

Global attractivity of a two-species competitive system with nonlinear inter-inhibition terms

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Abstract

Sufficient conditions are obtained for the global attractivity of the positive equilibrium and boundary equilibria of the following two-species competitive system with nonlinear inter-inhibition terms

$$\begin{aligned}\frac{dy_1(t)}{dt} &= y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 y_2}{1 + y_2} \right], \\ \frac{dy_2(t)}{dt} &= y_2(t) \left[r_2 - a_2 y_2 - \frac{b_2 y_1}{1 + y_1} \right],\end{aligned}$$

where $r_i, a_i, b_i, i = 1, 2$ are all positive constants. Our result shows that conditions which ensure the permanence of the system are almost enough to ensure the global stability of the system. The results not only improve but also complement the main results of Wang et al. [Q. L. Wang, Z. J. Liu, Z. X. Li, R. A. Cheke, *Int. J. Biomath.*, **7** (2014), 18 pages]. ©2016 All rights reserved.

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1. Introduction

The aim of this paper is to investigate the global dynamic behaviors of the following two-species competitive system with nonlinear inter-inhibition terms

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$$\begin{aligned}\frac{dy_1(t)}{dt} &= y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 y_2}{1 + y_2} \right], \\ \frac{dy_2(t)}{dt} &= y_2(t) \left[r_2 - a_2 y_2 - \frac{b_2 y_1}{1 + y_1} \right],\end{aligned}\tag{1.1}$$

where $r_i, a_i, b_i, i = 1, 2$ are all positive constants.

Recently, Wang et al. [23] studied the dynamic behaviors of the following two-species competitive system with nonlinear inter-inhibition terms

$$\begin{aligned}\frac{dy_1(t)}{dt} &= y_1(t) \left[r_1(t) - a_1(t) y_1 - \frac{b_1(t) y_2}{1 + y_2} \right], \\ \frac{dy_2(t)}{dt} &= y_2(t) \left[r_2(t) - a_2(t) y_2 - \frac{b_2(t) y_1}{1 + y_1} \right],\end{aligned}\tag{1.2}$$

where y_i ($i = 1, 2$) are the population densities of two competing species at time t ; $r_i(t)$ ($i = 1, 2$) are the intrinsic growth rates of species i ; a_i ($i = 1, 2$) are the rates of intraspecific competition of the first and second species, respectively; and $b_i(t)$ ($i = 1, 2$) are the rates of intraspecific competition of the first and second species, respectively. For more background of system (1.1), we refer the reader to [11, 19, 21–23, 28] and the references cited therein. For an almost periodic function $f(t)$, set $f^L = \inf_{t \in \mathbb{R}} f(t)$, $f^U = \sup_{t \in \mathbb{R}} f(t)$. Under the assumption $r_i(t), a_i(t)$ and $b_i(t), i = 1, 2$ are all positive almost periodic functions defined on $\mathbb{R}^+ = [0, +\infty)$. The authors investigated the existence and global asymptotic stability of positive almost periodic solutions of the system (1.2), they obtained the following results:

Theorem 1.1. *Any positive solution $(y_1(t), y_2(t))$ of system (1.2) satisfies*

$$\limsup_{t \rightarrow +\infty} y_i(t) \leq M_i = \frac{r_i^U}{a_i^L}, \quad i = 1, 2.$$

Theorem 1.2. *If the following assumptions*

$$r_1^L > (b_1^U - r_1^L)M_2, \quad r_2^L > (b_2^U - r_2^L)M_1\tag{1.3}$$

are satisfied, then any positive solution $(y_1(t), y_2(t))$ of system (1.2) satisfies

$$\liminf_{t \rightarrow +\infty} y_i(t) \geq m_i = \frac{r_i^L + (r_i^L - b_i^U)M_j}{a_i^U(1 + M_j)}, \quad i, j = 1, 2; \quad i \neq j.\tag{1.4}$$

Theorem 1.3. *If the almost periodic parameters $r_i(t), a_i(t), b_i(t)$ ($i = 1, 2$) of system (1.2) satisfies (1.3) and*

$$a_1^L - \frac{b_2^U}{(1 + m_1)^2} > 0, \quad a_2^L - \frac{b_1^U}{(1 + m_2)^2} > 0,\tag{1.5}$$

then system (1.2) has a globally asymptotically stable positive almost periodic solution.

