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Introduction to temporal topology

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Abstract

Temporal topology studies chronological variation of the topology of an object (set). In this paper, we will present the basic definitions related to temporal topology. In addition, the classical continuity and separations axioms known in general topology will be generalized to temporal topological spaces. Also, we will show that the subcategory of temporal T_{0Q}-spaces is reflective in the category of temporal topological spaces. The necessary illustrative examples to elaborate on the obtained results and relationships are provided.

Keywords: Topological space, operators of interior and closure, continuity, separation axioms, category of T_0-space.

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1. Introduction

The study of general topology as a branch of mathematics has been started since the 17$^{\text{th}}$ century. But the use of the term "topology" was used in 1874 by Johan Benedict Listing in the textbook "Vorsludier zur Topologie". Subsequently, the study of topology as well as topological spaces has undergone considerable advances and it generates a famous field of research until these days.

Euler, Lagrange, Cauchy, Bolzano, Riemann, Weirestrass, Poincaré, Frechet, Hausdorff, Kolmogorov, Alexandroff, Tychonoff and many other well-known mathematicians have left their mark in the history of topology. In [10], one can find many more details about the history of general topology.

General topology is interested in the geometric properties of an object (set) while neglecting the notion of the distance between the points (elements) of this object (set). We know the famous example of the cup which is continuously deformed into a torus while keeping the same topology of this object. This explains well what the topology of a set is about. Therefore, the continuous deformations of an object preserve the topology in the event of absence of uprooting or reattachment.

The idea for this paper comes from the observation in this last example. When building a cup, one can start with a ball of clay which is provided, of course, with a different topology than the final result

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which is the cup. Also, during the construction the cup can suffer work accidents which can tear or pierce it, that is to say, there is a change in the topology of this object over time. It can therefore be seen that the topology of the object changes over time. This is from where the idea of studying topologies on a set over time comes.

Studying the change of topology over time helps us to improve the application of topology theory in physical phenomena encountered in nature which are of course (naturally) time-related. This topology (which varies over time) on a non-empty set X will be represented by an application φ of $\mathbb{R}_+$ with value in $\mathcal{P}(\mathcal{P}(X))$, which will be called a temporal topology on X and (X, φ) will be said as a temporal topological space.

We draw the reader's attention to topological frames had been applied by many authors to address real-life problems. This matter can be noted in the published monographs, for more details, see [2, 4, 13, 15, 18–20]. It is well-known that some authors proposed expanding topological spaces for different purposes, theoretical and applied. These extensions had been exploited to handle some practical issues, for example, supra topology [7, 14], infra topology [8] and minimal structures [12], which enhances studying these structures and looking at master properties. Through the theoretical context, it was proposed novel criteria to produce separation axioms via topological spaces such as those defined in terms of limit points [5] and maps [17].

The rest of the paper is organized as follows. In the second segment of this article, we will give the basic definitions with some properties and characteristics. The third segment is interested in the axioms of separations in temporal topological spaces. Finally, in the last segment, observing from the point of view of category theory, we will present the collection of all temporal topological spaces, which satisfy the T_0 separation axioms as a reflective subcategory of the category of all temporal topological spaces. In closing, we outline the main contributions and indicate some possible directions for future work.

2. Main concepts via temporal topologies

In this segment, we will introduce the basic definitions of the temporal topology as well as generalizations of certain notions and properties known in the general topology and that we will use it thereafter.

Definition 2.1. Let X be a nonempty set and φ a map from $\mathbb{R}_+$ into $\mathcal{P}(\mathcal{P}(X))$. Then, φ will be called a temporal topology on X if we have $\varphi(t)$ defines a topology on X for every $t \in \mathbb{R}_+$. That is, for every $t \in \mathbb{R}_+$, we have:

- $\emptyset, X \in \varphi(t)$;
- $U, V \in \varphi(t)$ implies $U \cap V \in \varphi(t)$;
- if $(U_i)_{i \in I}$ is an collection of elements of $\varphi(t)$, then $\bigcup_{i \in I} U_i \in \varphi(t)$.

In this case, (X, φ) is said to be a temporal topological space.

Example 2.2. Let X be a nonempty set and $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ such that for every $n \in \mathbb{N}$ and every non negative real number t we have:

$$\varphi(t) = \{\emptyset, X\} \text{ if } t \in [2n, 2n+1[, \qquad \varphi(t) = \mathcal{P}(X) \text{ if } t \in [2n-1, 2n[.$$

Then, clearly (X, φ) is a temporal topological space.

Remark 2.3. A (classical) topological space (X, τ) could be seen as a (stationary) temporal topological space, it is sufficient to take $\varphi(t) = \tau$ for every $t \in \mathbb{R}_+$.

Definition 2.4. Let (X, φ) be a temporal topological space and A be a subset of X. Then, A is said to be:

(i) a stationary open set, for short an s-open, if $A \in \varphi(t)$ for all $t \in \mathbb{R}_+$;

(ii) a quasi-open set, for short a q-open, if there exists $t \in \mathbb{R}_+$ satisfying $A \in \varphi(t)$;
(iii) a stationary closed set, for short an s-closed, if its complement is s-open;
(iv) a quasi-closed set, for short a q-closed, if its complement is q-open.

Remark 2.5. Let (X, φ) be a temporal topological space. Then,

(i) it is clear that the collection of all s-open sets defines a topology on X;
(ii) in general, the collection of all q-open sets does not define a topology on X;
(iii) an s-open (resp. s-closed) is q-open (resp. q-closed);
(iv) a q-open (resp. q-closed) could be not s-open (resp. s-closed).

