



Directional differentiability of interval-valued functions

Yu-e Bao^{a,*}, Bo Zao^a, Eer-dun Bai^b

^aCollege of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia 028043, P. R. China.

^bCollege of Computer Science and Technology, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia 028043, P. R. China.

Abstract

In this paper, by using the generalized Hukuhara difference (gH -difference) of interval numbers, we introduce and study the directional differentiability problem of interval-valued function. Firstly, we put forward the concept of directional differentiability of interval-valued function, discuss the characterizations of directional differentiability of interval-valued function, and propose the sufficient condition of the directional differentiability of interval-valued function. Secondly, we discuss the relations among the directional derivative, derivative and partial derivative of interval-valued function, and prove that the derivative and partial derivative are both special directional derivatives. ©2016 All rights reserved.

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1. Introduction

Interval analysis (or interval mathematics) is a theory whose basic idea is to use interval variable to calculate instead of point variable. Theory of interval analysis was proposed systematically in 1966 by Moore [5]. From then on, theory and application of interval mathematics developed enormously with the joint effort of many scholars. The interval-valued function and its application receive lots of attention recently [2, 4, 8]. In papers [3, 6, 7] some concepts of differentiability of interval-valued function are introduced, problems about the optimization and differential equation of interval-value are discussed, and some worthy conclusion are obtained.

*Corresponding author

Email address: byebed@163.com (Yu-e Bao)

Particularly, in order to study the theory of differential equation of interval-valued function better, Stefanini and Bede discussed the generalized differentiability of interval-valued function by means of gh -difference, and obtained the concepts of gh -derivative and its differentiability of interval-valued function [7]. By using these concepts, Osuna-Gomez et al. [6] proposed the concepts of gh -partial derivative and its differentiability of multivariate interval-valued function, discussed the optimization problem about it, and got the optimal conditions which work more widely. Bede and Stefanini discussed the generalized differentiability of fuzzy valued function by means of gh -difference, obtained the concepts of gh -derivative and its differentiability of fuzzy valued function, and proved its relative properties [1].

In this paper, we put forward the concepts of gH -directional derivative and gH -directional differentiability of interval-valued functions from a subset of n -dimensional Euclidean space R^n into interval numbers space $[R]$, study its related properties, and lay a good foundation for the theory of gH -directional derivative and differentiability of fuzzy-valued functions from a subset of space into fuzzy numbers space.

2. Preliminaries

Let R be the real line. For $\underline{a}, \bar{a} \in R$ and $\underline{a} \leq \bar{a}$, we call $a = [\underline{a}, \bar{a}]$ an interval number, and denote by $[R]$ the set of all interval numbers of the real line R , i.e.,

$$[R] = \{[\underline{a}, \bar{a}] \mid \underline{a}, \bar{a} \in R, \underline{a} \leq \bar{a}\},$$

and $[R]$ is called interval numbers space. Let M be a nonempty subset of space R^n which is an n -dimensional Euclidean space. Let $F : M \rightarrow [R]$ be an interval-valued function $F(x) = [\underline{F}(x), \bar{F}(x)]$, where \underline{F} and \bar{F} are real-valued functions defined on M and satisfy $\underline{F}(x) \leq \bar{F}(x)$ for any $x \in M$.

For $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}] \in [R]$, we define $a = b$ iff $\underline{a} = \underline{b}$ and $\bar{a} = \bar{b}$. And the Hausdorff metric between interval numbers a and b is defined by

$$d_H(a, b) = \max\{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}.$$

Then $([R], d_H)$ is a complete metric space.