It brings to our attention that the authors had investigated the stability property of the system (1.2) by constructing some suitable Lyapunov function, generally speaking, the conditions obtained by using Lyapunov function are very complicated, the additional condition, to some extent, is necessary. But for the system itself, this condition may not be necessary. Also, the authors in [23] did not investigate the extinction property of the system (1.2), which is one of the most important topics in the study of population dynamics (see [2–5, 7, 8, 12, 13, 18, 20] and the references therein). Above

analysis motivated us to revisit the autonomous case of system(1.2), i.e., system (1.1).

From the point of view of biology, in the sequel, we shall consider (1.1) together with the initial conditions

$$y_i(0) > 0, \quad i = 1, 2. \tag{1.6}$$

Obviously, system(1.1) has a unique solution $(y_1(t), y_2(t))$ satisfying the initial condition (1.6). We easily prove $y_i(t) > 0$ for all $i = 1, 2$ in maximal interval of existence of the solution.

The aim of this paper is, by further developing the analysis technique of [1, 6, 9, 10, 14–17, 24–27, 29] and using the differential inequality theory, to investigate the global dynamic behaviors of the system (1.1). More precisely, we will prove the following results.

Theorem 1.4. *Assume that the following inequalities*

$$r_1(a_2 + r_2) > b_1r_2, \quad r_2(a_1 + r_1) > b_2r_1 \tag{1.7}$$

hold, and assume further that one of the following conditions holds,

(A)

$$a_2 - b_2 + r_2 \neq 0; \tag{1.8}$$

(B)

$$a_2 - b_2 + r_2 = 0, a_1r_2 - a_2r_1 > 0, \tag{1.9}$$

then system (1.1) admits a unique positive equilibrium (y_1^*, y_2^*) , which is globally attractive, that is, for any positive solution $(y_1(t), y_2(t))$ of system (1.1) with the initial condition (1.6), one has

$$\lim_{t \rightarrow +\infty} y_i(t) = y_i^*, \quad i = 1, 2.$$

Remark 1.5. For system (1.1), the first inequality $r_1^L > (b_1^U - r_1^L)M_2$ in (1.3) is equivalent to $r_1 > (b_1 - r_1)\frac{r_2}{a_2}$, or $r_1(a_2 + r_2) > b_1r_2$. Similarly, the second inequality in (1.3) is equivalent to $r_2(a_1 + r_1) > b_2r_1$. Therefore, Theorem 1.4 shows that for the autonomous case of system (1.2), the conditions which ensure the permanence of the system are almost enough to ensure the global stability of the system, only the degenerate case $a_2 - b_2 + r_2 = 0$ needs further consideration.

As a direct corollary of Theorem 1.4, we have,

Corollary 1.6. *Assume that the following inequalities*

$$r_1 > b_1, \quad r_2 > b_2$$

hold, then system (1.1) admits a unique positive equilibrium (y_1^*, y_2^*) , which is globally attractive, that is, for any positive solution $(y_1(t), y_2(t))$ of system (1.1) with the initial condition (1.6), one has:

$$\lim_{t \rightarrow +\infty} y_i(t) = y_i^*, \quad i = 1, 2.$$

Theorem 1.7. *Assume that the following inequalities*

$$r_1(a_2 + r_2) > b_1r_2, \quad r_2 - \frac{b_2m_1}{1 + m_1} < 0 \tag{1.10}$$

hold, where

$$m_1 = \frac{r_1 - \frac{b_1 M_2}{1+M_2}}{a_1}, \quad M_2 = \frac{r_2}{a_2},$$

then

$$\lim_{t \rightarrow +\infty} y_1(t) = \frac{r_1}{a_1}, \quad \lim_{t \rightarrow +\infty} y_2(t) = 0.$$

Remark 1.8. Condition (1.10) is equivalent to

$$\begin{aligned} r_1(a_2 + r_2) &> b_1 r_2, \\ (a_1 - b_1 + r_1)r_2^2 + (a_1 a_2 + a_2 r_1 + b_1 b_2 - b_2 r_1)r_2 - a_2 b_2 r_1 &< 0. \end{aligned} \tag{1.11}$$

One could easily see that if r_1 is large enough and r_2 is small enough, then (1.11) always holds, and the second species will be driven to extinction. That is, for system (1.1), the intrinsic growth rate plays important role on the persistent and extinction property of the species. Large intrinsic growth rate will improve the chance of the survival of the species.

