* $\zeta, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$ $(i = 1, 2, 3, \dots, w)$, $\Re(\zeta) > 0$, $\Re(\varepsilon) > 0$, $\Re(\varpi) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = z^{\zeta+\varepsilon} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\varepsilon + \varpi n + 1, 1)(1, 1) \\ (\delta, 1)(\varepsilon + \varpi n + \zeta + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

Proof. Consider the Riemann liouville fractional integral operator (1.9) with the multi-index functions defined in (2.6) and (1.3), we have

$$\begin{aligned} I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= \frac{\mathscr{E}_q^n}{\Gamma(\zeta)} \int_0^z (z-y)^{\zeta-1} y^{\varepsilon+\varpi n+q} dy \\ &= \frac{\mathscr{E}_q^n z^{\zeta-1}}{\Gamma(\zeta)} \int_0^z (1 - \frac{y}{z})^{\zeta-1} y^{\varepsilon+\varpi n+q} dy. \end{aligned} \tag{5.16}$$

Putting values $\frac{y}{z} = t \implies dy = zdt$, $y = z \implies t = 1$, and $y = 0 \implies t = 0$ in equation (5.16) we get

$$\begin{aligned} I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= \frac{\mathscr{E}_q^n z^{\zeta-1}}{\Gamma(\zeta)} \int_0^1 (1-t)^{\zeta-1} (zt)^{\varepsilon+\varpi n+q} z dt \\ &= \frac{\mathscr{E}_q^n z^{\zeta+\varepsilon+\varpi n+q}}{\Gamma(\zeta)} \int_0^1 (1-t)^{\zeta-1} t^{\varepsilon+\varpi n+q} dt. \end{aligned} \tag{5.17}$$

Using (1.1) and (1.2) in equation (5.17)

$$I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{\mathscr{E}_q^n z^{\zeta+\varepsilon+\varpi n+q}}{\Gamma(\zeta)} \frac{\Gamma(\varepsilon + \varpi n + q + 1)\Gamma(\zeta)}{\Gamma(\varepsilon + \varpi n + q + \zeta + 1)}. \tag{5.18}$$

Using (5.3) and (1.12) in equation (5.18) we obtain

$$I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{z^{\zeta+\varepsilon}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} \sum_{q=0}^{\infty} \frac{\Gamma(\lambda + hq)(-rz)^q}{\Gamma(\delta + q) \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})} \frac{\Gamma(\varepsilon + \varpi n + q + 1)}{\Gamma(\varepsilon + \varpi n + q + \zeta + 1)},$$

we get the required result

$$I_{0,z}^{\zeta}[y^{\varepsilon} E_{\frac{1}{\rho_j},\mu_j} y^{\varpi} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = z^{\zeta+\varepsilon} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\varepsilon + \varpi n + 1, 1)(1, 1) \\ (\delta, 1)(\varepsilon + \varpi n + \zeta + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

□

Similarly, we obtain the following result by applying the same pattern, wich is discussed in Theorem (5.7).

Theorem 5.8. *Let* $\zeta, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$ $(i = 1, 2, 3, \dots, w)$, $\Re(\zeta) > 0$, $\Re(\varepsilon) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$I_{z,\infty}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{-\varepsilon} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = z^{\zeta} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{-\varepsilon n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(-\zeta + \varepsilon n, -1)(1, 1) \\ (\delta, 1)(\varepsilon n, -1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

Theorem 5.9. *Let* $\zeta, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$ $(i = 1, 2, 3, \dots, w)$, $\Re(\zeta) > 0$, $p = \Re(\zeta) + 1$, $\Re(\varpi) > 0$, $\Re(\kappa) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$*;* $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= z^{-\zeta} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n+\kappa n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\varpi n + \kappa n + 1, 1)(1,1) \\ (\delta, 1)(\varpi n + \kappa n - \zeta + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

Proof. Consider the left-sided Riemann liouville differential operator (1.10) with multi-index function defined in (2.6) and (1.3), we have

$$\begin{aligned} D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= (\frac{d}{dz})^p \frac{\mathscr{E}_q^n}{\Gamma(p-\zeta)} \int_0^z (z-y)^{p-\zeta-1} y^{\varpi n+\kappa n+q} dy \\ &= (\frac{d}{dz})^p \frac{\mathscr{E}_q^n z^{p-\zeta-1}}{\Gamma(p-\zeta)} \int_0^z (1-\frac{y}{z})^{p-\zeta-1} y^{\varpi n+\kappa n+q} dy. \end{aligned} \tag{5.19}$$

