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New results on the oscillation of second-order damped neutral differential equations with several sub-linear neutral terms

A. A. El-Gaber[a,*], E. I. El-Saedy[a], M. M. A. El-Sheikh[a], S. A. A. El-Marouf[a,b]

[a]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El-Koom, Egypt.

[b]Department of Mathematics, Faculty of Science, Taibah University, Saudi Arabia.

Abstract

In this paper, we establish some new sufficient conditions which guarantee the oscillatory behavior of solutions of a class of second-order damped neutral differential equations with sub-linear neutral terms. Our criteria improve and complement related results in the literature. Two examples are given to justify our main results.

Keywords: Oscillation, second order damped differential equations, neutral differential equations, sub-linear neutral terms.

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1. Introduction

This article is devoted to studying the oscillatory behavior of solutions of a class of second-order damped neutral differential equations of the type

$$\left(a\left(t\right)\left(\varpi'\left(t\right)\right)^{\gamma}\right)'+h\left(t\right)\left(\varpi'\left(t\right)\right)^{\gamma}+f\left(t,y\left(\varphi\left(t\right)\right)\right)=0,\ t\geqslant t_0>0, \tag{1.1}$$

where $\varpi\left(t\right)=y\left(t\right)+\sum_{i=1}^{m}c_i\left(t\right)y^{\alpha_i}\left(\nu_i\left(t\right)\right), m>0$ is an integer. Throughout the paper, we use the following assumptions:

(A$_1$) $0<\alpha_i\leqslant 1$ for $i=1,2,\dots,m$, and γ are the ratios of odd positive integers;

(A$_2$) $a,h,c_i:[t_0,\infty)\to\mathbb{R}^+$ are continuous functions and $\lim_{t\to\infty}c_i\left(t\right)=0$ for $i=1,2,\dots,m$;

(A$_3$) $\nu_i,\varphi:[t_0,\infty)\to\mathbb{R}$ are continuous functions with $\nu_i\left(t\right)<t, \varphi\left(t\right)\leqslant t, \varphi'\left(t\right)>0$ and $\nu_i\left(t\right),\varphi\left(t\right)\to\infty$ as $t\to\infty$ for $i=1,2,\dots,m$;

(A$_4$) $f\left(t,y\right)\in C\left([t_0,\infty)\times\mathbb{R},\mathbb{R}\right)$, and there exists a function $g\left(t\right)\in C\left([t_0,\infty),(0,\infty)\right)$ such that $f\left(t,y\right)/y^{\beta}\geqslant g\left(t\right)$ where β is a ratio of odd positive integers.

*Corresponding author

Email address: amina.aboalnour@science.menofia.edu.eg (A. A. El-Gaber)

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We will be concerned in this work with nontrivial solutions satisfying $\sup\{y(t) : t \geqslant T \geqslant t_y\} > 0$. We mean by an oscillatory solution that nontrivial one which has an infinite number of zeros in the half-line $[t_0, \infty)$. Meanwhile we say that equation (1.1) is oscillatory if all its solutions are oscillatory.

In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [23, 24, 33]. Recently, there has been considerable interest in the study of the oscillation of second-order damped equations because of their numerous applications in the fields of science, engineering, and technology, etc (see [1, 6, 8, 9, 14–16, 31, 35–37]), and it has been studied extensively, see for instance [25, 28, 30] and the references cited therein. To the best of our knowledge, we note that most of the results obtained in the literature have been centered around the special un-damped case of Eq. (1.1), i.e., when $h(t) = 0$ (see [2, 3, 5, 7, 11, 13, 18, 21, 22, 26, 27, 29, 32, 34, 38, 40–42]). Moreover, there are relatively few results dealing with the oscillation of second order differential equations with sub-linear neutral terms (see [2, 4, 10–12]). Here, we mention some recent works which were concerned with some special cases of (1.1), and motivated this work.

Grammatikopoulos et al. [18] deduced that all solutions of the equation

$$(y(t) + b(t) y(t-\tau))'' + g(t) y(t-\nu) = 0$$

are oscillatory if

$$\int_{t_0}^{\infty} g(s)(1 - b(s-\nu))\, ds = \infty.$$

In [17], Grace and Lalli were able to improve and extend the results of [18] to the more general equation

$$\left(a(t)(y(t) + b(t) y(t-\nu))'\right)' + g(t) f(y(t-\nu)) = 0, \tag{1.2}$$

with

$$\frac{f(y)}{y} \geqslant k > 0 \quad \text{and} \quad \int_{t_0}^{\infty} \frac{ds}{a(s)} = \infty.$$

They proved that Eq. (1.2) is oscillatory if for some continuously differentiable function $U(t)$, one has

$$\int_{t_0}^{\infty} \left(U(s) g(s)(1 - b(s-\nu)) - \frac{(U'(s))^2 a(s-\nu)}{4kU(s)} \right) ds = \infty.$$

Agarwal et al. [3] and Baculíková et al. [5] discussed the second order nonlinear neutral differential equation

$$\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)' + g(t) y^{\beta}(\varphi(t)) = 0, \tag{1.3}$$

where $\varpi(t) = y(t) + b(t) y(\nu(t))$ with $0 \leqslant b(t) \leqslant b_0 < \infty$ and γ, β are the ratios of two positive odd integers. Recently, Baculíková [4] and Džurina et al. [12] discussed the second order nonlinear differential equation (1.3) with $\gamma = 1$, and several sub-linear neutral terms, i.e., $\varpi(t) = y(t) + \sum_{i=1}^{m} c_i(t) y^{\alpha_i}(\nu_i(t)), m > 0$ is an integer, $0 < \alpha_i \leqslant 1$ for $i = 1, 2, \ldots, m$ and β are the ratios of odd positive integers, where the conditions (A_2) and (A_3) hold. Liu et al. [34] and Wu et al. [42] considered the generalized Emden-Fowler equation with neutral type delay of the form

$$\left(a(t)\left|\varpi'(t)\right|^{\gamma-1} \varpi'(t)\right)' + g(t)\left|y(\varphi(t))\right|^{\beta-1} y(\varphi(t)) = 0,$$

where $\varpi(t) = y(t) + b(t) y(\nu(t)), a'(t) \geqslant 0, \varphi'(t) > 0$ and $0 \leqslant b(t) < 1, g(t) \geqslant 0$ in the two cases