Theorem 1.9. *Assume that the following inequalities*

$$r_1 - \frac{b_1 m_2}{1 + m_2} < 0, \quad r_2(a_1 + r_1) > b_2 r_1 \tag{1.12}$$

hold, where

$$m_2 = \frac{r_2 - \frac{b_2 M_1}{1+M_1}}{a_2}, \quad M_1 = \frac{r_1}{a_1},$$

then

$$\lim_{t \rightarrow +\infty} y_1(t) = 0, \quad \lim_{t \rightarrow +\infty} y_2(t) = \frac{r_2}{a_2}.$$

Remark 1.10. Condition (1.12) is equivalent to

$$\begin{aligned} (a_2 - b_2 + r_2)r_1^2 + (a_1 a_2 + a_1 r_2 + b_1 b_2 - b_1 r_2)r_1 - a_1 b_2 r_2 &< 0, \\ r_2(a_1 + r_1) &> b_2 r_1. \end{aligned} \tag{1.13}$$

One can easily see that if r_2 is large enough and r_1 is small enough, then (1.13) always holds, and the first species will be driven to extinction.

The rest of the paper is arranged as follows: We will introduce some useful lemmas in the next section, and then prove the main results in Section 3. Some numeric simulations are carried out in Section 4, and we end this paper by a brief discussion.

2. Lemmas

Now let us state several lemmas which will be useful in the proving of main results.

Lemma 2.1. *In addition to (1.7), further assume that (1.8) or (1.9) holds, then system (1.1) admits a unique positive equilibrium (y_1^*, y_2^*) .*

Proof. The positive equilibrium of system (1.1) satisfies the following equation

$$\begin{aligned} r_1 - a_1 y_1 - \frac{b_1 y_2}{1 + y_2} &= 0, \\ r_2 - a_2 y_2 - \frac{b_2 y_1}{1 + y_1} &= 0. \end{aligned} \tag{2.1}$$

Since we focus on the positive solution of the system (1.1), it implies that we only need to consider the case $y_1 > 0, y_2 > 0$. Hence, to ensure the first equality holds, y_1 should be lied in the interval $(0, \frac{r_1}{a_1})$. Similarly, to ensure the second equality holds, y_2 should be lied in the interval $(0, \frac{r_2}{a_2})$. In th following we will investigate the positive equilibrium of system (1.1) on the rectangle $(0, \frac{r_1}{a_1}) \times (0, \frac{r_2}{a_2})$.

From the second equation of system (2.1) one can obtain

$$y_2 = -\frac{b_2y_1 - r_2y_1 - r_2}{a_2(1 + y_1)}. \tag{2.2}$$

Substituting (2.2) into the first equation of (2.1) leads to

$$A_1y_1^2 + A_2y_1 + A_3 = 0,$$

where

$$\begin{aligned} A_1 &= a_1a_2 - a_1b_2 + a_1r_2, \\ A_2 &= a_1a_2 + a_1r_2 - a_2r_1 - b_1b_2 + b_1r_2 + b_2r_1 - r_1r_2, \\ A_3 &= -a_2r_1 + b_1r_2 - r_1r_2. \end{aligned}$$

Now let us consider the function

$$F(y_1) = A_1y_1^2 + A_2y_1 + A_3.$$

Since

$$F(0) = A_3 < 0,$$

and

$$F\left(\frac{r_1}{a_1}\right) = \frac{b_1(a_1r_2 - b_2r_1 + r_1r_2)}{a_1} > 0,$$

which means that $F(y_1) = 0$ has at least one solution on the interval $(0, \frac{r_1}{a_1})$.

Now let us show that under the assumption of the lemma, $F(y_1) = 0$ has at most one positive solution on the interval $(0, \frac{r_1}{a_1})$.

- (1) Assume that $a_2 - b_2 + r_2 > 0$, in this case, $F(+\infty) = F(-\infty) = +\infty$, since $F(0) < 0$, it follows that $F(y_1)$ has at least one solution on the interval $(-\infty, 0)$ and $(0, +\infty)$, respectively. Since $F(y_1) = 0$ has at most two solutions, it follows that $F(y_1) = 0$ has at most one solution on the interval $(0, \frac{r_1}{a_1})$;
- (2) Assume that $a_2 - b_2 + r_2 < 0$, in this case, $F(+\infty) = -\infty$, since $F(0) < 0, F(\frac{r_1}{a_1}) > 0$, it follows that $F(y_1)$ has at least one solution on the interval $(0, \frac{r_1}{a_1})$ and $(\frac{r_1}{a_1}, +\infty)$, respectively. Since $F(y_1) = 0$ has at most two solutions, it follows that $F(y_1) = 0$ has at most one solution on the interval $(0, \frac{r_1}{a_1})$;
- (3) Assume that $a_2 - b_2 + r_2 = 0$, in this case, $F(0) < 0$, also, from (1.9) one has $F(\frac{r_1}{a_1}) = a_1b_1r_2 - a_2b_1r_1 > 0$, since $F(y_1)$ is the linear function of y_1 , it immediately follows that $F(y_1) = 0$ has only one solution on the interval $(0, \frac{r_1}{a_1})$.