Theorem 2.1 ([4]). *Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function $F(x) = [\underline{F}(x), \bar{F}(x)]$, then $\lim_{x \rightarrow x_0} F(x)$ exists if and only if $\lim_{x \rightarrow x_0} \underline{F}(x)$ and $\lim_{x \rightarrow x_0} \bar{F}(x)$ exist as finite numbers, and*

$$\lim_{x \rightarrow x_0} F(x) = \left[\lim_{x \rightarrow x_0} \underline{F}(x), \lim_{x \rightarrow x_0} \bar{F}(x) \right].$$

Definition 2.2 ([6]). Let $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}] \in [R]$. The generalized H -difference (gH -difference) of two interval numbers a and b is defined as follows:

$$a -_g b = c \Leftrightarrow a = b + c \quad \text{or} \quad b = a + (-1)c.$$

For $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}] \in [R]$, the gH -difference of two interval numbers a and b always exists and it is equal to (see Proposition 4 in [4])

$$a -_g b = [\min\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}, \max\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}].$$

Definition 2.3 ([7]). Let $x_0 \in [a, b]$ and h such that $x_0 + h \in [a, b]$, the gH -derivative of the function $F : [a, b] \rightarrow ([R], d_H)$ at x_0 is defined as

$$F'(x_0) = \lim_{h \rightarrow 0} \frac{1}{h} [F(x_0 + h) -_g F(x_0)].$$

If $F'(x_0) \in [R]$, then we say that F is generalized H -differentiable (gH -differentiable for short) at x_0 .

Definition 2.4 ([6]). Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function and $x_0 = (x_1^{(0)}, \dots, x_n^{(0)})$ be a fixed element of M . We consider the interval-valued function

$$h_i(x_i) = F(x_1^{(0)}, \dots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \dots, x_n^{(0)}).$$

If h_i is gH -differentiable at $x_i^{(0)}$, then we say that F has the i -th partial gH -derivative at x_0 (denoted by $(\partial F / \partial x_i)(x_0)$) and $(\partial F / \partial x_i)(x_0) = h_i'(x_i^{(0)}) \in [R]$.

Definition 2.5 ([6]). Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function and let

$$x_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in M$$

be fixed. We say that F is gH -differentiable at x_0 , and we denote by $\nabla F(x_0)$, if all the partial gH -derivatives $(\partial F / \partial x_1)(x_0), \dots, (\partial F / \partial x_n)(x_0)$ exist on some neighborhoods of x_0 and are continuous at x_0 .

3. Directional derivatives of interval-valued functions

In [6], Osuna-Gmez et al. defined the partial gH -derivative and gH -differentiability of interval-valued function from a subset of space R^n into interval numbers space $[R]$ by using gH -difference of interval numbers. In the following, by using gH -difference of interval numbers, we defined the gH -directional derivative and gH -directional differentiability of interval-valued function from a subset of space R^n into interval numbers space $[R]$.

In this section, for $y \in R^n$, y_e denotes the unit vector of y . If $\cos \alpha_1, \cos \alpha_2, \dots, \cos \alpha_n$ are direction cosines of y , then $y_e = (\cos \alpha_1, \cos \alpha_2, \dots, \cos \alpha_n)$.

Definition 3.1. Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function and $x \in M$. If for $y \in R^n$, there exists $\delta > 0$ such that $x + hy_e \in M$ (resp. $x - hy_e \in M$) for any real number $h \in (0, \delta)$, and there exists $a^+ \in [R]$ (resp. $a^- \in [R]$) such that

$$\lim_{h \rightarrow 0^+} \frac{1}{h} [F(x + hy_e) -_g F(x)] = a^+ \left(\lim_{h \rightarrow 0^+} \frac{1}{h} [F(x) -_g F(x - hy_e)] = a^- \right).$$

Then we say F is right (resp. left) gH -differentiable in the direction y at x , and a^+ (resp. a^-) the right (resp. left) directional gH -derivative of F at x in the direction y and denote

$$F_{gH}^+(x, y) = a^+ (F_{gH}^-(x, y) = a^-).$$

If $F_{gH}^+(x, y) = F_{gH}^-(x, y)$, then we say F is gH -differentiable in the direction y at x , and denote

$$F_{gH}(x, y) = F_{gH}^+(x, y) = F_{gH}^-(x, y), \tag{3.1}$$

and call $F_{gH}(x, y)$ the directional gH -derivative of F at x in the direction y .