Notation 2.6. Let (X, φ) be a temporal topological space.

(i) $\mathcal{SO}(X, \varphi)$ (if not confusion, $\mathcal{SO}(X)$) denotes the collection of all s-open sets.
(ii) $\mathcal{QO}(X, \varphi)$ (if not confusion, $\mathcal{QO}(X)$) denotes the collection of all q-open sets.
(iii) $\mathcal{SF}(X, \varphi)$ (if not confusion, $\mathcal{SF}(X)$) denotes the collection of all s-closed sets.
(iv) $\mathcal{QF}(X, \varphi)$ (if not confusion, $\mathcal{QF}(X)$) denotes the collection of all q-closed sets.

Example 2.7. Let $X = \{a, b, c, d\}$ and $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ defined by

$$\varphi(0) = \{\emptyset, X, \{a, b\}\}, \quad \varphi(1) = \{\emptyset, X, \{b, c\}\}, \quad \text{and} \quad \varphi(t) = \{\emptyset, X\}, \quad \text{when} \quad t \in \mathbb{R}_+ \setminus \{0, 1\}.$$

Then, we have

- $\mathcal{SO}(X) = \{\emptyset, X\}$;
- $\mathcal{QO}(X) = \{\emptyset, X, \{a, b\}, \{b, c\}\}$;
- $\mathcal{SF}(X) = \{\emptyset, X\}$;
- $\mathcal{QF}(X) = \{\emptyset, X, \{c, d\}, \{a, d\}\}$.

It is clear that $\mathcal{QO}(X)$ does not define a topology on X.

Remark that $\mathcal{QO}(X)$ produces a minimal structure on X, in general.

Definition 2.8. Let (X, φ) be a temporal topological space and A a subset of X. Then

(i) $\underline{A}_S = \bigcup\{G \in \mathcal{SO}(X) : G \subseteq A\}$;
(ii) $\underline{A}_Q = \bigcup\{G \in \mathcal{QO}(X) : G \subseteq A\}$;
(iii) $\forall t \in \mathbb{R}_+$, $\underline{A}_t = \bigcup\{G \in \varphi(t) : G \subseteq A\}$.

Definition 2.9. Let (X, φ) be a temporal topological space and A a subset of X. Then

(i) $\overline{A}^S = \bigcap\{F \subseteq X : A \subseteq F \in \mathcal{SF}(X)\}$;
(ii) $\overline{A}^Q = \bigcap\{F \subseteq X : A \subseteq F \in \mathcal{QF}(X)\}$;
(iii) $\forall t \in \mathbb{R}_+$, $\overline{A}^t = \bigcap\{F \subseteq X : A \subseteq F \in \varphi(t)\}$.

One can see that $\underline{A}_S$ and $\overline{A}^S$ are respectively the interior and closure of A in a topological space $(X, \mathcal{SO}(X))$ and $\underline{A}_t$ and $\overline{A}^t$ are respectively the interior and closure of A in a topological space $(X, \varphi(t))$. The following example points out how these operators are calculated.

Example 2.10. We take the same temporal topological space built in Example 2.7. If $A = \{a, b, d\}$ and $B = \{d\}$, then

(i) $\underline{A}_S = \emptyset$ and $\overline{B}^S = X$;

(ii) $\underline{A}_Q = \{a, b\}$ and $\overline{B}^Q = B$;

(iii) $\underline{A}_0 = \{a, b\}$ and $\overline{B}^0 = \{c, d\}$;

(iv) $\underline{A}_1 = \emptyset$ and $\overline{B}^1 = \{a, d\}$.

We can see that $\overline{B}^Q$ is not q-closed.

Proposition 2.11. *Let A be a subset of a temporal topological space (X, φ). Then,*

(i) *A is s-open iff $A = \underline{A}_S$;*

(ii) *A is s-closed iff $A = \overline{A}^S$;*

(iii) *if A is q-open, then $A = \underline{A}_Q$;*

(iv) *if A is q-closed, then $A = \overline{A}^Q$.*

Proof. Straightforward. □

The converse of (iii) and (iv) of Proposition 2.11 need not be true as illustrated by the next example.

Example 2.12. Let $X = \{a, b, c, d\}$ and $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ defined by

$$\varphi(0) = \{\emptyset, X, \{a\}\}, \quad \varphi(1) = \{\emptyset, X, \{b\}\}, \quad \text{and} \quad \varphi(t) = \{\emptyset, X\}, \quad \text{when} \quad t \in \mathbb{R}_+ \setminus \{0, 1\}.$$

By taking $A = \{a, b\}$ and $B = \{c, d\}$ as subsets of X, we find $\underline{A}_Q = A$ and $\overline{B}^Q = B$. But A is neither q-open nor B is q-closed.

Proposition 2.13. *Let A be a subset of a temporal topological space (X, φ). Then, $x \in \overline{A}^S$ (resp., $x \in \overline{A}^Q$) iff $A \cap G \neq \emptyset$ for every s-open (resp., q-open) set G containing x.*

Proof.

Necessity: Let $x \in \overline{A}^S$ and let G be an s-open set such that $x \in G$. Suppose that $G \cap A = \emptyset$. Accordingly, we have $A \subseteq G^c$. This leads to that $\overline{A}^S \subseteq \overline{G^c}^S = G^c$. But this contradicts that $x \in \overline{A}^S$. Hence, $A \cap G \neq \emptyset$.