Above analysis shows that under the assumption of Lemma 2.1, $F(y_1) = 0$ has at most one positive solution on the interval $(0, \frac{r_1}{a_1})$. Therefore, $F(y_1) = 0$ has a unique positive solution on the interval $(0, \frac{r_1}{a_1})$. Set this solution as y_1^* . Also, from (2.2) we have

$$y_2'(y_1) = -\frac{b_2}{a_2(1 + x)^2} < 0.$$

Since $y_2(0) = \frac{r_2}{a_2}$ and $y_2(\frac{r_1}{a_1}) = \frac{a_1 r_2 - b_2 r_1 + r_1 r_2}{a_2(a_1 + r_1)} > 0$, it follows that $y_2(y_1^*) > 0$. Set $y_2(y_1^*) = y_2^*$. Then system (1.1) admits a unique positive equilibrium (y_1^*, y_2^*) . This ends the proof of Lemma 2.1. \square

As a direct corollary of Lemma 2.2 of [1] by Chen, we have,

Lemma 2.2. *If $a > 0, b > 0$ and $\dot{x} \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \leq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b}{a}.$$

3. Proof of the main results

Now we are in the position to prove the main results of this paper.

Proof of Theorem 1.4. It follows from (1.7) that there exists an $\varepsilon > 0$ small enough such that

$$r_1 > \frac{b_1(\frac{r_2}{a_2} + \varepsilon)}{1 + (\frac{r_2}{a_2} + \varepsilon)} + a_1\varepsilon, \quad r_2 > \frac{b_2(\frac{r_1}{a_1} + \varepsilon)}{1 + (\frac{r_1}{a_1} + \varepsilon)} + a_2\varepsilon. \tag{3.1}$$

Let $(y_1(t), y_2(t))$ be any positive solution of system (1.1) with initial condition (1.6). From system (1.1) it follows that

$$\frac{dy_i(t)}{dt} \leq y_i(t) [r_i - a_i y_i]. \tag{3.2}$$

Thus, as a direct corollary of Lemma 2.2, according to (3.1), one has

$$\limsup_{t \rightarrow +\infty} y_i(t) \leq \frac{r_i}{a_i}. \tag{3.3}$$

Hence, for small enough $\varepsilon > 0$, it follows from (3.3) that there exists a $T_1 > 0$ such that

$$y_i(t) < \frac{r_i}{a_i} + \varepsilon \stackrel{\text{def}}{=} M_i^{(1)}, \quad i = 1, 2. \tag{3.4}$$

For $t > T_1$, it follows from the first equation of system (1.1) that

$$\frac{dy_1(t)}{dt} \geq y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 M_2^{(1)}}{1 + M_2^{(1)}} \right]. \tag{3.5}$$

Thus, as a direct corollary of Lemma 2.2, according to (3.5), one has

$$\liminf_{t \rightarrow +\infty} y_1(t) \geq \frac{r_1 - \frac{b_1 M_2^{(1)}}{1 + M_2^{(1)}}}{a_1}. \tag{3.6}$$

Hence, for small enough $\varepsilon > 0$, satisfying (3.1), it follows from (3.1) and (3.6) that there exists a $T'_2 > 0$ such that

$$y_1(t) > \frac{r_1 - \frac{b_1 M_2^{(1)}}{1 + M_2^{(1)}}}{a_1} - \varepsilon \stackrel{\text{def}}{=} m_1^{(1)} \text{ for } t > T'_2. \tag{3.7}$$

Similarly, for above $\varepsilon > 0$, it follows from the second equation of system (1.1) that there exists a $T_2 > T'_2$ such that

$$y_2(t) > \frac{r_2 - \frac{b_2 M_1^{(1)}}{1 + M_1^{(1)}}}{a_2} - \varepsilon \stackrel{\text{def}}{=} m_2^{(1)} \text{ for } t > T_2.$$