$$\int_{t_0}^{\infty} \frac{dt}{a^{\frac{1}{\gamma}}(t)} = \infty \tag{1.4}$$

and
$$\int_{t_0}^{\infty} \frac{dt}{a^{\frac{1}{\gamma}}(t)} < \infty. \tag{1.5}$$
The authors in [42] were able to discuss all the possible cases $\gamma > \beta, \gamma = \beta$, and $\gamma < \beta$, while those in [34] were concerned only with the case $\gamma \geqslant \beta > 0$. Meanwhile, Sallam et al. [38] and Wang et al. [40] studied the nonlinear second order neutral delay differential equation
$$\left(a(t)\left|\varpi'(t)\right|^{\gamma-1}\varpi'(t)\right)' + f(t, y(\varphi(t))) = 0. \tag{1.6}$$
In [38], the authors studied Eq. (1.6) when $\varpi(t) = y(t) \pm b(t)y(\nu(t)), a(t) > 0, 0 \leqslant b(t) \leqslant 1, \gamma$ is a positive constant and the condition (A_4) is satisfied in all the three possible cases $\gamma > \beta, \gamma = \beta, \gamma < \beta$ and in the two cases (1.4) and (1.5), while the authors in [40] studied Eq. (1.6) when $\varpi(t) = y(t) + b(t)y(\nu(t))$ with γ is a positive constant and only with the condition (1.4) in the two cases $0 \leqslant b(t) < 1$ and $-1 < b(t) < 0$, but they considered the condition (A_4) with β as a positive constant satisfying $1 < \beta \leqslant \gamma$. On the other hand Eq. (1.1) can be considered as a natural generalization of the second order differential equation
$$(a(t)y'(t))' + h(t)y'(t) + g(t)f(y(t)) = 0,$$
which was studied by Agarwal et al. [1] and Rogovchenko et al. [35–37], under the conditions $a \in C^1([t_0, \infty), \mathbb{R}), h, g \in C(\mathbb{R}, \mathbb{R}), yf(y) > 0$, and $f'(y) \geqslant k > 0$. Also Eq. (1.1) can be considered as a natural generalization of the second order differential equation studied by Fu et al. [15] of the form
$$(a(t)y'(t))' + h(t)y'(t) + g(t)f(y(\nu(t))) = 0.$$
Meanwhile, Jadlovská [16] studied Eq. (1.1) with $f(t, y(\varphi(t))) = g(t)f(y(\varphi(t)))$, where $\varpi(t) = y(t) + b(t)y(\nu(t)), \gamma \geqslant 1$, is a quotient of positive odd integers, $0 \leqslant b(t) \leqslant 1, a(t), h(t) : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions. They assumed that $f \in C(\mathbb{R}, \mathbb{R})$, with $yf(y) > 0$ and $\frac{f(y)}{y^\beta} \geqslant k > 0$ with $y \neq 0$, k is a constant and β is a ratio of odd positive integers. The aim of this paper is to complement and extend some of the results given in [12, 16, 34, 38, 40, 42], by using some elementary inequalities and Riccati substitution. In this paper, we cover all possible cases $\gamma > \beta, \gamma = \beta$, and $\gamma < \beta$. So we think that our results are of high generality.

2. Preliminaries

We consider the notation
$$E(t) = \exp\left(-\int_{t_0}^{t} \frac{h(s)}{a(s)} ds\right), \Pi(t) = \int_{t_1}^{t} \left(\frac{E(s)}{a(s)}\right)^{\frac{1}{\gamma}} ds, t_1 \geqslant t_0 > 0. \tag{2.1}$$
We suppose that there exists a positive, continuous function $\rho : [t_0, \infty) \to \mathbb{R}^+$ decreasing to zero, and
$$\Psi(t) = 1 - \sum_{i=1}^{m} \alpha_i c_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1 - \alpha_i) c_i(t), \tag{2.2}$$
such that $\Psi(\varphi(t)) > 0$,
$$\Im(t) = \frac{\beta\varphi'(t)(\xi(\varphi(t)))^{\beta-1}}{M^{1-\frac{\beta}{\gamma}} a^{\frac{1}{\gamma}}(\varphi(t))\chi(t)}, \quad \xi(t) = \int_{t_1}^{t} a^{-\frac{1}{\gamma}}(s) ds, \tag{2.3}$$
and
$$\Omega(t) = \frac{\chi'(t)}{\chi(t)} - \frac{h(t)}{a(t)}, \tag{2.4}$$
where the parameter $\chi(t) \in C^1([t_0, \infty), \mathbb{R})$ will be determined later.

Lemma 2.1 ([20]). *If* r *is nonnegative, then*

$$r^{\alpha} \leqslant \alpha r + (1-\alpha) \text{ for } 0 < \alpha \leqslant 1. \tag{2.5}$$

Lemma 2.2. *Assume that*

$$\int_{t_0}^{\infty} \left(\frac{E(s)}{a(s)}\right)^{\frac{1}{\gamma}} ds = \infty \tag{2.6}$$

holds, where $E(t)$ *is defined by* (2.1). *If there exists a positive solution* $y(t)$ *of Eq.* (1.1), *then there exists* $T \in [t_0, \infty)$, *large enough, such that*

(i) $\varpi(t) > 0, \varpi'(t) > 0$, *and* $\left(a(t)(\varpi'(t))^{\gamma}\right)' < 0$;

(ii) $\frac{\varpi(t)}{\Pi(t)}$ *is decreasing.*

Proof. Since $y(t)$ is a positive solution of Eq. (1.1) on $[t_0, \infty)$, then by the assumption (A_3) there exists $t_1 \geqslant t_0$ such that $y(\nu_i(t)) > 0$ and $y(\varphi(t)) > 0$ on $[t_1, \infty)$. Then $\varpi(t) \geqslant y(t) > 0$, for $t \geqslant t_1$. Thus in view of (1.1), we have

$$\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)' + h(t)\left(\varpi'(t)\right)^{\gamma} = -f(t, y(\varphi(t))) \leqslant -g(t)\, y^{\beta}(\varphi(t)) < 0.$$

Therefore,

$$\left(\frac{a(t)}{E(t)}\left(\varpi'(t)\right)^{\gamma}\right)' = -\frac{f(t, y(\varphi(t)))}{E(t)} < 0.$$

Thus $\frac{a(t)}{E(t)}(\varpi'(t))^{\gamma}$ is decreasing. Now, to show that $\varpi'(t) > 0$ on $[t_1, \infty)$, suppose the contrary that there exists $t_2 \in [t_1, \infty)$ such that $\varpi'(t_2) < 0$. But since $\frac{a(t)}{E(t)}(\varpi'(t))^{\gamma}$ is decreasing, it follows for $t \geqslant t_2$, that

$$\frac{a(t)}{E(t)}\left(\varpi'(t)\right)^{\gamma} < \frac{a(t_2)}{E(t_2)}\left(\varpi'(t_2)\right)^{\gamma} = l < 0.$$

