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Some properties of $\mathcal{R}_i$-axioms via multi-set topological spaces

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Abstract

multi-set are sets that are allowed to have repeated members, that is a multi-set $\mathcal{M}$ on a set $\mathcal{U}$ is a count function $C_{\mathcal{M}}$ from $\mathcal{U}$ to non-negative numbers. This study focuses on introducing and analyzing two new classes of separation axioms named, $\mathcal{M}$-$\mathcal{R}_0$ and $\mathcal{M}$-$\mathcal{R}_1$ in the context of multi-set topological spaces by utilizing the concepts of distinct $\mathcal{M}$-singletons and m-closure operator, investigating certain properties and characterizing them with some illustrative examples. Relationships with other $\mathcal{M}$-separation axioms are explored, and it is demonstrated that M-R_0 and M-R_1 are special cases of $\mathcal{M}$-regularity. Furthermore, we show that in the context of compact $\mathcal{M}$-spaces, $\mathcal{M}$-$\mathcal{R}_1$ is equivalent to whole $\mathcal{M}$-regularity. Finally, the hereditary property of these classes is examined.

Keywords: m-sets, closed m-sets, m-singletons sets, m-closure operator, $\mathcal{M}$-topology, $\mathcal{M}$-$\mathcal{R}_i$ spaces, $i = 0, 1$.

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1. Introduction

The concept of a multi-set, abbreviated as m-sets, holds a significant presence in both mathematical and computer science fields. In mathematical terms, an m-set is viewed as an extension of an ordinary set which define a set as a well-defined family of distinct objects. When the possibility of repeated occurrences of any object is introduced, it gives rise to the structure known as an m-set (or bag) [1, 12, 22]. An m-set $\mathcal{M}$ over a set $\mathcal{U}$, conveniently represented as $\mathcal{M} = \{k_1/u_1, k_2/u_2, \ldots, k_n/u_n\}$, indicates that the object u_i occurs k_i times in $\mathcal{M}$, with each multiplicity k_i being a non-negative integer. In studies involving m-sets, the number of occurrences of an object u in a finite m-set $\mathcal{M}$ is termed its multiplicity or characteristic value, typically symbolized by $C_M(u)$. A natural and straightforward example is the m-set representing the prime factors of a non-negative integer n. For instance, the factorization of 360 as $(2^3)(3^2)(5^1)$ yields the m-set $\mathcal{M} = \{3/u, 2/v, 1/t\}$, where $C_{\mathcal{M}}(u) = 3, C_{\mathcal{M}}(v) = 2$, and $C_{\mathcal{M}}(t) = 1$.

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In the context of ordinary sets, each element appears only once in a set, implying that all mathematical objects are unique. However, in the real world, particularly in the physical realm, significant repetition is observed such as multiple hydrogen atoms, water molecules, and DNA strands [4]. Even seemingly identical objects like coins, electrons, or grains of sand exist separately. This challenges the classical set theory's assumption, leading to three possible relationships between physical objects: they can be different, the same but separate, or coincident and identical. To clarify, we consider two physical objects as identical if they physically coincide, and as the same or equal if they are indistinguishable but possibly separate. This led to the introduction of multi-set Theory.

The m-sets have wide-ranging applications in logic, philosophy, linguistics, physics, mathematics, and computer science [3, 20, 21]. Singh et al. [19] conducted a comprehensive survey on m-sets and their various applications. The utilization of m-sets in decision-making is exemplified in [23]. In mathematics, m-sets find widespread applications in various mathematical branches. Ibrahim et al. [11] have established algebraic structures for $\mathcal{M}$-space. The topological structure of the m-sets (or $\mathcal{M}$-topological space) was introduced by Girish-Sunil in 2012 [10]. They proposed the $\mathcal{M}$-topologies arising from m-set relations. These authors also delved into the exploration of some notions and associated properties within $\mathcal{M}$-topologies [8]. Subsequently, many authors study various concepts in $\mathcal{M}$-topology such as $\mathcal{M}$-compactness [16], $\mathcal{M}$-connectedness [14, 15], $\mathcal{M}$-proximity [13], rough m-sets [9], and $\mathcal{M}$-filters [24]. Tripathy et al. in [17, 18] introduced generalized closed sets and studied the notion of $\mathcal{M}$-ideals in $\mathcal{M}$-topological space respectively. ElSheikh et al. [5] proposed the various forms of generalized open m-sets. They also, introduced the concept of supra $\mathcal{M}$-topologies [7] and studied their respective properties. Then the same authors [6] defined the separation axioms $\mathcal{M}$-$T_i(i = 0, 1, 2, 3, 4, 5)$ and studied some of their properties. However, there are further developments that have not yet been achieved in these settings.