For $t > T_2$, it follows from the first equation of system (1.1) that

$$\frac{dy_1(t)}{dt} \leq y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 m_2^{(1)}}{1 + m_2^{(1)}} \right]. \tag{3.8}$$

Thus, as a direct corollary of Lemma 2.2, according to (3.8), one has

$$\limsup_{t \rightarrow +\infty} y_1(t) \leq \frac{r_1 - \frac{b_1 m_2^{(1)}}{1 + m_2^{(1)}}}{a_1}. \tag{3.9}$$

Hence, for $\varepsilon > 0$ satisfying (3.1), it follows from (3.9) that there exists a $T'_3 > 0$ such that

$$y_1(t) < \frac{r_1 - \frac{b_1 m_2^{(1)}}{1 + m_2^{(1)}}}{a_1} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_1^{(2)} \text{ for } t > T'_3.$$

Similarly, for above $\varepsilon > 0$, it follows from the second equation of system (1.1) that there exists a $T_3 > T'_3$ such that

$$y_2(t) < \frac{r_2 - \frac{b_2 m_1^{(1)}}{1 + m_1^{(1)}}}{a_2} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^{(2)} \text{ for } t > T_3.$$

For $t > T_3$, it follows from the first equation of system (1.1) that

$$\frac{dy_1(t)}{dt} \geq y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 M_2^{(2)}}{1 + M_2^{(2)}} \right].$$

Thus, as a direct corollary of Lemma 2.2, according to (3.8), one has

$$\liminf_{t \rightarrow +\infty} y_1(t) \geq \frac{r_1 - \frac{b_1 M_2^{(2)}}{1 + M_2^{(2)}}}{a_1}. \tag{3.10}$$

Hence, for $\varepsilon > 0$ satisfying (3.1), it follows from (3.10) that there exists a $T'_4 > 0$ such that

$$y_1(t) > \frac{r_1 - \frac{b_1 M_2^{(2)}}{1 + M_2^{(2)}}}{a_1} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_1^{(2)} \text{ for } t > T'_4.$$

Similarly, for above $\varepsilon > 0$, it follows from the second equation of system (1.1) that there exists a

$T_4 > T'_4$ such that

$$y_2(t) > \frac{r_2 - \frac{b_2 M_1^{(2)}}{1 + M_1^{(2)}}}{a_2} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_2^{(2)} \text{ for } t > T_4.$$

One can easily see that

$$\begin{aligned} M_1^{(2)} &= \frac{r_1 - \frac{b_1 m_2^{(1)}}{1 + m_2^{(1)}}}{a_1} + \frac{\varepsilon}{2} < \frac{r_1}{a_1} + \varepsilon = M_1^{(1)}, \\ M_2^{(2)} &= \frac{r_2 - \frac{b_2 m_1^{(1)}}{1 + m_1^{(1)}}}{a_2} + \frac{\varepsilon}{2} < \frac{r_2}{a_2} + \varepsilon = M_2^{(1)}, \\ m_1^{(2)} &= \frac{r_1 - \frac{b_1 M_2^{(2)}}{1 + M_2^{(2)}}}{a_1} - \frac{\varepsilon}{2} > \frac{r_1 - \frac{b_1 M_2^{(1)}}{1 + M_2^{(1)}}}{a_1} - \varepsilon = m_1^{(1)}, \\ m_2^{(2)} &= \frac{r_2 - \frac{b_2 M_1^{(2)}}{1 + M_1^{(2)}}}{a_2} - \frac{\varepsilon}{2} > \frac{r_2 - \frac{b_2 M_1^{(1)}}{1 + M_1^{(1)}}}{a_2} - \varepsilon = m_2^{(1)}. \end{aligned} \tag{3.11}$$

Repeating the above procedure, we get four sequences $M_i^{(n)}, m_i^{(n)}, i = 1, 2, n = 1, 2, \dots$, such that for $n \geq 2$

$$\begin{aligned} M_1^{(n)} &= \frac{r_1 - \frac{b_1 m_2^{(n-1)}}{1 + m_2^{(n-1)}}}{a_1} + \frac{\varepsilon}{n}, & M_2^{(n)} &= \frac{r_2 - \frac{b_2 m_1^{(n-1)}}{1 + m_1^{(n-1)}}}{a_2} + \frac{\varepsilon}{n}, \\ m_1^{(n)} &= \frac{r_1 - \frac{b_1 M_2^{(n)}}{1 + M_2^{(n)}}}{a_1} - \frac{\varepsilon}{n}, & m_2^{(n)} &= \frac{r_2 - \frac{b_2 M_1^{(n)}}{1 + M_1^{(n)}}}{a_2} - \frac{\varepsilon}{n}. \end{aligned} \tag{3.12}$$