Thus it follows by integration from t_2 to t, that

$$\varpi(t) < \varpi(t_2) + l^{\frac{1}{\gamma}} \int_{t_2}^{t} \left(\frac{E(s)}{a(s)}\right)^{\frac{1}{\gamma}} ds,$$

for $t \geqslant t_2$. This with (2.6), leads to $\lim_{t\to\infty} \varpi(t) = -\infty$, which contradicts the fact that $\varpi(t)$ is eventually positive. Therefore $\varpi'(t) > 0$. Moreover since from Eq. (1.1), we deduce that $\left(a(t)(\varpi'(t))^{\gamma}\right)' < 0$ and $\frac{a(t)}{E(t)}(\varpi'(t))^{\gamma}$ is decreasing, then we have

$$\varpi(t) > \int_{t_2}^{t} \left(\frac{a(s)}{E(s)}\left(\varpi'(s)\right)^{\gamma} \frac{E(s)}{a(s)}\right)^{\frac{1}{\gamma}} ds > \left(\frac{a(t)}{E(t)}\left(\varpi'(t)\right)^{\gamma}\right)^{\frac{1}{\gamma}} \Pi(t),$$

which yields

$$\left(\frac{\varpi(t)}{\Pi(t)}\right)' < 0.$$

Thus $\frac{\varpi(t)}{\Pi(t)}$ is decreasing for $t \geqslant t_2$. □

3. Main results

Theorem 3.1. *Let* (A_1)-(A_4), *and* (2.6) *hold. Furthermore suppose that* $1 < \beta \leqslant \gamma$. *If one has*

$$\int_{t_0}^{\infty} \left[\chi(t)\, g(t)\, \Psi^{\beta}(\varphi(t)) - \frac{(\Omega(t))^2}{4\Im(t)} \right] dt = \infty, \tag{3.1}$$

for any function $\chi(t) \in C^1([t_0,\infty),(0,\infty))$, *where* $\Psi(t)$, $\Im(t)$, *and* $\Omega(t)$ *are as defined in* (2.2), (2.3), *and* (2.4), *respectively, then every solution of Eq.* (1.1) *oscillates.*

Proof. Suppose the contrary that there exists a $t_1 \geqslant t_0$ such that $y(t) > 0$, $y(\nu_i(t)) > 0$, and $y(\varphi(t)) > 0$ for $t \geqslant t_1$ and $i = 1, 2, \ldots, m$. Now since $\varpi(t)$ is increasing, then from the definition of $\varpi(t)$, and (2.5), we have

$$\begin{aligned} y(t) &= \varpi(t) - \sum_{i=1}^{m} c_i(t)\, y^{\alpha_i}(\nu_i(t)) \geqslant \varpi(t) - \sum_{i=1}^{m} c_i(t)\, \varpi^{\alpha_i}(\nu_i(t)) \\ &\geqslant \varpi(t) - \sum_{i=1}^{m} c_i(t) \left(\alpha_i \varpi(\nu_i(t)) + (1-\alpha_i)\right) \\ &\geqslant \left(1 - \sum_{i=1}^{m} \alpha_i c_i(t)\right) \varpi(t) - \sum_{i=1}^{m} (1-\alpha_i)\, c_i(t) \\ &= \varpi(t) \left(1 - \sum_{i=1}^{m} \alpha_i c_i(t) - \frac{1}{\varpi(t)} \sum_{i=1}^{m} (1-\alpha_i)\, c_i(t)\right). \end{aligned} \tag{3.2}$$

But since $\varpi(t)$ is positive and increasing, while $\rho(t)$ is positive and decreasing to zero, there is a $t_2 \geqslant t_1$ such that

$$\varpi(t) \geqslant \rho(t) \ \text{ for } t \geqslant t_2. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$y(t) \geqslant \varpi(t) \left(1 - \sum_{i=1}^{m} \alpha_i c_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i)\, c_i(t)\right) = \Psi(t)\, \varpi(t). \tag{3.4}$$

This with (1.1) yields

$$\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)' + h(t)\left(\varpi'(t)\right)^{\gamma} + g(t)\, \Psi^{\beta}(\varphi(t))\, \varpi^{\beta}(\varphi(t)) \leqslant 0, \quad t \geqslant t_2, \tag{3.5}$$

or

$$\left(\frac{a(t)\left(\varpi'(t)\right)^{\gamma}}{E(t)}\right)' + \frac{g(t)\, \Psi^{\beta}(\varphi(t))\, \varpi^{\beta}(\varphi(t))}{E(t)} \leqslant 0. \tag{3.6}$$

Define

$$\Theta(t) = \chi(t) \frac{a(t)\left(\varpi'(t)\right)^{\gamma}}{\varpi^{\beta}(\varphi(t))}, \quad t \geqslant t_2. \tag{3.7}$$

Then $\Theta(t) > 0$ for $t \geqslant t_2$, and

$$\begin{aligned} \Theta'(t) = {} & \chi'(t) \frac{a(t)\left(\varpi'(t)\right)^{\gamma}}{\varpi^{\beta}(\varphi(t))} + \chi(t) \frac{\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)'}{\varpi^{\beta}(\varphi(t))} \\ & - \frac{\beta \chi(t)\, a(t)\left(\varpi'(t)\right)^{\gamma}}{\varpi^{2\beta}(\varphi(t))} \varpi^{\beta-1}(\varphi(t))\, \varpi'(\varphi(t))\, \varphi'(t). \end{aligned} \tag{3.8}$$

Since $a(t)(\varpi'(t))^{\gamma}$ is positive and decreasing, then there may exist a positive constant M such that for some $t_2 \geqslant t_1$, we have

$$a(t)\left(\varpi'(t)\right)^{\gamma} \leqslant M, \quad t \geqslant t_2. \tag{3.9}$$

Moreover, since $\varphi(t) \leqslant t$, then

$$a(t)\left(\varpi'(t)\right)^{\gamma} \leqslant a(\varphi(t))\left(\varpi'(\varphi(t))\right)^{\gamma}, \tag{3.10}$$

and

$$\varpi(t) = \varpi(t_1) + \int_{t_1}^{t} \frac{\left(a(s)(\varpi'(s))^{\gamma}\right)^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} ds > a^{\frac{1}{\gamma}}(t)\varpi'(t)\int_{t_1}^{t} a^{-\frac{1}{\gamma}}(s)\,ds = \xi(t)\,a^{\frac{1}{\gamma}}(t)\,\varpi'(t). \tag{3.11}$$