This paper is an attempt to explore the theoretical aspects of m-set theory by extending the study of separation axioms in $\mathcal{M}$-topologies by introducing new classes of separation properties, namely $\mathcal{M}$-R_0 and $\mathcal{M}$-R_1. We investigate their properties and characterizations, providing illustrated examples. We establish their relationships with $\mathcal{M}$-$T_i(i = 0, 1, 2)$ and demonstrate that the classes $\mathcal{M}$-R_0 and $\mathcal{M}$-R_1 are special cases of $\mathcal{M}$-regularity. Additionally, we prove the equivalence of $\mathcal{M}$-R_1 and $W\mathcal{M}$-regularity in the case of compact $\mathcal{M}$-spaces, investigating the hereditary property of $\mathcal{M}$-R_0 and $\mathcal{M}$-R_1. Finally, we delve into the interconnection between these classes and other $\mathcal{M}$-separation properties. Through our study, we significantly contribute to the understanding of these new classes in $\mathcal{M}$-topology, offering an extensive analysis of their properties, relationships, and implications.

2. Basic definitions and preliminaries

In all this document, U refers to a set of objects from which m-sets are constructed, $\mathcal{M}$ is an m-set on U, $[U]^w$ refers to the m-sets space which is the set of all m-sets whose members are from U such that no member occurs more than w times, and $(\mathcal{M}, \tau)$ refers to $\mathcal{M}$-topological space (or $\mathcal{M}$-TS) on $\mathcal{M}$.

Firstly, we will recall a set of fundamental definitions and key findings that will be employed throughout this article. For more information see [1, 2, 10, 12, 22].

Definition 2.1 ([1, 12])**.** Consider the set of objects $U = \{u_1, u_2, u_3, \ldots, u_n\}$. An m-set $\mathcal{M}$ on U is a count function $C_{\mathcal{M}}$ from U to the set of non-negative integers N such that for $u \in U$, $C_{\mathcal{M}}(u) \in N$, where $C_{\mathcal{M}}(u)$ refers to the number of times occurs u in $\mathcal{M}$. The set U is named the ground or generic set of the class of all m-sets containing elements from U.

In other words, the m-set $\mathcal{M}$ drawn from the set U, conveniently represented as

$$\mathcal{M} = \{k_1/u_1, k_2/u_2, \ldots, k_n/u_n\}.$$

Here k_i represents the number of occurrences of the element u_i, $i = 0, 1, 2, \ldots, n$ within $\mathcal{M}$. However, the element $u \in U$ will have $C_{\mathcal{M}}(u) = 0$ if it is not belong to $\mathcal{M}$. If M is an m-set with element u repeated k-times, it is symbolized by $u \in^k \mathcal{M}$ and the negation of this case is denoted by $u \notin^k \mathcal{M}$. An m-set $\mathcal{M}$ is named an empty-set if $C_{\mathcal{M}}(u) = 0$ for all $u \in U$.

Definition 2.2 ([12])**.** $\mathcal{M} \in [U]^w$ is named a finite m-set if the number of distinct elements and their occurrences in $\mathcal{M}$ is finite, otherwise it is called an infinite m-set. The support or root of an m-set $\mathcal{M}$, denoted by $\mathcal{M}^*$, is given by $\mathcal{M}^* = \{u \in U : C_{\mathcal{M}}(u) > 0\}$.

Remark 2.3. We provide some new concepts in $\mathcal{M}$-topology based on the count function. When $C_{\mathcal{M}}(u) = 1$ for all $u \in U$, the m-sets align structurally with the class of sets. Consequently, any findings and definitions established under this condition are inherently equivalent to corresponding results in ordinary set theory.