Obviously,

$$m_i^{(n)} < N_i(t) < M_i^{(n)}, \text{ for } t \geq T_{2n}, \quad i = 1, 2.$$

We claim that sequences $M_i^{(n)}, i = 1, 2$ are non-increasing, and sequences $m_i^{(n)}, i = 1, 2$ are non-decreasing. To prove this claim, we will carry out by the induction. Firstly, from (3.11) we have

$$M_i^{(2)} < M_i^{(1)}, \quad m_i^{(2)} > m_i^{(1)}, \quad i = 1, 2.$$

Let us assume now that our claim is true for n , that is,

$$M_i^{(n)} < M_i^{(n-1)}, \quad m_i^{(n)} > m_i^{(n-1)}, \quad i = 1, 2.$$

Again from the strictly increasing of function of $g(x) = \frac{x}{1+x}$, we immediately obtain

$$M_1^{(n+1)} = \frac{r_1 - \frac{b_1 m_2^{(n)}}{1 + m_2^{(n)}}}{a_1} + \frac{\varepsilon}{n+1} < \frac{r_1 - \frac{b_1 m_2^{(n-1)}}{1 + m_2^{(n-1)}}}{a_1} + \frac{\varepsilon}{n} = M_1^{(n)},$$

$$\begin{aligned}
 M_2^{(n+1)} &= \frac{r_2 - \frac{b_2 m_1^{(n)}}{1 + m_1^{(n)}}}{a_2} + \frac{\varepsilon}{n+1} < \frac{r_2 - \frac{b_2 m_1^{(n-1)}}{1 + m_1^{(n-1)}}}{a_2} + \frac{\varepsilon}{n} = M_2^{(n)}, \\
 m_1^{(n+1)} &= \frac{r_1 - \frac{b_1 M_2^{(n+1)}}{1 + M_2^{(n+1)}}}{a_1} - \frac{\varepsilon}{n+1} > \frac{r_1 - \frac{b_1 M_2^{(n)}}{1 + M_2^{(n)}}}{a_1} - \frac{\varepsilon}{n} = m_1^{(n)}, \\
 m_2^{(n+1)} &= \frac{r_2 - \frac{b_2 M_1^{(n+1)}}{1 + M_1^{(n+1)}}}{a_2} - \frac{\varepsilon}{n+1} > \frac{r_2 - \frac{b_2 M_1^{(n)}}{1 + M_1^{(n)}}}{a_2} - \frac{\varepsilon}{n} = m_2^{(n)}.
 \end{aligned}$$

Therefore,

$$\lim_{t \rightarrow +\infty} M_i^{(n)} = \bar{y}_i, \quad \lim_{t \rightarrow +\infty} m_i^{(n)} = \underline{y}_i, \quad i = 1, 2.$$

Letting $n \rightarrow +\infty$ in (3.12), we obtain

$$\begin{aligned}
 a_1 \bar{y}_1 &= r_1 - \frac{b_1 \underline{y}_2}{1 + \underline{y}_2}, & a_2 \underline{y}_2 &= r_2 - \frac{b_2 \bar{y}_1}{1 + \bar{y}_1}, \\
 a_1 \underline{y}_1 &= r_1 - \frac{b_1 \bar{y}_2}{1 + \bar{y}_2}, & a_2 \bar{y}_2 &= r_2 - \frac{b_2 \underline{y}_1}{1 + \underline{y}_1}.
 \end{aligned} \tag{3.13}$$

Equation (3.13) shows that $(\bar{y}_1, \underline{y}_2)$ and $(\underline{y}_1, \bar{y}_2)$ are solutions of (2.1). By Lemma 2.1, equation (2.1) has a unique positive solution $E^*(y_1^*, y_2^*)$. Hence, we conclude that

$$\bar{y}_i = \underline{y}_i = N_i^*, \quad i = 1, 2,$$

that is,

$$\lim_{t \rightarrow +\infty} y_i(t) = y_i^* \quad i = 1, 2.$$

Thus, the unique interior equilibrium $E^*(y_1^*, y_2^*)$ is globally attractive. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.7. Condition

$$r_1(a_2 + r_2) > b_1 r_2, \quad r_2 - \frac{b_2 m_1}{1 + m_1} < 0$$

implies that there exists a small enough ε such that

$$r_1 > \frac{b_1(\frac{r_2}{a_2} + \varepsilon)}{1 + (\frac{r_2}{a_2} + \varepsilon)} + a_1 \varepsilon, \quad r_2 - \frac{b_2 m_1^\varepsilon}{1 + m_1^\varepsilon} < 0$$

holds, where

$$m_1^\varepsilon = \frac{r_1 - \frac{b_1 M_2^\varepsilon}{1 + M_2^\varepsilon}}{a_1}, \quad M_2^\varepsilon = \frac{r_2}{a_2} + \varepsilon.$$