By substituting (3.5) and (3.11) into (3.8), we get

$$\begin{aligned}
\Theta'(t) &\leqslant \frac{\chi'(t)}{\chi(t)}\Theta(t) - \chi(t)\frac{h(t)(\varpi'(t))^{\gamma}}{\varpi^{\beta}(\varphi(t))} - \chi(t)g(t)\Psi^{\beta}(\varphi(t)) \\
&\quad - \beta\varphi'(t)\chi(t)\frac{a(t)(\varpi'(t))^{\gamma}}{\varpi^{2\beta}(\varphi(t))}\varpi'(\varphi(t))(\xi(\varphi(t)))^{\beta-1}a^{\frac{\beta-1}{\gamma}}(\varphi(t))\left(\varpi'(\varphi(t))\right)^{\beta-1} \\
&\leqslant \left[\frac{\chi'(t)}{\chi(t)} - \frac{h(t)}{a(t)}\right]\Theta(t) - \chi(t)g(t)\Psi^{\beta}(\varphi(t)) \\
&\quad - \beta\varphi'(t)\chi(t)(\xi(\varphi(t)))^{\beta-1}\frac{a(t)(\varpi'(t))^{\gamma}}{\varpi^{2\beta}(\varphi(t))}a^{\frac{\beta-1}{\gamma}}(\varphi(t))\left(\varpi'(\varphi(t))\right)^{\beta}.
\end{aligned} \tag{3.12}$$

It follows from (3.9), (3.10), and (3.12) that

$$\begin{aligned}
\Theta'(t) &\leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \Omega(t)\Theta(t) - \frac{\beta\varphi'(t)(\xi(\varphi(t)))^{\beta-1}}{(a(\varphi(t)))^{\frac{1}{\gamma}}}\chi(t)\frac{a(t)(\varpi'(t))^{\gamma}}{\varpi^{2\beta}(\varphi(t))}a^{\frac{\beta}{\gamma}}(t)\left(\varpi'(t)\right)^{\beta} \\
&= -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \Omega(t)\Theta(t) - \frac{\beta\varphi'(t)(\xi(\varphi(t)))^{\beta-1}a^{\frac{\beta}{\gamma}}(t)}{(a(\varphi(t)))^{\frac{1}{\gamma}}\chi(t)a(t)}\Theta^2(t)\frac{1}{(\varpi'(t))^{\gamma-\beta}} \\
&\leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \Omega(t)\Theta(t) - \frac{\beta\varphi'(t)(\xi(\varphi(t)))^{\beta-1}}{M^{1-\frac{\beta}{\gamma}}(a(\varphi(t)))^{\frac{1}{\gamma}}\chi(t)}\Theta^2(t) \\
&= -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \Omega(t)\Theta(t) - \Im(t)\Theta^2(t),
\end{aligned}$$

where

$$\Im(t) = \frac{\beta\varphi'(t)(\xi(\varphi(t)))^{\beta-1}}{M^{1-\frac{\beta}{\gamma}}(a(\varphi(t)))^{\frac{1}{\gamma}}\chi(t)}.$$

By completing the squares, we obtain

$$\Theta'(t) \leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \frac{(\Omega(t))^2}{4\Im(t)} - \left[\sqrt{\Im(t)}\Theta(t) - \frac{\Omega(t)}{2\sqrt{\Im(t)}}\right]^2 \leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t)) + \frac{(\Omega(t))^2}{4\Im(t)}.$$

Integrating from t_2 to t, we get

$$0 < \Theta(t) \leqslant \Theta(t_2) - \int_{t_2}^{t}\left[\chi(s)g(s)\Psi^{\beta}(\varphi(s)) - \frac{(\Omega(s))^2}{4\Im(s)}\right] ds.$$

This is a contradiction with (3.1), and so the proof is completed. □

Remark 3.2. Although Theorem 3.1 depends on the technique of Theorem 4 of [40], however the authors in [40] dealt with the undamped case.

Theorem 3.3. *Assume that* (A_1)-(A_4) *and* (2.6) *hold. If for any function* $\chi(t) \in C^1([t_0,\infty),(0,\infty))$, *and a positive number* M, *we have*

$$\int_{t_0}^{\infty}\left[\chi(t)\, g(t)\, \Psi^{\beta}(\varphi(t)) - \frac{M^{\frac{\gamma-\beta}{\gamma}}\chi(t)\,(\Omega(t))^2\, a^{\frac{\beta}{\gamma}}(\varphi(t))}{4\beta\left(\frac{\varphi(t)}{2}\right)^{\beta-1}\varphi'(t)}\right]dt = \infty, \tag{3.13}$$

where $\Psi(t)$ *is defined by* (2.2), *and* $\Omega(t)$ *is defined by* (2.4), $\gamma \geqslant \beta > 1$, *and* $a'(t) > 0$, *then Eq.* (1.1) *is oscillatory.*

Proof. Suppose the contrary that there exists a $T_1 \geqslant t_0$ such that $y(t) > 0$, $y(\nu_i(t)) > 0$, and $y(\varphi(t)) > 0$ for $t \geqslant T_1$ and $i = 1, 2, \dots, m$. Now as in the proof of Theorem 3.1, we have the inequality (3.5), by which with the positivity of $a'(t)$, we get by Lemma 2.2 (i) that

$$\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)' < 0, t \geqslant T_1.$$

This implies that there exists some $T_2 > T_1$ such that $\varpi''(t) < 0$, for $t \geqslant T_1$, i.e., $\varpi'(t)$ is eventually decreasing. Therefore from the mean value theorem, we have

$$\varpi(t) - \varpi(T_1) = (t - T_1)\,\varpi'(\zeta) > (t - T_1)\,\varpi'(t), \quad \zeta \in (T_1, t),$$

i.e.,

$$\varpi(t) > \frac{t}{2}\varpi'(t), \ \text{ for } t \geqslant T_2 > 2T_1. \tag{3.14}$$

Now define $\Theta(t)$ as in Theorem 3.1, then $\Theta(t) > 0$. Moreover from (3.5), (3.7), (3.10), and (3.14), we have

$$\begin{aligned}
\Theta'(t) &= \frac{\chi'(t)}{\chi(t)}\Theta(t) + \chi(t)\frac{\left(a(t)(\varpi'(t))^{\gamma}\right)'}{\varpi^{\beta}(\varphi(t))} - \frac{\beta\chi(t)\, a(t)\,(\varpi'(t))^{\gamma}}{\varpi^{2\beta}(\varphi(t))}\varpi^{\beta-1}(\varphi(t))\,\varpi'(\varphi(t))\,\varphi'(t)\\
&\leqslant \left[\frac{\chi'(t)}{\chi(t)} - \frac{h(t)}{a(t)}\right]\Theta(t) - \chi(t)\, g(t)\,\Psi^{\beta}(\varphi(t))\\
&\quad -\chi(t)\frac{\beta a(t)\,(\varpi'(t))^{\gamma+\beta}\left(\frac{\varphi(t)}{2}\right)^{\beta-1}\varphi'(t)}{\varpi^{2\beta}(\varphi(t))}\left(\frac{a(t)}{a(\varphi(t))}\right)^{\frac{\beta}{\gamma}}\\
&\leqslant \Omega(t)\,\Theta(t) - \chi(t)\, g(t)\,\Psi^{\beta}(\varphi(t)) - \frac{\beta\left(\frac{\varphi(t)}{2}\right)^{\beta-1}\varphi'(t)}{\chi(t)\, a^{\frac{\gamma-\beta}{\gamma}}(t)\, a^{\frac{\beta}{\gamma}}(\varphi(t))}\Theta^2(t)\frac{1}{(\varpi'(t))^{\gamma-\beta}}.
\end{aligned} \tag{3.15}$$