Definition 2.4 ([12])**.** For two m-sets $\mathcal{M}, \mathcal{N} \in [U]^w$, we have:

- $\mathcal{M} \subseteq \mathcal{N}$ if $C_{\mathcal{M}}(u) \leqslant C_{\mathcal{N}}(u)$ for all $u \in U$;
- $D = \mathcal{M} \cup \mathcal{N}$ if $C_D(u) = \max\{C_{\mathcal{M}}(u), C_{\mathcal{N}}(u)\}$ for all $u \in U$;
- $D = \mathcal{M} \cap \mathcal{N}$ if $C_D(u) = \min\{C_{\mathcal{M}}(u), C_{\mathcal{N}}(u)\}$ for all $u \in U$;
- $D = \mathcal{M} \oplus \mathcal{N}$ if $C_D(u) = \min\{C_{\mathcal{M}}(u) + C_{\mathcal{N}}(u), w\}$ for all $u \in U$;
- $D = \mathcal{M} \ominus \mathcal{N}$ if $C_D(u) = \max\{C_{\mathcal{M}}(u) - C_{\mathcal{N}}(u), 0\}$ for all $u \in U$, where $\oplus$ and $\ominus$, represent m-set addition and m-set subtraction respectively.

Definition 2.5 ([12])**.** Consider $\mathcal{M} \in [U]^w$ and $H \subseteq \mathcal{M}$. The complement H^c of H in $[U]^w$is an element of $[U]^w$such that $H^c = \mathcal{M} \ominus H$.

Definition 2.6 ([10])**.** A sub m-set H of $\mathcal{M} \in [U]^w$ is named:

- a whole sub m-set of $\mathcal{M}$ if and only if $C_{\mathcal{M}}(u) = C_H(u)$ for all $u \in H^*$;
- a partial whole sub m-set of $\mathcal{M}$ if and only if $C_{\mathcal{M}}(u) = C_H(u)$ for some $u \in H^*$;
- a full sub m-set of $\mathcal{M}$ if and only if $H^* = \mathcal{M}^*$ with $C_H(u) \leqslant C_{\mathcal{M}}(u)$ for all $u \in H^*$.

Note. Evidently, $\emptyset$ could be a whole sub m-set of any m-set, however it does not achieve both of the other two types in the case of m-set is nonempty.

Based on the previously defined sub m-sets, Girish and John [10] also introduced power sub m-sets in the following manner.

Definition 2.7. For an m-set $M \in [U]^w$, we have:

- the set of all sub m-sets of $\mathcal{M}$, symbolized by $P(\mathcal{M})$, is named a power m-set of $\mathcal{M}$;
- the set of all whole sub m-sets of $\mathcal{M}$ is named a power whole m-set of $\mathcal{M}$ and is symbolized by $PW(\mathcal{M})$;
- the set of all full sub m-sets of $\mathcal{M}$, symbolized by $PF(\mathcal{M})$, is named a power full m-set of $\mathcal{M}$.

Note. In ordinary set theory, Cantor's power set theorem does not hold for m-sets. However, feasible to define a power m-set of a finite m-set M in a manner that maintains Cantor's power set theorem.

Definition 2.8 ([10])**.** Consider $\mathcal{M} \in [U]^w$. The power m-set $P(\mathcal{M})$ of $\mathcal{M}$ is the set of all sub m-sets of $\mathcal{M}$. We have $H \in P(\mathcal{M})$ if and only if $H \subseteq \mathcal{M}$. If $H = \emptyset$, then $H \in^1 P(M)$ and if $H \neq \emptyset$, then $H \in^k P(\mathcal{M})$, where $k = \prod_z \binom{|[\mathcal{M}]_z|}{|[H]_z|}$, the product $\prod_z$ is taken over by distinct elements z of the m-set H and $|[\mathcal{M}]z| = m$ if and only if $z \in^m \mathcal{M}$ and $|[H]z| = r$ if and only if $z \in^r H$, then $\binom{|[\mathcal{M}]_z|}{|[H]_z|} = \binom{m}{r} = \frac{m!}{r!(m-r)!}$.

The power set of an m-set is the support set of the power m-set and is denoted by $P^*(\mathcal{M})$.

Note. A power m-set is itself an m-set, but its support set is an ordinary set consisting of elements that are m-sets.

Definition 2.9 ([10])**.** Let $\mathcal{M} \in [U]^w$ and $\tau \subseteq P^*(\mathcal{M})$. Then τ is named a multi-set topology (or $\mathcal{M}$-topology) of $\mathcal{M}$ if τ satisfies the next properties.

- (i) $\mathcal{M}$ and ϕ are in τ.
- (ii) The m-set union of the elements of any sub-collection of τ is in τ.
- (iii) The m-set intersection of the elements of any finite sub-collection of τ is in τ.