Similar to the analysis of (3.2)-(3.7), there exists a $T > 0$ such that

$$y_i(t) < \frac{r_i}{a_i} + \varepsilon = M_i^\varepsilon, \quad i = 1, 2$$

and

$$y_1(t) > \frac{r_1 - \frac{b_1 M_2^\varepsilon}{1 + M_2^\varepsilon}}{a_1} - \varepsilon \stackrel{\text{def}}{=} m_1^\varepsilon. \tag{3.14}$$

From the second equation of system (1.1) and (3.14), we have

$$\frac{dy_2(t)}{dt} < y_2(t) \left[r_2 - \frac{b_2 m_1^\varepsilon}{1 + m_1^\varepsilon} \right].$$

Hence,

$$y_2(t) = y_2(T) \exp\left\{ \int_T^t \left[r_2 - \frac{b_2 m_1^\varepsilon}{1 + m_1^\varepsilon} \right] dt \right\} \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{3.15}$$

For small enough $\varepsilon_1 > 0$, it follows from (3.15) that there exists a $T_1 > T$ such that

$$0 < y_2(t) < \varepsilon_1 \text{ for all } t \geq T_1. \tag{3.16}$$

It follows from (3.16) and the first equation of (1.1) that

$$\frac{dy_1(t)}{dt} > y_1(t) \left[r_1 - a_1 y_1 - \frac{b_1 \varepsilon_1}{1 + \varepsilon_1} \right]. \tag{3.17}$$

Applying Lemma 2.2 to (3.17), it follows that

$$\lim_{t \rightarrow +\infty} y_1(t) \geq \frac{r_1 - \frac{b_1 \varepsilon_1}{1 + \varepsilon_1}}{a_1}.$$

Setting $\varepsilon \rightarrow 0$ leads to

$$\lim_{t \rightarrow +\infty} y_1(t) \geq \frac{r_1}{a_1}.$$

This together with (3.3) shows that

$$\lim_{t \rightarrow +\infty} y_1(t) = \frac{r_1}{a_1}.$$

This completes the proof of Theorem 1.7. □

Proof of Theorem 1.9. Since the proof of Theorem 1.9 is similar to that of the proof of Theorem 1.7, we omit the detail here. □

4. Numeric simulations

Now let us consider the following examples.

Example 4.1.

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_1(t) \left[3 - y_1 - \frac{3y_2}{1 + y_2} \right], \\ \frac{dy_2(t)}{dt} &= y_2(t) \left[3 - y_2 - \frac{3y_1}{1 + y_1} \right]. \end{aligned} \tag{4.1}$$

Corresponding to system (1.1), one has

$$r_1 = r_2 = b_1 = b_2 = 3, \quad a_1 = a_2 = 1,$$

and so,

$$r_1(a_2 + r_2) = 12 > 9 = b_1r_2, r_2(a_1 + r_1) = 12 > 9 = b_2r_1, \tag{4.2}$$

and

$$a_2 - b_2 + r_2 = 1 \neq 0. \tag{4.3}$$

Equations (4.2) and (4.3) show that all the conditions of Theorem 1.4 hold, and it follows from Theorem 1.4 that system (4.1) admits a unique positive equilibrium which is globally attractive. Fig. 1 also supports these findings. However, by simple computation, one can easily obtain that

$$M_1 = M_2 = 3$$

and

$$m_1 = m_2 = 3 - \frac{3 \times 3}{1 + 3} = \frac{3}{4},$$

and so,

$$a_1 - \frac{b_2}{(1 + m_1)^2} = 1 - \frac{3}{(1 + \frac{3}{4})^2} = -\frac{1}{49} < 0, \quad a_2 - \frac{b_1}{(1 + m_2)^2} = -\frac{1}{49} < 0.$$

Which means that Theorem C of [23] could not be applied to system (4.1), and one can only obtain the persistent property of system (4.1) from [23]. Obviously, we improve the main results of [23] by deleting unnecessary conditions.