Now, from the fact that $a(t)(\varpi'(t))^{\gamma}$ is positive and decreasing, there exists a $T_4 > T_3$ sufficiently large such that $a(t)(\varpi'(t))^{\gamma} \leqslant M, t \geqslant T_4$, where M is defined in (3.9), and therefore

$$(\varpi'(t))^{\gamma-\beta} \leqslant \left(\frac{M}{a(t)}\right)^{\frac{\gamma-\beta}{\gamma}}, \quad t \geqslant T_4. \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$\Theta'(t) \leqslant \Omega(t)\,\Theta(t) - \chi(t)\, g(t)\,\Psi^{\beta}(\varphi(t)) - \frac{\beta\left(\frac{\varphi(t)}{2}\right)^{\beta-1}\varphi'(t)}{\chi(t)\, a^{\frac{\beta}{\gamma}}(\varphi(t))\, M^{\frac{\gamma-\beta}{\gamma}}}\Theta^2(t).$$

By completing the squares, we get

$$\Theta'(t) \leqslant -\chi(t)\, g(t)\, \Psi^{\beta}(\varphi(t)) + \frac{M^{\frac{\gamma-\beta}{\gamma}} \chi(t)\, a^{\frac{\beta}{\gamma}}(\varphi(t))\, (\Omega(t))^2}{4\beta \left(\frac{\varphi(t)}{2}\right)^{\beta-1} \varphi'(t)}.$$

Integrating from T_4 to t, we have

$$\Theta(t) \leqslant \Theta(T_4) - \int_{T_4}^{t} \left[\chi(s)\, g(s)\, \Psi^{\beta}(\varphi(s)) - \frac{M^{\frac{\gamma-\beta}{\gamma}} \chi(s)\, a^{\frac{\beta}{\gamma}}(\varphi(s))\, (\Omega(s))^2}{4\beta \left(\frac{\varphi(s)}{2}\right)^{\beta-1} \varphi'(s)}\right] ds. \tag{3.17}$$

Let $t \to \infty$ in (3.17), and using (3.13), then $\Theta(t)$ will be eventually negative, which is a contradiction, and so the proof is completed. □

Remark 3.4. In the special case $f(t, y(\varphi(t))) = g(t)\, |y(\varphi(t))|^{\beta-1} y(\varphi(t))$ and $h(t) = 0$, Theorem 3.3 improves and extends Theorem 2.1 of [34].

Theorem 3.5. *Suppose that* (A_1)-(A_4) *and* (2.6) *hold. Furthermore suppose that* $a'(t) > 0$. *If there exists a function* $\chi(t) \in C^1([t_0, \infty), (0, \infty))$ *such that*

$$\int_{t_0}^{\infty} \left[\chi(t)\, g(t)\, \Psi^{\beta}(\varphi(t)) - \frac{(\chi'(t)\, a(t) - \chi(t)\, h(t))^{\lambda+1} a(\lambda(t))}{(\lambda+1)^{\lambda+1} (q\chi(t)\, \varphi'(t))^{\lambda} (a(t))^{\lambda+1}}\right] dt = \infty, \tag{3.18}$$

where $\Psi(t)$ *is defined by* (2.2),

$$\lambda = \min\{\gamma, \beta\}, \quad \lambda(t) = \begin{cases} \varphi(t), & \beta \geqslant \gamma, \\ t, & \gamma > \beta, \end{cases} \quad \text{and} \quad q = \begin{cases} 1, & \gamma = \beta, \\ 0 < q \leqslant 1, & \gamma \neq \beta, \end{cases}$$

then Eq. (1.1) *is oscillatory.*

Proof. Suppose for the contrary that Eq. (1.1) has a non-oscillatory solution $y(t) > 0$, for sufficiently large t. The case of $y(t) < 0$ can be similarly treated. Now in view of (A_3), there may exist $t_1 \geqslant t_0$ such that $y(t) > 0$, $y(\nu_i(t)) > 0$, and $y(\varphi(t)) > 0$ for $t \geqslant t_1$ and $i = 1, 2, \dots, m$. It is not difficult to see that $y(t) > 0$ for $t \geqslant t_1$, but since from Lemma 2.2 (i), and the definition of $\varpi(t)$, we get (3.4), from which and (1.1), we arrive at (3.5). Now since $\varphi(t) \leqslant t$, we have (3.10). Put

$$\Upsilon(t) = \frac{a(t)\, (\varpi'(t))^{\gamma}}{\varpi^{\beta}(\varphi(t))}, \quad t \geqslant T.$$

Then $\Upsilon(t) > 0$, and

$$\begin{aligned} \Upsilon'(t) &= \frac{\left(a(t)\, (\varpi'(t))^{\gamma}\right)'}{\varpi^{\beta}(\varphi(t))} - \frac{\beta\varphi'(t)\, \varpi'(\varphi(t))\, a(t)\, (\varpi'(t))^{\gamma}}{\varpi^{\beta+1}(\varphi(t))} \\ &\leqslant -g(t)\, \Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)} \Upsilon(t) - \frac{\beta\varphi'(t)\, \varpi'(\varphi(t))\, a(t)\, (\varpi'(t))^{\gamma}}{\varpi^{\beta+1}(\varphi(t))}. \end{aligned} \tag{3.19}$$

Now, we consider the possible cases for (3.19).