Sufficiency: Let the sufficient part be holds. Suppose, to the contrary, that $x \notin \overline{A}^S$. This means we can find an s-closed set H containing A such that $x \notin H$. Accordingly, we have H^c as an s-open set containing x and its intersection with A is the empty set. This is a contradiction. This finishes the proof that $x \in \overline{A}^S$.

The case between brackets can be proved in a similar way. □

Definition 2.14. Let $(X, \varphi), (Y, \psi)$ be two temporal topological spaces and f a map from X to Y. Then, f is said to be s-continuous (resp. q-continuous) if the inverse image, by f, of every element of $\mathcal{SO}(Y)$ (resp. $\mathcal{QO}(Y)$) is an element of $\mathcal{SO}(X)$ (resp. $\mathcal{QO}(X)$).

Remark 2.15. s-continuous does not imply q-continuous and q-continuous does not imply s-continuous. That is, these types of continuity are independent of each other. The next examples are provided to clarify this fact.

Example 2.16. Let $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ be defined as given in Example 2.7. If we define the map f from (X, φ) into itself by

$$f(a) = f(c) = a \;\text{ and }\; f(b) = c,$$

then we get f is s-continuous but not q-continuous.

Example 2.17. Let $X = \{a, b, c\}$ and $\psi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ defined by

$$\psi(0) = \{\emptyset, X, \{b\}, \{a, b\}\}, \quad \psi(1) = \{\emptyset, X, \{b\}, \{b, c\}\}, \quad \text{and} \quad \psi(t) = \{\emptyset, \{b\}, X\}, \quad \text{and} \quad t \in \mathbb{R}_+ \setminus \{0, 1\}.$$

We define the map f from (X, ψ) into itself by

$$f(a) = f(b) = b \ \text{ and } \ f(c) = c,$$

then we obtain f is q-continuous but not s-continuous.

The decomposition theorem for these continuity is given in the following.

Proposition 2.18. *Let (X_1, φ_1), (X_2, φ_2), (X_3, φ_3) be three temporal topological spaces and $f : (X_1, \varphi_1) \longrightarrow (X_2, \varphi_2)$, $g : (X_2, \varphi_2) \longrightarrow (X_3, \varphi_3)$ two maps. Then, if f, g are s-continuous (resp. q-continuous), then $g \circ f$ is also s-continuous (resp. q-continuous).*

Proof. Obvious. □

Lemma 2.19. *Let A be a subset of a temporal topological space (X, φ). Then,*

(i) $\underline{A}_S = \overline{A^c}^S$ *and* $\overline{A}^S = \underline{A^c}_S$;
(ii) $\underline{A}_Q = \overline{A^c}^Q$ *and* $\overline{A}^Q = \underline{A^c}_Q$.

Proof. The proof of $\underline{A}_S = \overline{A^c}^S$ is given as

$$\underline{A}_S = \cup\{G \in \mathcal{SO}(X) : G \subseteq A\} = \cap\{G^c \in \mathcal{SF}(X) : A^c \subseteq G^c\} = \overline{A^c}^S.$$

The other cases are proved following similar technique. □

Theorem 2.20. *Let $f : (X, \varphi_1) \longrightarrow (Y, \varphi_2)$ be a map such that A and B are subsets of X and Y, respectively. Then the following properties are equivalent:*

(i) f *is s-continuous;*
(ii) *the inverse image of each element of $\mathcal{SF}(Y)$ is an element of $\mathcal{SF}(X)$;*
(iii) $\overline{f^{-1}(B)}^S \subseteq f^{-1}(\overline{B}^S)$;
(iv) $f(\overline{A}^S) \subseteq \overline{f(A)}^S$; *and*
(v) $f^{-1}(\underline{B}_S) \subseteq \underline{f^{-1}(B)}_S$.

Proof.

(i)⇒(ii): Let B be an element of $\mathcal{SF}(Y)$. Then B^c is element of $\mathcal{SO}(Y)$. By hypothesis, $f^{-1}(B^c) = (f^{-1}(B))^c$ is an element of $\mathcal{SF}(X)$, which automatically means that $f^{-1}(B)$ is element of $\mathcal{SO}(X)$.

(ii)⇒(iii): For any subset B of Y, we have that $\overline{B}^S$ is an s-closed subset. Since $f^{-1}(\overline{B}^S)$ is an element of $\mathcal{SF}(X)$, $\overline{f^{-1}(B)}^S \subseteq \overline{f^{-1}(\overline{B}^S)}^S = f^{-1}(\overline{B}^S)$.

(iii)⇒(iv): Let A be a subset of X. Then $\overline{A}^S \subseteq \overline{f^{-1}(f(A))}^S \subseteq f^{-1}(\overline{f(A)}^S)$. Therefore, $f(\overline{A}^S) \subseteq f(f^{-1}(\overline{f(A)}^S)) \subseteq \overline{f(A)}^S$.

(iv)⇒(v): Let B be a subset of Y. By Lemma 2.19, we obtain that $f(\underline{X - (f^{-1}(B))}_S) = f(\overline{((f^{-1}(B))^c)}^S)$. By (iv), $f(\overline{((f^{-1}(B))^c)}^S) \subseteq \overline{f(f^{-1}(B))^c}^S = \overline{f(f^{-1}(B^c))}^S \subseteq \overline{Y - B}^S = Y - \underline{B}_S$. Therefore $\underline{X - (f^{-1}(B))}_S \subseteq f^{-1}(Y - \underline{B}_S) = X - f^{-1}(\underline{B}_S)$. Thus $f^{-1}(\underline{B}_S) \subseteq \underline{f^{-1}(B)}_S$.