The pair $(\mathcal{M}, \tau)$ is named an $\mathcal{M}$-topological space (or $\mathcal{M}$-TS). Each element in τ is named an open m-set. A sub m-set H of an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ is named a closed m-set if the m-set $H^c = \mathcal{M} \ominus H$ is open. In discrete $\mathcal{M}$-TS, every m-set is an open m-set as well as a closed m-set.

Definition 2.10 ([8])**.** For an $\mathcal{M}$-space $(\mathcal{M}, \tau)$ and a sub m-set H of $\mathcal{M}$, the class $\tau_H = \{G \cap H : G \in \tau\}$ is an $\mathcal{M}$-topology on H, called the subspace $\mathcal{M}$-topology.

Definition 2.11 ([8])**.** For a sub m-set H in an $\mathcal{M}$-space $(\mathcal{M}, \tau)$, the m-interior $\text{int}_M(H)$ of H is the m-set union of all open m-sets contained in H. That is, $\text{int}_m(H) = \cup\{G \subseteq \mathcal{M} : G \in \tau,\ G \subseteq H\}$ and $C_{\text{int}_m(H)}(u) = \max\{C_G(u) : G \subseteq H\}$.

On the other hand, the m-closure $\text{cl}_m(H)$ of H in $(\mathcal{M}, \tau)$, is the m-set intersection of all closed m-sets containing H. That is, $\text{cl}_m(H) = \cap\{G \subseteq \mathcal{M} : G \text{ is a closed m-set}, H \subseteq G\}$ and $C_{\text{cl}_m(H)}(u) = \min\{C_G(u) : H \subseteq G\}$.

Definition 2.12 ([3])**.** An m-set $\mathcal{M}$ is named simple if all its elements are the same. For example $\{k/u\}$. Moreover, k/u is named simple multipoint (briefly, m-point).

Definition 2.13 ([8])**.** For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, $u \in^k \mathcal{M}$, and $H \subseteq \mathcal{M}$, the m-set H is named a neighborhood of k/u if there is an open m-set N such that $u \in^k N$ and $C_N(v) = C_H(v)$ for all $v \neq u$ that is, $\mathcal{N}(k/u) = \{H \subseteq M : \exists\ N \in \tau \text{ such that } u \in^k N,\ C_N(v) = C_H(v),\ \forall\ v \neq u\}$ is the class of all τ-neighborhood of k/u.

Definition 2.14 ([6])**.** For $u, v \in U$ and $\mathcal{M} \in [U]^w$, we have:

(i) $u \in^k \mathcal{M}$ means that $C_{\mathcal{M}}(u) = k$, so $\{k/u\}$ is named whole $\mathcal{M}$-singleton sub m-set of $\mathcal{M}$ and $\{m/u\}$ is named M-singleton, where $0 < m < k$. This approach contains all m-points which considered as a sub m-set of $\mathcal{M}$;

(ii) the two $\mathcal{M}$-singletons $\{k/u\}$ and $\{m/v\}$ are called distinct if $u \neq v$;

(iii) for an m-set $\mathcal{M}$, if $u \in^m \mathcal{M}$ and $u \in^n \mathcal{M}$, then, $m = n$.

Theorem 2.15 ([8])**.** *For an $\mathcal{M}$-space $(\mathcal{M}, \tau)$, $H \subseteq \mathcal{M}$, and $\{k/u\} \subseteq \mathcal{M}$, we have:*

(1) $\{k/u\} \subseteq \text{cl}_m(H) \Longleftrightarrow$ *every open m-set* F *containing* $\{k/u\}$ *intersects* H*;*

(2) *a sub m-set of an $\mathcal{M}$-space $(\mathcal{M}, \tau)$ is an open m-set $\Longleftrightarrow$ it is a neighborhood of each of its elements with some multiplicity.*

Definition 2.16 ([6])**.** An $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ is named:

(i) $\mathcal{M}$-T_0 iff for any two $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $u \neq v$, there is τ-open m-set that contains one of the m-sets $\{k/u\}, \{m/v\}$ but not the other;

(ii) $\mathcal{M}$-T_1 iff for any two M-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $u \neq v$, there are $G, H \in \tau$ such that $\{k/u\} \subseteq H, \{m/v\} \not\subseteq H$ and $\{k/u\} \not\subseteq G, \{m/v\} \subseteq G$;

(iii) $\mathcal{M}$-T_2 iff for any two $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq M$ with $u \neq v$, there are $G, H \in \tau$ such that $\{k/u\} \subseteq G, \{m/v\} \subseteq H$ and $G \cap H = \emptyset$;