Theorem 3.2. Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$. If function F is gH -differentiable in the direction y at x , and there exists $\delta > 0$ such that

$$\begin{aligned} \underline{F}(x + hy_e) - \underline{F}(x) &\leq \overline{F}(x + hy_e) - \overline{F}(x), \\ \underline{F}(x) - \underline{F}(x - hy_e) &\leq \overline{F}(x) - \overline{F}(x - hy_e) \end{aligned}$$

for any $h \in (0, \delta)$, then the directional derivatives of $\overline{F}(x)$ and $\underline{F}(x)$ exist in direction y at x , and

$$F_{gH}(x, y) = [\underline{F}(x, y), \overline{F}(x, y)],$$

where $\underline{F}(x, y)$ and $\overline{F}(x, y)$ are the direction derivatives of $\underline{F}(x)$ and $\overline{F}(x)$ in direction y at x , respectively.

Proof. Necessity.

Let $F(x) = [\underline{F}(x), \overline{F}(x)]$ be gH -differentiable in the direction y at x , then there exists $\delta > 0$ such that $x + hy_e \in M, x - hy_e \in M$ for any $h \in (0, \delta)$, and there exists $F_{gH}(x, y) \in [R]$ which satisfies

$$\lim_{h \rightarrow 0^+} \frac{F(x + hy_e) -_g F(x)}{h} = \lim_{h \rightarrow 0^+} \frac{F(x) -_g F(x - hy_e)}{h} = F_{gH}(x, y).$$

Therefore, we have

$$\begin{aligned} &\lim_{h \rightarrow 0^+} \left| \frac{\underline{F}(x + hy_e) - \underline{F}(x)}{h} - \underline{F}_{gH}^+(x, y) \right| \\ &= \lim_{h \rightarrow 0^+} \left| \frac{\min\{\underline{F}(x + hy_e) - \underline{F}(x), \overline{F}(x + hy_e) - \overline{F}(x)\}}{h} - \underline{F}_{gH}^+(x, y) \right| \\ &\leq \lim_{h \rightarrow 0^+} \max\left\{ \left| \frac{\min\{\underline{F}(x + hy_e) - \underline{F}(x), \overline{F}(x + hy_e) - \overline{F}(x)\}}{h} - \underline{F}_{gH}^+(x, y) \right|, \right. \\ &\quad \left. \left| \frac{\max\{\underline{F}(x + hy_e) - \underline{F}(x), \overline{F}(x + hy_e) - \overline{F}(x)\}}{h} - \overline{F}_{gH}^+(x, y) \right| \right\} \\ &= d_H\left(\frac{F(x + hy_e) -_g F(x)}{h}, F_{gH}^+(x, y)\right) = 0. \end{aligned}$$

We can obtain

$$\lim_{h \rightarrow 0^+} \left| \frac{\underline{F}(x + hy_e) - \underline{F}(x)}{h} - \underline{F}_{gH}^+(x, y) \right| = 0.$$

Therefore, the right derivative of $\underline{F}(x)$ exists in direction y at x and $\underline{F}^+(x, y) = \underline{F}_{gH}^+(x, y)$.

The proof of $\underline{F}^-(x, y) = \underline{F}_{gH}^-(x, y)$ can be completed with the same argument.