(v)⇒(i): Take B as an s-open subset of Y. Then $f^{-1}(B) = f^{-1}(\underline{B}_S) \subseteq \underline{f^{-1}(B)}_S$. Since $f^{-1}(B)$ is s-open, $\underline{f^{-1}(B)}_S \subseteq f^{-1}(B)$. Therefore, $f^{-1}(B)$ is an s-open subset of X. Thus, f is s-continuous, as required. □

It is easy to show the next result, so we omit the proof.

Theorem 2.21. *Let* $f : (X, \varphi_1) \longrightarrow (Y, \varphi_2)$ *be a map such that* A *and* B *are subsets of* X *and* Y*, respectively. Then,* f *is* q*-continuous iff the inverse image of each element of* $\mathcal{QF}(Y)$ *is an element of* $\mathcal{QF}(X)$*.*

By the next example, we confirm that the following conditions do not guarantee q-continuity for a map $f : (X, \varphi_1) \longrightarrow (Y, \varphi_2)$, where A and B are subsets of X and Y, respectively.

(i) $\overline{f^{-1}(B)}^{Q} \subseteq f^{-1}(\overline{B}^{Q})$;
(ii) $f(\overline{A}^{Q}) \subseteq \overline{f(A)}^{Q}$; and
(iii) $f^{-1}(\underline{B}_{Q}) \subseteq \underline{f^{-1}(B)}_{Q}$.

Example 2.22. Let $X = \{x, y, z\}$, $Y = \{a, b, c, d\}$. Let $\varphi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(X))$ and $\psi : \mathbb{R}_+ \to \mathcal{P}(\mathcal{P}(Y))$ be defined by

$$\varphi(0) = \{\emptyset, X, \{x\}\}, \quad \varphi(1) = \{\emptyset, X, \{y\}\}, \quad \text{and} \quad \varphi(t) = \{\emptyset, X\}, \quad \text{when} \quad t \in \mathbb{R}_+ \setminus \{0, 1\},$$
$$\psi(0) = \{\emptyset, Y, \{a\}\}, \quad \psi(1) = \{\emptyset, Y, \{c\}\}, \quad \psi(2) = \{\emptyset, Y, \{d\}\}, \quad \text{and} \quad \psi(t) = \{\emptyset, Y\}, \quad \text{when} \quad t \in \mathbb{R}_+ \setminus \{0, 1, 2\}.$$

We define the map f from (X, φ) into (Y, ψ) by $f(x) = a$, $f(y) = b$, and $f(z) = c$, then we obtain f is q-continuous. By taking $B = \{b\}$ as a subset of Y, we find $\overline{f^{-1}(B)}^{Q} = \overline{f^{-1}(\{b\})}^{Q} = \overline{\{y\}}^{Q} = \{y, z\}$, whereas $f^{-1}(\overline{B}^{Q}) = f^{-1}(\{b\}) = \{y\}$. Also, if we take $B = \{c\}$, we obtain $f^{-1}(\underline{B}_{Q}) = \{z\}$, whereas $\underline{f^{-1}(B)}_{Q} = \emptyset$.

3. Separation axioms in temporal topological spaces

In this part we are interested in the notion of separation axioms in temporal topological spaces. Separation axioms is a field of work for many mathematicians such as Hausdorff, Kolmogorov, Frechet, Tychonoff, and many others.

Here, we will generalize the classical separation axioms T_0, T_1, T_2 to temporal topological spaces and we will study their properties and the relationships between them.

Definition 3.1. Let (X, φ) be a temporal topological space. Then, we say that (X, φ) is a

1. temporal T_{0S}-space if $\forall x \neq y \in X$, $\exists\, O \in \mathcal{SO}(X)$ such that $|\{x, y\} \cap O| = 1$;
2. temporal T_{0Q}-space if $\forall x \neq y \in X$, $\exists\, O \in \mathcal{QO}(X)$ such that $|\{x, y\} \cap O| = 1$;
3. temporal T_{0I}-space if $\forall t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_0-space;
4. temporal T_{0P}-space if $\exists t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_0-space.

As we have seen previously that a topological space (X, τ) can be considered as a temporal topological space by taking $\varphi(t) = \tau$ for any non negative real number t, we can see that all the previous definitions coincide with the definition of Kolmogorv spaces when we delete the factor time.

Proposition 3.2. *The following implication holds:* $T_{0S} \Longrightarrow T_{0I} \Longrightarrow T_{0P} \Longrightarrow T_{0Q}$.

Proof. It is straightforward. □

Example below shows that all implications in Proposition 3.2 are not equivalent.

Example 3.3.

1. Let $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \{\emptyset, X, \{a, b\}, \{c\}\}$, $\varphi(1) = \{\emptyset, X, \{b, c\}, \{a\}\}$, and $\varphi(t) = \{\emptyset, X\}$ if $t \notin \{0, 1\}$. Then, (X, φ) is a temporal T_{0Q}-space but it is not a temporal T_{0P}-space.
2. Let $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \mathcal{P}(X)$, and $\varphi(t) = \{\emptyset, X\}$ if $t \neq 0$. Then, (X, φ) is a temporal T_{0P}-space but it is not a temporal T_{0I}-space.