(iv) $\mathcal{M}$-regular (or $\mathcal{M}$-R_2) iff for any closed m-set F with $\{k/u\} \nsubseteq F$, there are $G, H \in \tau$ such that $\{k/u\} \subseteq G, F \subseteq H$ and $G \cap H = \emptyset$;

(v) $\mathcal{M}$-normal (or $\mathcal{M}$-R_3) iff for any closed m-sets F_1, F_2 with $F_1 \cap F_2 = \emptyset$, there are $G, H \in \tau$ such that $F_1 \subseteq G, F_2 \subseteq H$ and $G \cap H = \emptyset$;

(vi) $\mathcal{M}$-T_3 (resp. $\mathcal{M}$-T_4) iff it is both $\mathcal{M}$-regular (resp. $\mathcal{M}$-normal) and $\mathcal{M}$-T_1.

Remark 2.17 ([6]). $\mathcal{M}$-$T_4 \Longrightarrow \mathcal{M}$-$T_3 \Longrightarrow \mathcal{M}$-$T_2 \Longrightarrow \mathcal{M}$-$T_1 \Longrightarrow \mathcal{M}$-$T_0$.

Definition 2.18 ([16]). For $\mathcal{M} \in [U]^w$ and an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ on $\mathcal{M}$, then:

(i) the class $\mathcal{C} = \{N_i : N_i \subseteq \mathcal{M},\ i \in J\}$ is named a cover of $\mathcal{M}$ if $\mathcal{M} \subseteq \cup\{N : N \in \mathcal{C}\}$;

(ii) $(\mathcal{M}, \tau)$ is named a compact $\mathcal{M}$-space if for any open cover $\mathcal{C}$ of $\mathcal{M}$ there is a finite sub-cover of $\mathcal{C}$ covering $\mathcal{M}$.

Theorem 2.19 ([16]). *Every closed whole sub m-set of a compact $\mathcal{M}$-space $(\mathcal{M}, \tau)$ is compact.*

3. Basic properties of multi-set R_0 and R_1 spaces

In this part, we will introduce and discuss two new classes of separations properties named, $\mathcal{M}$-$R_i, i = 0, 1$ in $\mathcal{M}$-topologies, investigating certain properties and characterizing them.

Definition 3.1. An $\mathcal{M}$-TS (M, τ) is named:

(i) $\mathcal{M}$-R_0 iff for any open m-set F and any $\{k/u\} \subseteq F$, $cl_m(\{k/u\}) \subseteq F$;

(ii) $\mathcal{M}$-R_1 iff for any two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$, there are $G, H \in \tau$ such that $\{k/u\} \subseteq G, \{m/v\} \subseteq H$ and $G \cap H = \emptyset$.

Example 3.2. Consider the m-set $\mathcal{M} = \{2/u, 3/v, 1/t\}$ and an $\mathcal{M}$-topology τ on $\mathcal{M}$, where $\tau = \{\emptyset, \mathcal{M}, \{2/u\}, \{3/v\}, \{1/t\}, \{2/u, 3/v\}, \{2/u, 1/t\}, \{3/v, 1/t\}\}$. One can verify that $(\mathcal{M}, \tau)$ is an $\mathcal{M}$-R_0 space. Indeed, any open m-set in $(\mathcal{M}, \tau)$ contains the m-closure of all its m-points.

Example 3.3. For an m-set $\mathcal{M} = \{2/u, 3/v, 1/t\}$ and an $\mathcal{M}$-topology τ on $\mathcal{M}$, where $\tau = \{\emptyset, \mathcal{M}, \{3/v\}, \{2/u, 1/t\}\}$, one can verify that $(\mathcal{M}, \tau)$ is an $\mathcal{M}$-R_1 space. Indeed, for any two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$, which have different m-closure, there are disjoint open m-sets $G, H \in \tau$ containing $\{k/u\}, \{m/v\}$, respectively.

In the following, we investigate some characterizations of $\mathcal{M}$-R_i spaces, $i = 0, 1$.

Theorem 3.4. *For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, the next items are equivalent:*

(1) $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_0;

(2) $\{k/u\} \subseteq cl_m(\{m/v\})$ *iff* $\{m/v\} \subseteq cl_m(\{k/u\})$ *for all* $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$.

Proof.