Case 1: $\gamma = \beta$. From (3.10), it is clear that

$$\Upsilon'(t) \leqslant -g(t)\, \Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)} \Upsilon(t) - \frac{\gamma\varphi'(t)}{(a(\varphi(t)))^{\frac{1}{\gamma}}} \Upsilon^{\frac{\gamma+1}{\gamma}}(t), \quad t \geqslant T. \tag{3.20}$$

Case 2: $\gamma < \beta$. Since $\varpi(t)$ is increasing on $[T, \infty)$, then there may exist a constant $q_1 > 0$ such that

$$\begin{aligned} \Upsilon'(t) &\leqslant -g(t)\Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)}\Upsilon(t) - \frac{\beta\varphi'(t)}{(a(\varphi(t)))^{\frac{1}{\gamma}}}\left[\varpi(\varphi(t))\right]^{\frac{\beta-\gamma}{\gamma}}\Upsilon^{\frac{\gamma+1}{\gamma}}(t) \\ &\leqslant -g(t)\Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)}\Upsilon(t) - \frac{\gamma\varphi'(t)q_1}{(a(\varphi(t)))^{\frac{1}{\gamma}}}\Upsilon^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \tag{3.21}$$

Case 3: $\gamma > \beta$. From the fact that $\left(a(t)(\varpi'(t))^{\gamma}\right)' < 0$, and $a'(t) > 0$, we get $\varpi''(t) < 0$, and so $\varpi'(t)$ is decreasing. Thus, there exists a positive constant q_2, such that

$$\begin{aligned} \Upsilon'(t) &\leqslant -g(t)\Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)}\Upsilon(t) - \frac{\beta\varphi'(t)}{(a(t))^{\frac{1}{\beta}}}\left[\varpi'(t)\right]^{\frac{\beta-\gamma}{\beta}}\Upsilon^{\frac{\beta+1}{\beta}}(t) \\ &\leqslant -g(t)\Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)}\Upsilon(t) - \frac{\beta\varphi'(t)q_2}{(a(t))^{\frac{1}{\beta}}}\Upsilon^{\frac{\beta+1}{\beta}}(t). \end{aligned} \tag{3.22}$$

Now from (3.20), (3.21), and (3.22) it follows that for any $\gamma > 0$, and $\beta > 0$,

$$\Upsilon'(t) \leqslant -g(t)\Psi^{\beta}(\varphi(t)) - \frac{h(t)}{a(t)}\Upsilon(t) - \frac{\lambda q\varphi'(t)}{(a(\lambda(t)))^{\frac{1}{\lambda}}}\Upsilon^{\frac{\lambda+1}{\lambda}}(t), \quad t \geqslant T. \tag{3.23}$$

Multiplying (3.23) by $\chi(t)$ and integrating it from T to t, we obtain

$$\begin{aligned} &\int_T^t \chi(s)g(s)\Psi^{\beta}(\varphi(s))\,ds \\ &\leqslant \chi(T)\Upsilon(T) + \int_T^t \left[\left(\chi'(s) - \frac{\chi(s)h(s)}{a(s)}\right)\Upsilon(s) - \frac{\lambda q\chi(s)\varphi'(s)}{(a(\lambda(s)))^{\frac{1}{\lambda}}}\Upsilon^{\frac{\lambda+1}{\lambda}}(s)\right]ds. \end{aligned} \tag{3.24}$$

Applying the following inequality of [39],

$$D\Upsilon - F\Upsilon^{\frac{\lambda+1}{\lambda}} \leqslant \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}D^{\lambda+1}F^{-\lambda}, \tag{3.25}$$

with $F > 0$ and $\lambda > 0$, then (3.24) will take the form

$$\int_T^t \left[\chi(s)g(s)\Psi^{\beta}(\varphi(s)) - \frac{(\chi'(s)a(s) - \chi(s)h(s))^{\lambda+1}a(\lambda(s))}{(\lambda+1)^{\lambda+1}(q\chi(s)\varphi'(s))^{\lambda}(a(s))^{\lambda+1}}\right]ds \leqslant \chi(T)\Upsilon(T).$$

Letting $t \to \infty$ in the above inequality, we get a contradiction with (3.18). □

Remark 3.6. The above theorem includes Theorem 1 of [42] in the case $f(t, y(\varphi(t))) = g(t)|y(\varphi(t))|^{\beta-1}$ $y(\varphi(t))$, and $h(t) = 0$.

Theorem 3.7. *Let the conditions* (A_1)-(A_4) *and* (2.6) *hold. Furthermore assume that there exists a positive continuously differentiable function* $\chi(t)$ *such that, for all sufficiently large* $t_1 \geqslant t_0$

$$\begin{aligned} &\limsup_{t\to\infty}\{\chi(t)E(t)\int_t^{\infty}\frac{g(s)\Psi^{\beta}(\varphi(s))}{E(s)}ds \\ &+ \int_{t_1}^t \left[g(s)\chi(s)\Psi^{\beta}(\varphi(s)) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{\chi(s)a(\varphi(s))(\Omega(s))^{\gamma+1}}{(\beta\varphi'(s)\psi(s))^{\gamma}}\right]ds\} = \infty, \end{aligned} \tag{3.26}$$

where

$$\psi(t)=\begin{cases} d_1, & d_1 \textit{ is some positive constant if } \beta>\gamma,\\ 1, & \textit{if } \beta=\gamma,\\ d_2\left(\int_{t_0}^{t} a^{\frac{1}{\gamma}}(s)\,ds\right), & d_2 \textit{ is some positive constant if } \beta<\gamma,\end{cases}$$

and $\gamma \geqslant 1$*. Then, Eq.* (1.1) *is oscillatory, where* $E(t)$, $\Psi(t)$, *and* $\Omega(t)$ *be as in* (2.1), (2.2), *and* (2.4).

Proof. Suppose to the contrary that $y(t)$ is a non-oscillatory solution of Eq. (1.1). We may assume that there exists a $t_1 \geqslant t_0$ such that $y(t)>0$, $y(\nu_i(t))>0$, and $y(\varphi(t))>0$ for $t \geqslant t_1$ and $i=1,2,\dots,m$. It is not difficult to see that $\varpi(t)>0$ for $t\geqslant t_1$. But since from Lemma 2.2 (i) with the definition of $\varpi(t)$, we get (3.4). Moreover by substituting (3.4) into (1.1), we arrive at (3.5), and using the assumption (A_4), we get (3.6). Integrating inequality (3.6) from t to ∞ and using the fact that $\varpi(t)$ is increasing, we have

$$\frac{a(t)}{E(t)}\left(\varpi'(t)\right)^{\gamma} \geqslant \int_t^{\infty} \frac{g(s)\Psi^{\beta}(\varphi(s))\varpi^{\beta}(\varphi(s))}{E(s)}ds \geqslant \varpi^{\beta}(\varphi(t))\int_t^{\infty}\frac{g(s)\Psi^{\beta}(\varphi(s))}{E(s)}ds.$$

Defining $\Theta(t)$ as in Theorem 3.1, we get

$$\Theta(t)=\chi(t)\frac{a(t)\left(\varpi'(t)\right)^{\gamma}}{\varpi^{\beta}(\varphi(t))} \geqslant \chi(t)E(t)\int_t^{\infty}\frac{g(s)\Psi^{\beta}(\varphi(s))}{E(s)}ds. \tag{3.27}$$