3. Let $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \{\emptyset, X, \{a, b\}, \{a\}\}$, $\varphi(1) = \{\emptyset, X, \{b, c\}, \{c\}\}$, and $\varphi(t) = \mathcal{P}(X)$ if $t \notin \{0, 1\}$. Then, (X, φ) is a temporal T_{0I}-space but it is not a temporal T_{0S}-space.

Proposition 3.4. *Let (X, φ) be a temporal topological space. Then, the following assertions are equivalent:*

1. *(X, φ) is a temporal T_{0S}-space;*
2. *if $\overline{\{x\}}^S = \overline{\{y\}}^S$, then $x = y$.*

Proof.
$1 \to 2$: Let $x \neq y \in X$. Then, there exists $O \in \mathcal{SO}(X)$ such that $|\{x, y\} \cap O| = 1$. Without loss of generality, take $x \in O$. This implies that $x \notin \overline{\{y\}}^S$. In contrast, $y \in \overline{\{y\}}^S$. This unmistakably leads to that $\overline{\{x\}}^S \neq \overline{\{y\}}^S$. Hence, $\overline{\{x\}}^S = \overline{\{y\}}^S$ implies $x = y$.

$2 \to 1$: Let $x \neq y \in X$. By hypothesis, $\overline{\{x\}}^S \neq \overline{\{y\}}^S$. Then there is $z \in X$ such that $z \in \overline{\{x\}}^S$ and $z \notin \overline{\{y\}}^S$, or $z \notin \overline{\{x\}}^S$ and $z \in \overline{\{y\}}^S$. Say, $z \in \overline{\{x\}}^S$ and $z \notin \overline{\{y\}}^S$. This implies that there is $O \in \mathcal{SO}(X)$ containing z such that $O \cap \{y\} = \emptyset$ and $O \cap \{x\} = \{x\}$. Hence, $|\{x, y\} \cap O| = 1$, which proves that (X, φ) is a temporal T_{0S}-space. □

Following similar arguments displayed in the proof of Proposition 3.4, the next propositions are proved.

Proposition 3.5. *Let (X, φ) be a temporal topological space. Then, the following assertions are equivalent:*

1. *(X, φ) is a temporal T_{0Q}-space;*
2. *if $\overline{\{x\}}^Q = \overline{\{y\}}^Q$, then $x = y$.*

Proposition 3.6. *Let (X, φ) be a temporal topological space. Then,*

1. *(X, φ) is a temporal T_{0P}-space iff $\overline{\{x\}}^t = \overline{\{y\}}^t$ implies $x = y$ for some $t \in \mathbb{R}$;*
2. *(X, φ) is a temporal T_{0I}-space iff $\overline{\{x\}}^t = \overline{\{y\}}^t$ implies $x = y$ for every $t \in \mathbb{R}$.*

Definition 3.7. Let (X, φ) be a temporal topological space. Then, we say that (X, φ) is a

1. temporal T_{1S}-space if $\forall x \neq y \in X$, $\exists\, O \in \mathcal{SO}(X)$ such that $\{x, y\} \cap O = \{x\}$;
2. temporal T_{1Q}-space if $\forall x \neq y \in X$, $\exists\, O \in \mathcal{QO}(X)$ such that $\{x, y\} \cap O = \{x\}$;
3. temporal T_{1I}-space if $\forall t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_1-space;
4. temporal T_{1P}-space if $\exists t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_1-space.

Proposition 3.8. *The following implication holds:* $T_{1S} \Longrightarrow T_{1I} \Longrightarrow T_{1P} \Longrightarrow T_{1Q}$.

Proof. It is straightforward. □

Example below elucidates that all implications in Proposition 3.8 cannot be converse.

Example 3.9.

1. Let $X = \{a, b\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \{\emptyset, X, \{a\}\}$, $\varphi(1) = \{\emptyset, X, \{b\}\}$, and $\varphi(t) = \{\emptyset, X\}$ if $t \notin \{0, 1\}$. Then, (X, φ) is a temporal T_{1Q}-space but it is not a temporal T_{1P}-space.
2. Lat $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \mathcal{P}(X)$ and $\varphi(t) = \{\emptyset, X\}$ if $t \neq 0$. Then, (X, φ) is a temporal T_{1P}-space but it is not a temporal T_{1I}-space.
3. Let $X = \mathbb{N}$, $\tau = \{\emptyset, \mathbb{N}, \{0\}, \{1\}\} \cup \{A \cup \{0, 1\} : \ A \subseteq \mathbb{N}\}$ and τ_{co} is the co-finite topology of $\mathbb{N}$. We take also $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \tau$, $\varphi(1) = \tau_{co}$, and $\varphi(t) = \mathcal{P}(X)$ if $t \notin \{0, 1\}$. Since $\mathcal{SO}(\mathbb{N}) = \{\emptyset\} \cup \{A \cup \{0, 1\}$ such that A^c is finite$\}$, we cannot separate 0 and 1 by s-open sets and then (X, φ) is a temporal T_{1I}-space but it is not a temporal T_{1S}-space.

Proposition 3.10. *If* X *is finite, then we have equivalence between* T_{1S} *and* T_{1I}.