(1) $\Longrightarrow$ (2). Assume that $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0 with $\{k/u\} \nsubseteq cl_m(\{m/v\})$, there is an open m-set K containing $\{k/u\}$ such that $K \cap \{m/v\} = \emptyset$. That is, $\{m/v\} \nsubseteq K$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0 and $\{k/u\} \subseteq K$, we get $cl_m(\{k/u\}) \subseteq K$, but $\{m/v\} \nsubseteq K$. Hence $\{m/v\} \nsubseteq cl_m(\{k/u\})$. The other implication is similar.

(2) $\Longrightarrow$ (1). Assume that H is an open sub m-set in $(\mathcal{M}, \tau)$ with $\{k/u\} \subseteq H$. To prove that $cl_m(\{k/u\}) \subseteq H$, let $\{m/v\} \nsubseteq H$, then $\{k/u\} \nsubseteq cl_m(\{m/v\})$ and from (2) we get $\{m/v\}\} \nsubseteq cl_m(\{k/u\})$. Therefore $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0. □

Theorem 3.5. *For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, the next items are equivalent:*

(1) $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_0;

(2) *for any two distinct* $\mathcal{M}$*-singletons* $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ *with* $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$*, we have* $cl_m(\{k/u\}) \cap cl_m(\{m/v\}) = \emptyset$.

Proof.

(1) $\Longrightarrow$ (2). Suppose that $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0 and $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ are two distinct $\mathcal{M}$-singletons with $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$, there is $\{n/t\} \subseteq \mathcal{M}$ such that $\{n/t\} \subseteq cl_m(\{k/u\})$ and $\{n/t\} \nsubseteq cl_m(\{m/v\})$. Suppose $\{k/u\} \subseteq cl_m(\{m/v\})$, then $cl_m(\{k/u\}) \subseteq cl_m(\{m/v\})$, which implies $\{n/t\} \subseteq cl_m(\{m/v\})$. This is a contradiction. Hence $\{k/u\} \nsubseteq cl_m(\{m/v\})$ and so, there is an open m-set H containing $\{k/u\}$ such that $H \cap \{m/v\} = \emptyset$, and the result holds.

(2) $\Longrightarrow$ (1). Consider two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$ and let G be an open m-set in $(\mathcal{M}, \tau)$ with $\{k/u\} \subseteq G$. To show that $cl_m(\{k/u\}) \subseteq G$, suppose that $\{m/v\} \nsubseteq G$. By assumption $cl_m(\{k/u\}) \cap cl_m(\{m/v\}) = \emptyset$ implies that $\{m/v\} \nsubseteq cl_m(\{k/u\})$. This means that $cl_m(\{k/u\}) \subseteq G$. Therefore, $(\mathcal{M}, \tau)$ is M-R_0. □

Based on the aforementioned theorems, it is possible to confirm the next corollary.

Corollary 3.6. *An* $\mathcal{M}$*-TS* $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$*-*R_0 *iff for any closed m-set* G *in* $(\mathcal{M}, \tau)$ *with* $\{k/u\} \nsubseteq G$*, we have* $cl_m(\{k/u\}) \cap G = \emptyset$.

Proposition 3.7. *If* $(\mathcal{M}, \tau_1)$ *is* $\mathcal{M}$*-*R_0 *and* $\tau_1 \leqslant \tau_2$*, then* $(\mathcal{M}, \tau_2)$ *is also an* $\mathcal{M}$*-*R_0 *space.*

Proof. It is immediate. □

Definition 3.8. For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ and a sub m-set F in $(\mathcal{M}, \tau)$, the multi-set kernel (or $\mathcal{M}$-kernel) of F symbolized by MK(F) is the m-set given by $MK(F) = \cap\{G \in \tau : F \subseteq G\}$ with $C_{MK(F)}(u) = \min\{C_G(u) : F \subseteq G,\ G \in \tau\}$. In particular, the $\mathcal{M}$-kernel of $\{k/u\} \subseteq M$ is the m-set $MK(\{k/u\}) = \cap\{H \in \tau : \{k/u\} \subseteq H\}$.

Lemma 3.9. *For an* $\mathcal{M}$*-TS* $(\mathcal{M}, \tau)$ *and a sub m-set* F *in* $(\mathcal{M}, \tau)$*, we have* $MK(F) = \cup\{\{k/u\} \subseteq \mathcal{M} : cl_M(\{k/u\}) \cap F \neq \emptyset\}$.