On the other hand, by (3.1) we have

$$\underline{F}_{gH}^+(x, y) = \underline{F}_{gH}^-(x, y) = \underline{F}_{gH}(x, y).$$

Therefore, the derivative of $\underline{F}(x)$ exists in direction y at x , and

$$\underline{F}_{gH}(x, y) = \underline{F}(x, y). \tag{3.2}$$

Likewise, we can also obtain

$$\overline{F}_{gH}(x, y) = \overline{F}(x, y). \tag{3.3}$$

So by (3.2) and (3.3), we have

$$F_{gH}(x, y) = [\underline{F}(x, y), \overline{F}(x, y)].$$

□

Theorem 3.3. *Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$. If the directional derivative of $\underline{F}(x)$ and $\overline{F}(x)$ exist in direction y at x_0 , then $F(x)$ is gH -differentiable in direction y at x_0 and*

$$F_{gH}(x_0, y) = [\min\{\underline{F}(x_0, y), \overline{F}(x_0, y)\}, \max\{\underline{F}(x_0, y), \overline{F}(x_0, y)\}],$$

where $\underline{F}(x_0, y)$ and $\overline{F}(x_0, y)$ are the direction derivatives of $\underline{F}(x)$ and $\overline{F}(x)$ in direction y at x_0 , respectively.

Proof.

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F(x_0 + hy_e) -_g F(x_0)}{h} &= \lim_{h \rightarrow 0^+} \left[\frac{1}{h} \min\{\underline{F}(x_0 + hy_e) - \underline{F}(x_0), \overline{F}(x_0 + hy_e) - \overline{F}(x_0)\}, \right. \\ &\quad \left. \frac{1}{h} \max\{\underline{F}(x_0 + hy_e) - \underline{F}(x_0), \overline{F}(x_0 + hy_e) - \overline{F}(x_0)\} \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{1}{h} \min\{\underline{F}(x_0 + hy_e) - \underline{F}(x_0), \overline{F}(x_0 + hy_e) - \overline{F}(x_0)\} \\ &= \frac{1}{2h} ((\underline{F}(x_0 + hy_e) - \underline{F}(x_0)) + (\overline{F}(x_0 + hy_e) - \overline{F}(x_0))) \\ &\quad - |(\underline{F}(x_0 + hy_e) - \underline{F}(x_0)) - (\overline{F}(x_0 + hy_e) - \overline{F}(x_0))| \\ &= \frac{1}{h} \max\{\underline{F}(x_0 + hy_e) - \underline{F}(x_0), \overline{F}(x_0 + hy_e) - \overline{F}(x_0)\} \\ &= \frac{1}{2h} ((\underline{F}(x_0 + hy_e) - \underline{F}(x_0)) + (\overline{F}(x_0 + hy_e) - \overline{F}(x_0))) \\ &\quad - |(\underline{F}(x_0 + hy_e) - \underline{F}(x_0)) - (\overline{F}(x_0 + hy_e) - \overline{F}(x_0))|. \end{aligned}$$

Therefore, by Theorem 2.1 we have

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F(x_0 + hy_e) -_g F(x_0)}{h} &= \lim_{h \rightarrow 0^+} \left[\frac{1}{2h} \{(\underline{F}^+(x_0, y) + \overline{F}^+(x_0, y)) - |\underline{F}^+(x_0, y) - \overline{F}^+(x_0, y)|\}, \right. \\ &\quad \left. \frac{1}{2h} \{(\underline{F}^+(x_0, y) + \overline{F}^+(x_0, y)) + |\underline{F}^+(x_0, y) - \overline{F}^+(x_0, y)|\} \right] \\ &= [\min\{\underline{F}^+(x_0, y), \overline{F}^+(x_0, y)\}, \max\{\underline{F}^+(x_0, y), \overline{F}^+(x_0, y)\}]. \end{aligned}$$

Therefore, the right derivative of $F(x)$ exists in direction y at x and

$$F_{gH}^+(x, y) = [\min\{\underline{F}^+(x_0, y), \overline{F}^+(x_0, y)\}, \max\{\underline{F}^+(x_0, y), \overline{F}^+(x_0, y)\}].$$