Proof. Obviously we have T_{1S} implies T_{1I}. Conversely, since X is finite, $(X, \varphi(t))$ is a classical T_1-space implies that $\varphi(t)$ is the discreet topology on X, which signify that $\mathcal{SO}(X) = \mathcal{P}(X)$. Thus X is a temporal T_{1S}-space. □

Proposition 3.11. *Let* (X, φ) *be a temporal topological space. Then, we have*

1. (X, φ) *is a temporal* T_{1P}*-space if and only if* $\exists t \in \mathbb{R}_+$ *such that* $\overline{\{x\}}^t = \{x\}, \forall x \in X$;
2. (X, φ) *is a temporal* T_{1I}*-space if and only if* $\overline{\{x\}}^t = \{x\}\ \forall x \in X, \forall t \in \mathbb{R}_+$;
3. (X, φ) *is a temporal* T_{1S}*-space if and only if* $\overline{\{x\}}^S = \{x\}, \forall x \in X$;
4. (X, φ) *is a temporal* T_{1Q}*-space if and only if* $\overline{\{x\}}^Q = \{x\}, \forall x \in X$.

Proof. The proof of the first three items could be deduced directly from the fact that a classical topological space is a T_1-space if and only if singletons are closed.

4. Suppose that (X, φ) is a temporal T_{1Q}-space. Let $x \in X$ and $x \neq y \in X$. Then, since (X, φ) is a temporal T_{1Q}-space, there exists $O \in \mathcal{QO}(X)$ such that $y \in O$ and $x \in O^c \in \mathcal{QF}(X)$. So that, $\overline{\{x\}}^Q \subseteq O^c$, which implies that $y \notin \overline{\{x\}}^Q$ and we deduce that $\overline{\{x\}}^Q = \{x\}$. Conversely, $\overline{\{x\}}^Q = \{x\}$ and $y \neq x$ implies the existence of $O \in \mathcal{QO}(X)$, which contains x and does not contain y. Thus, (X, φ) is a temporal T_{1Q}-space. □

Definition 3.12. Let (X, φ) be a temporal topological space. Then, we say that (X, φ) is a

1. temporal T_{2S}-space if $\forall x \neq y \in X,\ \exists\, O_1, O_2 \in \mathcal{SO}(X)$ such that $O_1 \cap O_2 = \emptyset$, $x \in O_1$, and $y \in O_2$;
2. temporal T_{2Q}-space if $\forall x \neq y \in X,\ \exists\, O_1, O_2 \in \mathcal{QO}(X)$ such that $O_1 \cap O_2 = \emptyset$, $x \in O_1$, and $y \in O_2$;
3. temporal T_{2I}-space if $\forall t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_2-space;
4. temporal T_{2P}-space if $\exists t \in \mathbb{R}_+$, $(X, \varphi(t))$ is a classical T_2-space.

Proposition 3.13. *The following implication holds:* $T_{2S} \Longrightarrow T_{2I} \Longrightarrow T_{2P} \Longrightarrow T_{2Q}$.

Proof. It is straightforward. □

By the next example, it is illustrated that all implications in Proposition 3.13 cannot be reversed.

Example 3.14.

1. Let $X = \{a, b\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \{\emptyset, X, \{a\}\}$, $\varphi(1) = \{\emptyset, X, \{b\}\}$, and $\varphi(t) = \{\emptyset, X\}$ if $t \notin \{0, 1\}$. Then, (X, φ) is a temporal T_{2Q}-space but it is not a temporal T_{2P}-space.
2. Let $X = \{a, b, c\}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \mathcal{P}(X)$, and $\varphi(t) = \{\emptyset, X\}$ if $t \neq 0$. Then, (X, φ) is a temporal T_{2P}-space but it is not a temporal T_{2I}-space.
3. Let $X = \{(a, b) \in \mathbb{R}\ :\ b \geqslant 0\}$ and τ_1, τ_2 the two topologies defined on X by:
   - τ_1 is the subspace topology induced on X by the product topology $\tau_u \times \tau_u$ on $\mathbb{R}^2$, where τ_u denotes the standard topology on $\mathbb{R}$.
   - τ_2 is the topology X defined by its basis which contains all open discs contained in X and all sets of the form $U_{a,r} = B((a, 0), r) \cap X \setminus \{(x, 0)\ :\ x \in]a - r, a + r[\setminus \{a\}\}$ such that $r \in \mathbb{R}_+^*$.

   We take also $\varphi : \mathbb{R}_+ \longrightarrow \mathcal{P}(\mathcal{P}(X))$ defined by $\varphi(0) = \tau_1$, $\varphi(1) = \tau_2$ and $\varphi(t) = \mathcal{P}(X)$ if $t \notin \{0, 1\}$. In this case, the nontrivial elements of $\mathcal{SO}(X)$ are all open discs contained in X then we can not separate two distinct points of the form $(y, 0)$ using s-open sets. So that (X, φ) is a temporal T_{2Q}-space, a temporal T_{2P}-space and a temporal T_{2I}-space but it is not a temporal T_{2S}-space.

Definition 3.15. Let (X, φ) be a temporal topological space. A subset A of X is said to be an s-neighbourhood (resp., a q-neighbourhood) of $x \in X$ providing that there is an s-open (resp., a q-open) set G such that $x \in G \subseteq A$.