Putting $\frac{y}{z} = t \implies dy = zdt$, $y = z \implies t = 1$, and $y = 0 \implies t = 0$ in equation (5.19) we get

$$\begin{aligned} D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= (\frac{d}{dz})^p \frac{\mathscr{E}_q^n z^{p-\zeta-1}}{\Gamma(p-\zeta)} \int_0^1 (1-t)^{p-\zeta-1} (zt)^{\varpi n+\kappa n+q} z dt \\ &= (\frac{d}{dz})^p \frac{\mathscr{E}_q^n z^{p-\zeta+\varpi n+\kappa n+q}}{\Gamma(p-\zeta)} \int_0^1 (1-t)^{p-\zeta-1} t^{\varpi n+\kappa n+q} dt. \end{aligned} \tag{5.20}$$

Using (1.1) and (1.2) in equation (5.20),

$$D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) = \frac{\mathscr{E}_q^n}{\Gamma(p-\zeta)} \frac{\Gamma(\varpi n+\kappa n+q+1)\Gamma(p-\zeta)}{\Gamma(\varpi n+\kappa n+q+p-\zeta+1)} (\frac{d}{dz})^p z^{p-\zeta+\varpi n+\kappa n+q}. \tag{5.21}$$

Now taking the pth-derivative of equation (5.21), then

$$\begin{aligned} D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z) &= \frac{\mathscr{E}_q^n z^{-\zeta+\varpi n+\kappa n+q} \Gamma(\varpi n+\kappa n+q+1)}{\Gamma(\varpi n+\kappa n+q+p-\zeta+1)} \\ &\quad \times \frac{\Gamma(\varpi n+\kappa n+q+p-\zeta+1)}{\Gamma(\varpi n+\kappa n+q-\zeta+1)}. \end{aligned} \tag{5.22}$$

Using (5.3) and (1.12) in equation (5.22) we obtain

$$D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= \frac{z^{-\zeta}\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n+\kappa n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} \sum_{q=0}^{\infty} \frac{\Gamma(\lambda + hq)(-rz)^q}{\Gamma(\delta+q) \prod_{i=1}^{w} \Gamma(\beta_i q + \eta_i + \frac{1+p}{2})} \frac{\Gamma(\varpi n+\kappa n+q+1)}{\Gamma(\varpi n+\kappa n+q-\zeta+1)},$$

then we have desired result

$$D_{0,z}^{\zeta}[E_{\frac{1}{\rho_j},\mu_j} y^{\varpi+\kappa} \mathbb{J}_{(\eta_i)_w,h,p,\delta}^{(\beta_i)_w,\lambda,c} y](z)$$
$$= z^{-\zeta} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{\varpi n+\kappa n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{matrix} (\lambda, h)(\varpi n + \kappa n + 1, 1)(1,1) \\ (\delta, 1)(\varpi n + \kappa n - \zeta + 1, 1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{matrix} \middle| -zc \right].$$

□

Similarly, we obtain the following result by applying same pattern, which is discussed in Theorem (5.9).

Theorem 5.10. *Let* $\zeta, \beta_i, \eta_i, \lambda, \delta, h, c \in \mathbb{C}$, $(i = 1, 2, 3, \ldots, w)$, $\Re(\zeta) > 0$, $p = [\Re(\zeta)] + 1$, $\Re(\sigma) > 0$, $\Re(\tau) > 0$ *be such that* $\sum_{i=1}^{w} \Re(\beta_i) > \max\{0; \Re(h) - 1\}$; $h > 0$, $\Re(\eta_i) > 0$, $\Re(\lambda) > 0$, *and* $\Re(\delta) > 0$, $y > 0$, *then the following relation holds*

$$\begin{aligned} &D^{\zeta}_{z,\infty}[y^{-\sigma} E_{\frac{1}{\rho_j},\mu_j} y^{-\tau} \mathbb{J}^{(\beta_i)_w,\lambda,c}_{(\eta_i)_w,h,p,\delta} y](z) \\ &= z^{-\zeta-\sigma} \frac{\Gamma(\delta)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{z^{-\tau n}}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{n}{\rho_j})} {}_3\psi_{2+w} \left[\begin{array}{c} (\lambda, h)(\zeta + \sigma + \tau n, -1)(1, 1) \\ (\delta, 1)(\sigma + \tau n, -1)(\eta_i + \frac{1+p}{2}, \beta_i)|_{i=1}^{w} \end{array} \middle| -zc \right]. \end{aligned}$$

6. Conclusion

In this research paper work, we introduced the extension and generalization of Mittag Leffler function and Bessel-Maitland function by utilizing the generalized Pochhammer symbol and discussed its relations with other existing versions of such type of functions. Moreover, we analyzed the fractional integrals and differential behavior with the product of multi-index functions, and obtained results shows the strengthened class of Wright functions. We concluded that the fractional properties with such type of functions to deal the Wright type of functions, and also these multi-dimension functions improved the generalization of fractional operators by means of its kernel, which opens the new horizon of fractional filed in term of multi-index operators.

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