Similarly, we can show that the left gH -derivative of $F(x)$ exists in direction y at x and

$$F_{gH}^-(x_0, y) = [\min\{\underline{F}^-(x_0, y), \overline{F}^-(x_0, y)\}, \max\{\underline{F}^-(x_0, y), \overline{F}^-(x_0, y)\}].$$

So by

$$\underline{F}(x_0, y) = \underline{F}^-(x_0, y) = \underline{F}^+(x_0, y), \quad \overline{F}(x_0, y) = \overline{F}^-(x_0, y) = \overline{F}^+(x_0, y),$$

we can obtain that the function $F(x)$ is differentiable in the direction y at x_0 , and

$$F_{gH}(x_0, y) = [\min\{\underline{F}(x_0, y), \overline{F}(x_0, y)\}, \max\{\underline{F}(x_0, y), \overline{F}(x_0, y)\}].$$

□

Example 3.4. Let $M = [-2.2] \times [-2.2]$ and interval-valued function $F : M \rightarrow ([R], d_H)$ is given by

$$F(x) = [x_1^2 - x_2^2, 9 - x_1x_2 + x_2^2], \quad x = (x_1, x_2) \in M.$$

Then $F(x)$ is differentiable in the direction $y = (1, 3)$ at $x_0 = (0, 1)$, and

$$F_{gH}(x_0, y) = [-6/\sqrt{10}, 5/\sqrt{10}].$$

Proof. (1) By using Definition 3.1

$$\begin{aligned} \underline{F}(x) &= x_1^2 - x_2^2, & \overline{F}(x) &= 9 - x_1x_2 + x_2^2, & y_e &= (1/\sqrt{10}, 3/\sqrt{10}), \\ x_0 + hy_e &= (0, 1) + h(1/\sqrt{10}, 3/\sqrt{10}) = (h/\sqrt{10}, 1 + 3h/\sqrt{10}), \\ x_0 - hy_e &= (0, 1) - h(1/\sqrt{10}, 3/\sqrt{10}) = (-h/\sqrt{10}, 1 - 3h/\sqrt{10}), \\ \underline{F}(x_0 + hy_e) - \underline{F}(x_0) &= -6/\sqrt{10} - 8h^2/10, \\ \overline{F}(x_0 + hy_e) - \overline{F}(x_0) &= 5h/\sqrt{10} + 6h^2/10. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_{gH}^+(x_0, y) &= \lim_{h \rightarrow 0^+} \frac{F(x_0 + hy_e) -_g F(x_0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{[-6h/\sqrt{10} - 8h^2/10, 5h/\sqrt{10} + 6h^2/10]}{h} \\ &= \lim_{h \rightarrow 0^+} [-6/\sqrt{10} - 8h/10, 5/\sqrt{10} + 6h/10] = [-6/\sqrt{10}, 5/\sqrt{10}]. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\underline{F}(x_0 - hy_e) + \underline{F}(x_0) &= -6h/\sqrt{10} + 8h^2/10, \\ \overline{F}(x_0) - \overline{F}(x_0 - hy_e) &= 5h/\sqrt{10} - 6h^2/10. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_{gH}^-(x_0, y) &= \lim_{h \rightarrow 0^+} \frac{F(x_0) -_g F(x_0 - hy_e)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{[-6h/\sqrt{10} + 8h^2/10, 5h/\sqrt{10} - 6h^2/10]}{h} = [-6/\sqrt{10}, 5/\sqrt{10}]. \end{aligned}$$

So $F_{gH}^+(x_0, y) = F_{gH}^-(x_0, y) = [-6/\sqrt{10}, 5/\sqrt{10}]$. By Definition 3.1, $F(x)$ is gH -differentiable in the direction y at x_0 , and

$$F_{gH}(x_0, y) = [-6/\sqrt{10}, 5/\sqrt{10}].$$

(2) By using Theorem 3.2

$$\begin{aligned} \underline{F}_{x_1}(x_0) &= 2x_1 \big|_{(0,1)} = 0, & \underline{F}_{x_2}(x_0) &= -2x_2 \big|_{(0,1)} = -2, \\ \overline{F}_{x_1}(x_0) &= -x_1 \big|_{(0,1)} = 0, & \overline{F}_{x_2}(x_0) &= (-x_1 + 2x_2) \big|_{(0,1)} = 2, \\ y_e &= (\cos \alpha, \cos \beta) = (1/\sqrt{10}, 3/\sqrt{10}), \end{aligned}$$

where $\cos \alpha, \cos \beta$ are direction cosines of $y = (1, 3)$.