Proof. Assume that $\{k/u\} \subseteq MK(F)$ with $cl_m(\{k/u\}) \cap F = \emptyset$, we have $F \subseteq [cl_m(\{k/u\})]^c = H \in \tau$ and $\{k/u\} \nsubseteq H$. This contradicts that $\{k/u\} \subseteq MK(F)$. Hence $cl_m(\{k/u\}) \cap F \neq \emptyset$. On the other hand, Suppose that $cl_m(\{k/u\}) \cap F \neq \emptyset$ and $\{k/u\} \nsubseteq MK(F)$. This means that there is an open m-set K with $F \subseteq K$ and $\{k/u\} \nsubseteq K$. Now let $\{m/v\} \subseteq cl_m(\{k/u\}) \cap F \neq \emptyset$, then K is an open m-set containing $\{m/v\}$ but $\{k/u\} \nsubseteq K$. This is a contradiction. Therefore, $\{k/u\} \subseteq MK(F)$. □

Lemma 3.10. *For an* $\mathcal{M}$*-TS* $(\mathcal{M}, \tau)$ *and* $\{k/u\} \subseteq \mathcal{M}$*, we have* $\{m/v\} \subseteq cl_m(\{k/u\}) \Longleftrightarrow \{k/u\} \subseteq cl_m(\{m/v\})$.

Proof. It is straightforward. □

Proposition 3.11. *For an* $\mathcal{M}$*-TS* $(\mathcal{M}, \tau)$*, the following items are equivalent.*

(i) $(\mathcal{M}, \tau)$ *is* M-R_0.
(ii) $cl_m(\{k/u\}) \subseteq MK(\{k/u\})$ *for any* $\{k/u\} \subseteq \mathcal{M}$.

Proof. It stems from Definition 3.1 and Lemma 3.10. □

Theorem 3.12. *For an* $\mathcal{M}$*-TS* $(\mathcal{M}, \tau)$*, the following items are equivalent.*

(1) $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$*-*R_0.
(2) $H = MK(H)$ *for any closed m-set* H *in* $(\mathcal{M}, \tau)$.
(3) *If* H *is a closed m-set with* $\{k/u\} \subseteq H$*, then* $MK(\{k/u\}) \subseteq H$.
(4) $MK(\{k/u\}) \subseteq cl_m(\{k/u\})$ *for any* $\{k/u\} \subseteq M$.

Proof.

(1) $\Longrightarrow$ (2). Assume that H is closed m-set and $\{k/u\} \nsubseteq H$, we have $\{k/u\} \subseteq H^c$ that is an open m-set that contains $\{k/u\}$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0, we have $cl_m\{k/u\} \subseteq H^c$ implies that $cl_m(\{k/u\}) \cap H = \emptyset$. By Lemma 3.9, $\{k/u\} \nsubseteq MK(H)$. Hence $H = MK(H)$.

(2) $\Longrightarrow$ (3). It follows from fact that, $G \subseteq K \Longrightarrow MK(G) \subseteq MK(K)$.

(3) $\Longrightarrow$ (4). It is clear.

(4) $\Longrightarrow$ (1). Assume that $\{k/u\}$, $\{m/v\} \subseteq \mathcal{M}$ are two distinct $\mathcal{M}$-singletons with $\{k/u\} \subseteq cl_m(\{m/v\})$. By Lemma 3.9, we get $\{m/v\} \subseteq MK(\{k/u\})$. Since $\{k/u\} \subseteq cl_m(\{k/u\})$ which is a closed m-set. From (4) we get, $\{m/v\} \subseteq MK(\{k/u\}) \subseteq cl_m(\{k/u\})$. That is, $\{m/v\} \subseteq cl_m(\{k/u\})$. Hence (M, τ) is $\mathcal{M}$-R_0. $\square$

From above theorem, Proposition 3.11, and Lemma 3.10, one can verify the next result.

Corollary 3.13. *An $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0 iff $cl_m(\{k/u\}) = MK(\{k/u\})$ for any $\{k/u\} \subseteq \mathcal{M}$.*

Lemma 3.14. *For any two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\}$ in $(\mathcal{M}, \tau)$,*

$$MK(\{k/u\}) \neq MK(\{m/v\}) \Leftrightarrow cl_m(\{k/u\}) \neq cl_m(\{m/v\}).$$

Proof.