This with (3.5) leads to

$$\begin{aligned}\Theta'(t)&=\left(a(t)\left(\varpi'(t)\right)^{\gamma}\right)'\frac{\chi(t)}{\varpi^{\beta}(\varphi(t))}+\left(\frac{\chi(t)}{\varpi^{\beta}(\varphi(t))}\right)'a(t)\left(\varpi'(t)\right)^{\gamma}\\ &\leqslant \frac{-\chi(t)}{\varpi^{\beta}(\varphi(t))}\left(h(t)\left(\varpi'(t)\right)^{\gamma}+g(t)\Psi^{\beta}(\varphi(t))\varpi^{\beta}(\varphi(t))\right)\\ &\quad +a(t)\left(\varpi'(t)\right)^{\gamma}\left(\frac{\chi'(t)}{\varpi^{\beta}(\varphi(t))}-\frac{\beta\chi(t)\varpi^{\beta-1}(\varphi(t))\varpi'(\varphi(t))\varphi'(t)}{\varpi^{2\beta}(\varphi(t))}\right)\\ &\leqslant \frac{-\chi(t)}{\varpi^{\beta}(\varphi(t))}\left(h(t)\left(\varpi'(t)\right)^{\gamma}+g(t)\Psi^{\beta}(\varphi(t))\varpi^{\beta}(\varphi(t))\right)\\ &\quad +a(t)\left(\varpi'(t)\right)^{\gamma}\left(\frac{\chi'(t)}{\varpi^{\beta}(\varphi(t))}-\frac{\beta\chi(t)\varpi'(\varphi(t))\varphi'(t)}{\varpi^{\beta+1}(\varphi(t))}\right)\\ &\leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t))-\frac{h(t)}{a(t)}\Theta(t)+\frac{\chi'(t)}{\chi(t)}\Theta(t)-\beta\chi(t)\frac{a(t)\left(\varpi'(t)\right)^{\gamma}\varpi'(\varphi(t))\varphi'(t)}{\varpi^{\beta+1}(\varphi(t))}.\end{aligned} \tag{3.28}$$

Moreover, since $a(t)\left(\varpi'(t)\right)^{\gamma}$ is decreasing, we have

$$\frac{\varpi'(\varphi(t))}{\varpi'(t)} > \left(\frac{a(t)}{a(\varphi(t))}\right)^{\frac{1}{\gamma}}.$$

Thus the inequality (3.28) becomes

$$\Theta'(t) \leqslant -\chi(t)g(t)\Psi^{\beta}(\varphi(t))+\Omega(t)\Theta(t)-\frac{\beta\chi(t)\varphi'(t)}{a^{\frac{1}{\gamma}}(\varphi(t))}\left(\frac{\Theta(t)}{\chi(t)}\right)^{\frac{\gamma+1}{\gamma}}\varpi^{\frac{\beta-\gamma}{\gamma}}(\varphi(t)).$$

Now we have the three possible cases.

Case I: $\beta>\gamma$. In this case, since $\varpi'(t)>0$ for $t\geqslant t_0$, then there may exist $t_1\geqslant t_0$ such that $\varpi(\varphi(t))\geqslant d$ for $t\geqslant t_1$. Then it follows that

$$\varpi^{\frac{\beta-\gamma}{\gamma}}(\varphi(t)) \geqslant d^{\frac{\beta-\gamma}{\gamma}} = d_1.$$

Case II: $\beta = \gamma$. In this case, we see that $\varpi^{\frac{\beta-\gamma}{\gamma}}(\varphi(t)) = 1$.
Case III: $\beta < \gamma$. Since $a(t)(\varpi'(t))^{\gamma}$ is decreasing, there may exist a constant M such that

$$a(t)(\varpi'(t))^{\gamma} \leqslant M$$

for $t \geqslant t_0$. By integrating from t_0 to t, we get

$$\varpi(t) \leqslant \varpi(t_0) + \int_{t_0}^{t} \left(\frac{M}{a(s)}\right)^{\frac{1}{\gamma}} ds.$$

Hence, there may exist $t_1 \geqslant t_0$ and a constant M_1 depending on M such that

$$\varpi(t) \leqslant M_1 \int_{t_0}^{t} a^{-\frac{1}{\gamma}}(s)\, ds \ \text{ for } \ t \geqslant t_1,$$

and thus

$$\varpi^{\frac{\beta-\gamma}{\gamma}}(\varphi(t)) \geqslant M_1^{\frac{\beta-\gamma}{\gamma}} \left(\int_{t_0}^{t} a^{-\frac{1}{\gamma}}(s)\, ds\right)^{\frac{\beta-\gamma}{\gamma}} = d_2 \left(\int_{t_0}^{t} a^{-\frac{1}{\gamma}}(s)\, ds\right)^{\frac{\beta-\gamma}{\gamma}},$$

for some positive constant d_2.

Using the conclusions of these three cases and the definition of $\psi(t)$, we get

$$\Theta'(t) \leqslant -\chi(t) g(t) \Psi^{\beta}(\varphi(t)) + \Omega(t)\Theta(t) - \frac{\beta\varphi'(t)\psi(t)}{(\chi(t) a(\varphi(t)))^{\frac{1}{\gamma}}} \Theta^{\frac{\gamma+1}{\gamma}}(t)$$

for $t \geqslant t_1 > t_0$. Now setting

$$D = \Omega(t), \quad F = \frac{\beta\varphi'(t)\psi(t)}{(\chi(t) a(\varphi(t)))^{\frac{1}{\gamma}}},$$

and using the inequality (3.25), we obtain

$$\Theta'(t) \leqslant -\chi(t) g(t) \Psi^{\beta}(\varphi(t)) + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\chi(t) a(\varphi(t)) (\Omega(t))^{\gamma+1}}{(\beta\varphi'(t)\psi(t))^{\gamma}}.$$

Integrating from t_1 to t, we have

$$\Theta(t) \leqslant \Theta(t_1) - \int_{t_1}^{t} \left[\chi(s) g(s) \Psi^{\beta}(\varphi(s)) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\chi(s) a(\varphi(s)) (\Omega(s))^{\gamma+1}}{(\beta\varphi'(s)\psi(s))^{\gamma}}\right] ds.$$

Taking into account (3.27), we get

$$\begin{aligned} \Theta(t_1) \geqslant{}& \chi(t) E(t) \int_{t}^{\infty} \frac{g(s)\Psi^{\beta}(\varphi(s))}{E(s)} ds \\ &+ \int_{t_1}^{t} \left[\chi(s) g(s) \Psi^{\beta}(\varphi(s)) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\chi(s) a(\varphi(s)) (\Omega(s))^{\gamma+1}}{(\beta\varphi'(s)\psi(s))^{\gamma}}\right] ds. \end{aligned}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction with the condition (3.26). This completes the proof. □

Remark 3.8. Theorem 3.7 improves and extends Theorem 1 of [16].