Therefore, we have

$$\begin{aligned} \underline{F}(x_0, y) &= \underline{F}_{x_1}(x_0) \cos \alpha + \underline{F}_{x_2}(x_0) \cos \beta = -6/\sqrt{10}, \\ \overline{F}(x_0, y) &= \overline{F}_{x_1}(x_0) \cos \alpha + \overline{F}_{x_2}(x_0) \cos \beta = 5/\sqrt{10}. \end{aligned}$$

So by Theorem 3.2, we have $F(x)$ is gH -differentiable in the direction y at x_0 , and

$$F_{gH}(x_0, y) = [-6/\sqrt{10}, 5/\sqrt{10}].$$

□

Theorem 3.5. *Let $F : (a, b) \rightarrow ([R], d_H)$ be an interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$. Then $F(x)$ is gH -differentiable in the direction $y = 1$ at x if and only if $F(x)$ is gH -differentiable at x and*

$$F'(x) = F_{gH}(x, 1).$$

Proof. Necessity.

Let $F(x)$ be gH -differentiable in the direction $y = 1$ at x , then there exists $\delta > 0$ such that $x + h \in (a, b)$, $x - h \in (a, b)$, for any $h \in (0, \delta)$. There also exists $F_{gH}(x, 1) \in [R]$ such that

$$\lim_{h \rightarrow 0^+} \frac{F(x+h) -_g F(x)}{h} = \lim_{h \rightarrow 0^+} \frac{F(x+h) -_g F(x)}{h} = F_{gH}(x, 1).$$

On the other hand, we obtain

$$\begin{aligned} &-(F(x) -_g F(x-h)) \\ &= -[\min\{\underline{F}(x) - \underline{F}(x-h), \overline{F}(x) - \overline{F}(x-h)\}, \max\{\underline{F}(x) - \underline{F}(x-h), \overline{F}(x) - \overline{F}(x-h)\}] \\ &= [-\max\{\underline{F}(x) - \underline{F}(x-h), \overline{F}(x) - \overline{F}(x-h)\}, -\min\{\underline{F}(x) - \underline{F}(x-h), \overline{F}(x) - \overline{F}(x-h)\}] \\ &= [\min\{\underline{F}(x-h) - \underline{F}(x), \overline{F}(x-h) - \overline{F}(x)\}, \max\{\underline{F}(x-h) - \underline{F}(x), \overline{F}(x-h) - \overline{F}(x)\}]. \end{aligned}$$

Therefore, we have

$$\lim_{h \rightarrow 0^+} \frac{F(x) -_g F(x-h)}{h} = \lim_{h \rightarrow 0^+} \frac{F(x-h) -_g F(x)}{(-h)} = F_{gH}(x, 1).$$

So

$$\lim_{h \rightarrow 0} \frac{F(x+h) -_g F(x)}{h} = F_{gH}(x, 1).$$

Thus, by Definition 2.2, we know that $F(x)$ is gH -differentiable at x , and

$$F'(x) = F_{gH}(x, 1).$$

Sufficiency.