Proposition 3.16. *Let (X, φ) be a temporal topological space. Then, we have*

1. (X, φ) *is a temporal* T_{2P}*-space if and only if* $\exists t \in \mathbb{R}_+$ *such that* $\{x\} = \cap\{H : H$ *where* H *is a closed neighbourhood in* $\varphi(t)$ *of* $x\}$, $\forall x \in X$;
2. (X, φ) *is a temporal* T_{2I}*-space if and only if* $\{x\} = \cap\{H : H$ *where* H *is a closed neighbourhood of* $x\}$, $\forall x \in X$, $\forall t \in \mathbb{R}_+$;
3. (X, φ) *is a temporal* T_{2S}*-space if and only if* $\{x\} = \cap\{H : H$ *where* H *is an s-closed neighbourhood of* $x\}$, $\forall x \in X$;
4. (X, φ) *is a temporal* T_{2Q}*-space if and only if* $\{x\} = \cap\{H : H$ *where* H *is a* q*-closed neighbourhood of* $x\}$, $\forall x \in X$.

Proof. For the sake of abbreviating, we only prove (4).

$\Rightarrow$ Suppose that (X, φ) is a temporal T_{2Q}-space. Let $x \in X$ and $x \neq y \in X$. Then, since (X, φ) is a temporal T_{2Q}-space, there exists disjoint q-open sets O_1 and O_2 containing x and y, respectively. This implies that $x \in O_1 \subseteq \overline{O_1}^Q \subseteq \overline{O_2^c}^Q = O_2^c$. That is, O_2^c is a closed q-neighbourhood of x. So that $\{x\} = \cap\{H : H$ where H is a q-closed neighbourhood of $x\}$, $\forall x \in X$.

$\Leftarrow$ Let $x \neq y \in X$. Then, by hypothesis, there is a q-closed neighbourhood H of x such that $y \notin H$. So, there is a q-open set G such that $x \in G \subseteq H$. Now, remark that H^c and G are q-open sets containing y and x, respectively. Hence, (X, φ) is a temporal T_{2Q}-space. □

Proposition 3.17. *Let (X, φ) be a temporal topological space. Then, the following implications hold:*

$$\begin{array}{ccccccc}
T_{2S} & \Longrightarrow & T_{2I} & \Longrightarrow & T_{2P} & \Longrightarrow & T_{2Q} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_{1S} & \Longrightarrow & T_{1I} & \Longrightarrow & T_{1P} & \Longrightarrow & T_{1Q} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_{0S} & \Longrightarrow & T_{0I} & \Longrightarrow & T_{0P} & \Longrightarrow & T_{0Q}
\end{array}$$

Examples 3.3, 3.9, and 3.14 show that all horizontal implications are not equivalent. Also, vertical implications are not equivalent from classical topology. It suffices to take the same classical counterexamples such as the Seirpinski space, which is a Kolmogorov space and not a Frechet space and the co-finite topology on infinite set, which gives a Frechet topological space which is not Hausdorff.

We close this section by the next result which points out that the previous temporal spaces are topological properties with respect to s-continuous and q-continuous maps.

Proposition 3.18. *Let $f : (X, \varphi) \longrightarrow (Y, \psi)$ be an injective s-continuous (resp., injective q-continuous) map. If (Y, ψ) is temporal T_{jS} (resp., temporal T_{jQ}), then (X, φ) is temporal T_{jS} (resp., temporal T_{jQ}).*

4. Temporal T_{0Q}-spaces in the category of temporal topological spaces

On one hand, the collection of all temporal topological spaces forms the objects of a category denoted by TTop in which q-continuous maps are the arrows. On the other hand, the collection of all T_{0Q}-space regarded as a full subcategory of TTop that we will denote it by TTop_{0Q}. The goal of this segment is the study of the reflectivity of TTop_{0Q} in TTop.

By Maclane [16], to show that the full subcategory TTop_{0Q} is reflective in the category TTop, it will be sufficient to prove that for every object (X, φ) in TTop, there exists an object $(T_{0Q}(X), \widetilde{\varphi})$ in TTop_{0Q} and an arrow μ_X from (X, φ) to $(T_{0Q}(X), \widetilde{\varphi})$ in TTop such that for each object (Y, ψ) in TTop_{0Q} and each arrow $f : (X, \varphi) \longrightarrow (Y, \psi)$ in TTop, there exists a unique arrow $\widetilde{f}$ from $(T_{0Q}(X), \widetilde{\varphi})$ to (Y, ψ) in TTop_{0Q} rended commutative the following diagram:

$$\begin{tikzcd}
(X,\varphi) \arrow[rr, "\mu_X"] \arrow[dr, "f"'] & \arrow[d, phantom, "\nabla"] & (T_{0Q}(X),\widetilde{\varphi}) \arrow[dl, "\tilde{f}"] \\
& (Y,\psi) &
\end{tikzcd}$$

Let (X,φ) be a temporal topological space. We define on X the binary relation $\sim$ by

$$x \sim y \text{ if and only if } \overline{\{x\}}^{Q} = \overline{\{y\}}^{Q}.$$

It is clear that $\sim$ is an equivalence relation. We denote the quotient set by $X/\sim$ and we denote the canonical surjection from X to $X/\sim$ (s.t. $\mu_X(x) = \bar{x}$ for every $x \in X$) by μ_X. We also define the map $\widetilde{\varphi}$ by $\widetilde{\varphi}(t) = \left\{A \subseteq X/\sim \mid \mu_X^{-1}(A) \in \varphi(t)\right\}$ for every non negative real number t.

It is obvious to see that $(X/\sim,\widetilde{\varphi})$ is a temporal topological space.