Theorem 3.9. *Let the conditions* (A_1)-(A_4) *hold. Suppose further that* (2.6) *holds and* $\gamma = \beta$. *If there exists a positive function* $\chi(t) \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\chi(s) g(s) \Psi^{\beta}(\varphi(s)) - \frac{a(\varphi(s)) (\Omega(s))^{\beta+1} \chi(s)}{(\beta+1)^{\beta+1} (\varphi'(s))^{\beta}}\right] ds = \infty,$$

where $\Psi(t)$ *is defined by* (2.2), *and* $\Omega(t)$ *is defined by* (2.4), *then every solution of Eq.* (1.1) *is oscillatory.*

Proof. The proof follows the lines of the proof of Theorem 3.5. And so it is omitted. □

4. Examples

Example 4.1. Consider the differential equation

$$\begin{aligned}&\left(t\left[\left(y(t)+\frac{1}{t}y^{\frac{3}{5}}\left(\frac{t}{5}\right)+\frac{1}{t^2}y^{\frac{1}{7}}\left(\frac{t}{7}\right)\right)'\right]^5\right)'\\&+\left[\left(y(t)+\frac{1}{t}y^{\frac{3}{5}}\left(\frac{t}{5}\right)+\frac{1}{t^2}y^{\frac{1}{7}}\left(\frac{t}{7}\right)\right)'\right]^5+\frac{\vartheta}{t^3}y^3(t)=0,\quad t\geqslant 3.\end{aligned}\tag{4.1}$$

Here $a(t)=t, a'(t)=1>0, h(t)=1, c_1(t)=\frac{1}{t}, c_2(t)=\frac{1}{t^2}$. Thus clearly $\lim_{t\to\infty} c_i(t)=0$. Moreover $\nu_1(t)=\frac{t}{5}, \nu_2(t)=\frac{t}{7}, \alpha_1=\frac{3}{5}, \alpha_2=\frac{1}{7}$, and $f(t,y(\varphi(t)))=\frac{\vartheta}{t^3}y^3(t)$, i.e., $g(t)=\frac{\vartheta}{t^3}, \vartheta>0, \varphi(t)=t, \beta=3, \gamma=5$. It is not difficult to see that $E(t)=\frac{3}{t}$, and so (2.6) is satisfied. Choosing $\rho(t)=\frac{1}{t}$, then $\rho(t)\to 0$ as $t\to\infty$ and

$$\Psi(t)=1-\frac{3}{5t}-\frac{1}{7t^2}-t\left(\frac{2}{5t}+\frac{6}{7t^2}\right)=\frac{3}{5}-\frac{51}{35t}-\frac{1}{7t^2}=\frac{21t^2-51t-5}{35t^2}>0,\quad t\geqslant 3.$$

Choosing $\chi(t)=t^2$, we have $\Omega(t)=\frac{\chi'(t)}{\chi(t)}-\frac{h(t)}{a(t)}=\frac{1}{t}$ and

$$\int_{t_0}^{\infty}\left[\chi(t)g(t)\Psi^{\beta}\varphi(t)-\frac{M^{\frac{\gamma-\beta}{\gamma}}\chi(t)(\Omega(t))^2 a^{\frac{\beta}{\gamma}}\varphi(t)}{4\beta\left(\frac{\varphi(t)}{2}\right)^{\beta-1}\varphi'(t)}\right]dt=\int_3^{\infty}\left(\frac{\vartheta}{t}\left[\frac{3}{5}-\frac{51}{35t}-\frac{1}{7t^2}\right]^3-\frac{M^{\frac{2}{5}}}{3t^{\frac{7}{5}}}\right)dt=\infty.$$

So by Theorem 3.3, every solution of Eq. (4.1) is oscillatory.

Example 4.2. Consider the differential equation

$$\begin{aligned}&\left(\frac{1}{t^2}\left(y(t)+\frac{1}{t}y^{\frac{1}{3}}\left(\frac{t}{3}\right)+\frac{1}{t^2}y^{\frac{3}{5}}\left(\frac{t}{5}\right)\right)'\right)'\\&+\frac{1}{t^3}\left(y(t)+\frac{1}{t}y^{\frac{1}{3}}\left(\frac{t}{3}\right)+\frac{1}{t^2}y^{\frac{3}{5}}\left(\frac{t}{5}\right)\right)'+\frac{\vartheta}{t^3}y(t)=0,\quad t\geqslant 3.\end{aligned}\tag{4.2}$$

Here $a(t)=\frac{1}{t^2}, h(t)=\frac{1}{t^3}, \beta=1, c_1(t)=\frac{1}{t}, c_2(t)=\frac{1}{t^2}$. Thus clearly $\lim_{t\to\infty} c_i(t)=0$. Moreover $\nu_1(t)=\frac{t}{3}, \nu_2(t)=\frac{t}{5}, \alpha_1(t)=\frac{1}{3}, \alpha_2=\frac{3}{5}$, and $f(t,y(\varphi(t)))=\frac{\vartheta}{t^3}y(t)$, i.e., $g(t)=\frac{\vartheta}{t^3}, \vartheta>0, \varphi(t)=t$. It is not difficult to see that $E(t)=\frac{3}{t}$, and so (2.6) is satisfied. Letting $\rho(t)=\frac{1}{t}$, then $\rho(t)\to 0$ as $t\to\infty$ and

$$\Psi(t)=1-\frac{1}{3t}-\frac{3}{5t^2}-t\left[\frac{2}{3t}+\frac{2}{5t^2}\right]=\frac{1}{3}-\frac{11}{15t}-\frac{3}{5t^2}=\frac{5t^2-11t-9}{15t^2}>0\quad\text{for}\quad t\geqslant 3.$$

Choosing $\chi(t)=t^2$, we see that

$$\Omega(t)=\frac{1}{t},$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left[\chi(s)g(s)\Psi^{\beta}(\varphi(s))-\frac{a(\varphi(s))(\Omega(s))^{\beta+1}\chi(s)}{(\beta+1)^{\beta+1}(\varphi'(s))^{\beta}}\right]ds\\&=\limsup_{t\to\infty}\int_3^{t}\left[\frac{\vartheta}{3s}-\frac{11\vartheta}{15s^2}-\frac{3\vartheta}{5s^3}-\frac{1}{4s^2}\right]ds=\infty.\end{aligned}$$

Then by Theorem 3.9 every solution of Eq. (4.2) is oscillatory.

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