If $F(x)$ is gH -differentiable at x , then there exists $\delta > 0$ such that $x + h \in (a, b)$, $x - h \in (a, b)$, for any $h \in (0, \delta)$, and $F'(x) \in [R]$ such that

$$\lim_{h \rightarrow 0^+} \frac{F(x+h) -_g F(x)}{h} = \lim_{h \rightarrow 0^+} \frac{F(x+h) -_g F(x)}{h} = F'(x).$$

Therefore, $F_{gH}^+(x, 1) = F_{gH}^-(x, 1) = F'(x, 1)$. So the gH -derivative of $F(x)$ in direction $y = 1$ at x exists, and $F'(x) = F_{gH}(x, 1)$. □

Denote $e_i = (a_1, a_2, \dots, a_j, \dots, a_n) (i = 1, 2, \dots, n)$ with

$$a_j = \begin{cases} 1 & j = i, \\ 0 & j \neq i, \end{cases} \quad (i = 1, 2, \dots, n).$$

Theorem 3.6. *Let $F : M \rightarrow ([R], d_H)$ be an interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$. $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}) \in M$. If $F(x)$ is gH -differentiable in the direction e_i at x_0 , then $F(x)$ has the partial gH -derivative at x_0 for x_i , and $(\partial F / \partial x_i)(x_0) = F_{gH}(x_0, e_i)$.*

Proof. Let $h_i(x_i) = F(x_1^{(0)}, \dots, x_{(i-1)}^{(0)}, x_i, x_{(i+1)}^{(0)}, x_n^{(0)})$, then

$$\frac{h_i(x_i + h) -_g h_i(x_i)}{h} = \frac{F(x_0 + he_i) -_g F(x_0)}{h},$$

$$\frac{h_i(x_i) -_g h_i(x_i - h)}{h} = \frac{F(x_0) -_g F(x_0 - he_i)}{h}.$$

Since the gH -derivative of $F(x)$ in direction e_i at x_0 exists, i.e., there exists $F_{gH}(x_0, e_i) \in [R]$ such that

$$\lim_{h \rightarrow 0^+} \frac{F(x_0 + he_i) -_g F(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{F(x_0) -_g F(x_0 - he_i)}{h} = F_{gH}(x_0, e_i).$$

Therefore, we have

$$\lim_{h \rightarrow 0^+} \frac{h_i(x_i + h) -_g h_i(x_i)}{h} = \lim_{h \rightarrow 0^+} \frac{h_i(x_i) -_g h_i(x_i - h)}{h} = F_{gH}(x_0, e_i).$$

By Theorem 3.5, we have $h_i(x_i)$ is gH -differentiable at $x_i^{(0)}$. So by Definition 2.4, we can know that $F(x)$ has the partial gH -derivative at x_0 for x_i , and $(\partial F / \partial x_i)(x_0) = F_{gH}(x_0, e_i)$. □

Corollary 3.7. *Let $F : M \rightarrow [R]$ be an interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$. $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}) \in M$. If the gH -derivatives of $F(x)$ exist on some neighborhoods of x_0 in the direction $e_i (i = 1, 2, \dots, n)$ and are continuous at x_0 , then $F(x)$ is gH -differentiable at x_0 , and $\nabla F(x_0) = \{F_{gH}(x_0, e_1), F_{gH}(x_0, e_2), \dots, F_{gH}(x_0, e_n)\}$.*

Proof. Since $F_{gH}(x_0, e_i)$ of $F(x)$ exists on some neighborhoods of x_0 and is continuous at x_0 , by Theorem 3.6, we can know that all $\partial F / \partial x_i |_{x=x_0} (i = 1, 2, \dots, n)$ exist on some neighborhoods of x_0 and

$$(\partial F / \partial x_i)(x_0) = F_{gH}(x_0, e_i) (i = 1, 2, \dots, n)$$

are continuous at x_0 .

Therefore, by Definition 2.5 we can know that $F(x)$ is gH -differentiable at x_0 and

$$\nabla F(x_0) = \{F_{gH}(x_0, e_1), F_{gH}(x_0, e_2), \dots, F_{gH}(x_0, e_n)\}.$$

□

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