Proposition 4.1. μ_X *is* q*-continuous and* s*-continuous.*

Proof. It is sufficient to see that for each $A \in \widetilde{\varphi}(t)$, then, by definition of $\widetilde{\varphi}$, we have $\mu_X^{-1}(A) \in \varphi(t)$. □

Lemma 4.2. *If* $A \in \mathcal{OQ}(X)$*, then* $\mu_X(A) \in \mathcal{OQ}(X/\sim)$*.*

Proof. It will be clear if we use the fact $\mu_X^{-1}(\mu_X(A)) = A$. □

Proposition 4.3. $(X/\sim,\widetilde{\varphi})$ *is a temporal* T_{0Q}*-space.*

Proof. Let $\bar{x} \neq \bar{y} \in X/\sim$. Then, $\overline{\{x\}}^{Q} \neq \overline{\{y\}}^{Q}$. So, there exists $A \in \mathcal{OQ}(X)$ such that $A \cap \{x,y\} \in \{\{x\},\{y\}\}$. Now, using the preceding lemma, we can see that $\mu_X(A) \in \mathcal{OQ}(X/\sim)$ and $\mu_X(A) \cap \{\bar{x},\bar{y}\} \in \{\{\bar{x}\},\{\bar{y}\}\}$, which completes the proof. □

Theorem 4.4. *TTop*$_{0Q}$ *is reflective in TTop.*

Proof. We have to prove that $(X/\sim,\widetilde{\varphi})$ is the reflection of (X,φ) in the subcategory TTop$_{0Q}$. Using Propositions 4.1 and 4.3, we see that $(X/\sim,\widetilde{\varphi})$ represents an object of TTop$_{0Q}$ and the map μ_X is an arrow in TTop. Let (Y,ψ) be a temporal T_{0Q}-space and $f : (X,\varphi) \longrightarrow (Y,\psi)$ a q-continuous map. Then, we have to find the unique map $\widetilde{f} : (X/\sim,\widetilde{\varphi}) \longrightarrow (Y,\psi)$, which is q-continuous and rended commutative the following diagram:

$$\begin{tikzcd}
(X,\varphi) \arrow[rr, "\mu_X"] \arrow[dr, "f"'] & \arrow[d, phantom, "\nabla"] & (X/\sim,\widetilde{\varphi}) \arrow[dl, "\tilde{f}"] \\
& (Y,\psi) &
\end{tikzcd}$$

The map $\widetilde{f}$ must satisfies $\widetilde{f}(\bar{x}) = f(x)$ for each $x \in X$, which implies directly the uniqueness (if there exists). Now, we should prove that $\widetilde{f}$ is well defined and q-continuous.

$\widetilde{f}$ **is well defined:** By contradiction suppose that $\bar{x} = \bar{y} \in X/\sim$ and $f(x) \neq f(y)$. Since (Y,ψ) is a temporal T_{0Q}-space, then there exists $O \in \mathcal{QO}(Y)$ such that $|O \cap \{f(x),f(y)\}| = 1$. Using the fact that f is q-continuous, we deduce that $f^{-1}(O) \in \mathcal{QO}(X)$ and $|f^{-1}(O) \cap \{x,y\}| = 1$. So that $\overline{\{x\}} \neq \overline{\{y\}}$, which gives a contradiction with $\bar{x} = \bar{y}$. Finally, $\bar{x} = \bar{y}$ implies $f(x) = f(y)$ and then $\widetilde{f}$ is well defined.

$\widetilde{f}$ **is q-continuous:** Let $O \in \mathcal{QO}(Y)$. We have $\mu_X^{-1}\left(\widetilde{f}^{-1}(O)\right) = (\widetilde{f} \circ \mu_X)^{-1}(O) = f^{-1}(O)$. Since f is q-continuous, then $f^{-1}(O) \in \mathcal{QO}(X)$, which proves that $\widetilde{f}^{-1}(O) \in \mathcal{QO}(X/\sim)$. Therefor $\widetilde{f}$ is q-continuous.

Conclusion: $(X/\sim,\widetilde{\varphi})$ is the T_{0Q}-reflection T_{0Q} of (X,φ) and then TTop$_{0Q}$ is a reflective subcategory of TTop. □

5. Conclusion and upcoming work

In the presented article we have introduced a new formulation to define a topology namely, "temporal topology". This formula produces topological structures that vary over time, which opens a door to apply topologies in phenomena encountered in nature that are of course (naturally) time-related. As we have suggested the set of time (domain) is the positive real numbers with zero; however, one can deal with subsets of this domain by taking into account the topologies produced outside these subsets are the discrete or indiscrete topologies. The choice of these topologies discrete or indiscrete is based on the nature of phenomena (or problems) under study.

At first, we have exhibited the main notions of temporal topological spaces, for example, we have presented four versions of interior and closure operators. Then, we have debated two types of continuity and concluded some of their descriptions. After that, we have provided four versions for each classical T_i-space when $i = 0, 1, 2$ and revealed the relationships between them with the aid of some illustrative counterexamples. Ultimately, we have studied subcategory of temporal T_{0Q}-spaces and proved that it is reflective.

Last but not least, the new temporal topology deserves further and deeper investigation. Especially interesting in this context there will be the analysis of the topological properties of those structures or their generalizations. So we plan to familiarize other topological concepts such as regularity and normality, covering properties, connectedness, etc. Moreover, we shall study the ideas displayed herein in the structures of temporal infra-topological and temporal supra-topological spaces following approach documented in [1, 3, 6, 9, 11].

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