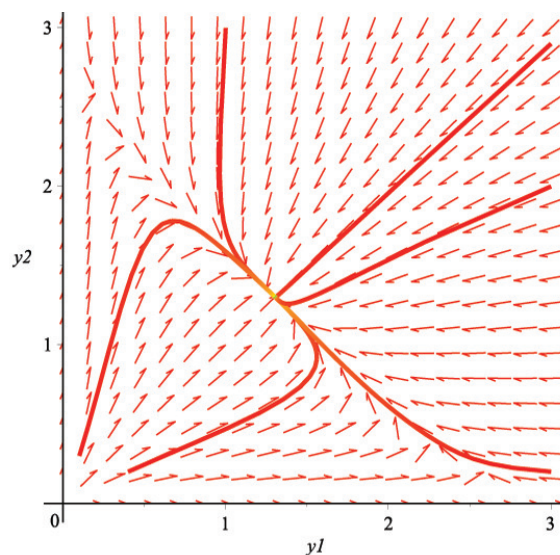


Figure 1: Dynamic behavior of system (4.1) with the initial condition $(y_1(0), y_2(0)) = (0.4, 0.2), (1, 3), (3, 2.9), (3, 0.2), (3, 2)$ and $(0.1, 3)$, respectively.

Example 4.2.

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_1(t) \left[3 - y_1 - \frac{3y_2}{1 + y_2} \right], \\ \frac{dy_2(t)}{dt} &= y_2(t) \left[3 - y_2 - \frac{8y_1}{1 + y_1} \right]. \end{aligned} \tag{4.4}$$

Corresponding to system (1.1), one has

$$r_1 = r_2 = b_1 = 3, \quad a_1 = a_2 = 1, \quad b_2 = 8$$

and so,

$$r_1(a_2 + r_2) = 12 > 9 = b_1r_2, \tag{4.5}$$

and

$$(a_1 - b_1 + r_1)r_2^2 + (a_1a_2 + a_2r_1 + b_1b_2 - b_2r_1)r_2 - a_2b_2r_1 = -3 < 0. \tag{4.6}$$

Equations (4.5) and (4.6) show that all the conditions of (1.11) hold, it follows from Theorem 1.7 that

$$\lim_{t \rightarrow +\infty} y_1(t) = 3, \quad \lim_{t \rightarrow +\infty} y_2(t) = 0.$$

Fig. 2 also supports these findings.

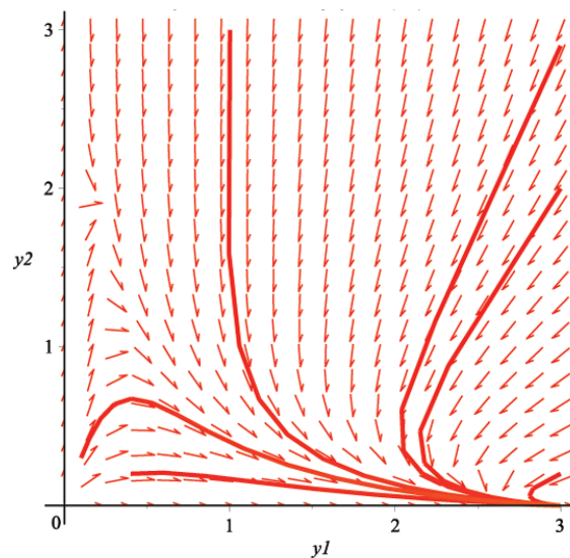


Figure 2: Dynamic behavior of system (4.4) with the initial condition $(y_1(0), y_2(0)) = (0.4, 0.2), (1, 3), (3, 2.9), (3, 0.2), (3, 2)$ and $(0.1, 3)$, respectively.

5. Discussion

In this paper, we revisit the dynamic behaviors of a two-species competitive system with nonlinear inter-inhibition terms, which was proposed by Wang et al. [23]. By using the iterative method, we

are able to obtain some more deep results. More precisely, we show that for the autonomous case of system (1.2), conditions which ensure the permanence of the system are almost enough to ensure the global stability of the system. We also investigate the extinction property of the system (1.1). Our results indicate that if the growth rate of the species is small enough and the rate of the intraspecific is large enough, then the species will be driven to extinction.

We mention here that a suitable population model should incorporate some past state of the species, and this will lead to a system with delay. Whether the delay has positive or negative influence on the dynamic behaviors of the system is still unknown, we leave this for future investigation.

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