Necessity. Assume that $MK(\{k/u\}) \neq MK(\{m/v\})$, there is $\{n/t\} \subseteq \mathcal{M}$ such that $\{n/t\} \subseteq MK(\{k/u\})$ and $\{n/t\} \nsubseteq MK(\{m/v\})$. If $\{n/t\} \subseteq MK(\{k/u\})$, by Lemma 3.9, we get $\{k/u\} \cap cl_M(\{n/t\}) \neq \emptyset$ implies $\{k/u\} \subseteq cl_m(\{n/t\})$ that is, $cl_m(\{k/u\}) \subseteq cl_M(\{n/t\})$. Similarly, if $\{n/t\} \subseteq MK(\{m/v\})$, we get $\{m/v\} \subseteq cl_m(\{n/t\})$. Since $cl_m(\{k/u\}) \subseteq cl_m(\{n/t\})$ and $\{m/v\} \nsubseteq cl_m(\{n/t\})$, we get $\{m/v\} \nsubseteq cl_m(\{k/u\})$. Hence $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$.

Converse. Suppose $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$, there is $\{n/t\} \subseteq \mathcal{M}$ such that $\{n/t\} \subseteq cl_m(\{k/u\})$ and $\{n/t\} \subseteq cl_m(\{m/v\})$. So, there is an open m-set containing $\{n/t\}$ and so $\{k/u\}$ but not $\{m/v\}$. Therefore $\{m/v\} \nsubseteq MK(\{k/u\}$. Hence $MK(\{k/u\}) \neq MK(\{m/v\})$. $\square$

Theorem 3.15. *For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, the following items are equivalent.*

(i) *$(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0.*
(ii) *For any two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $MK(\{k/u\}) \neq MK(\{m/v\})$, we have $MK(\{k/u\}) \cap MK(\{m/v\}) = \emptyset$.*

Proof.

Necessity. Assume that $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0 and $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ are two distinct $\mathcal{M}$-singletons with $MK(\{k/u\}) \neq MK(\{m/v\}$. From Lemma 3.14, $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$. Suppose that $cl_m(\{k/u\}) \cap cl_m(\{m/v\}) \neq \emptyset$. This means there is $\{n/t\} \subseteq MK(\{k/u\} \cap MK(\{m/v\})$. If $\{n/t\} \subseteq MK(\{k/u\})$, by Lemma 3.10, $\{k/u\} \subseteq cl_m(\{n/t\})$ and so, $\{k/u\} \subseteq cl_m(\{n/t\})$. Clearly $\{k/u\} \subseteq cl_m(\{k/u\})$. By Theorem 3.5, we have $cl_m(\{k/u\}) = cl_m(\{n/t\}$. Similarly, if $\{n/t\} \subseteq MK(\{m/v\})$, we get $cl_m(\{m/v\}) = cl_m\{n/t\} = cl_m(\{k/u\})$. This is a contradiction. Hence $\{k/u\} \cap cl_m(\{n/t\} = \emptyset$.

Converse. Suppose that $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ are two distinct $\mathcal{M}$-singletons with $cl_m(\{k/u\} \neq cl_m(\{m/v\}$. From Lemma 3.14, $MK(\{k/u\}) \neq MK(\{m/v\})$ implies that $\{k/u\} \cap cl_m(\{m/v\} = \emptyset$. Assume that $cl_m(\{k/u\}) \cap cl_m(\{m/v\}) \neq \emptyset$, there is $\{n/t\} \subseteq \mathcal{M}$ such that $\{n/t\} \subseteq cl_m(\{k/u\})$ and $\{n/t\} \subseteq cl_m(\{m/v\})$. So, by Lemma 3.10, $\{k/u\} \subseteq MK(\{n/t\})$, $\{m/v\} \subseteq MK(\{n/t\})$. By Lemma 3.9, $MK(\{k/u\}) \cap MK(\{n/t\}) \neq \emptyset$ and $MK(\{m/v\}) \cap MK(\{n/t\}) \neq \emptyset$ and by hypothesis, $MK(\{k/u\})) = MK(\{n/t\})$ and $MK(\{m/v\}) = MK(\{n/t\}) = MK(\{k/u\})$. So, $MK(\{k/u\}) \cap MK(\{m/v\}) \neq \emptyset$. This is a contradiction. Hence $cl_m(\{k/u\}) \cap cl_m(\{m/v\}) = \emptyset$. By Theorem 3.5, $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0. $\square$

Theorem 3.16. *For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, the following items are equivalent.*

(i) $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_1.

(ii) *For any two distinct* $\mathcal{M}$*-singletons* $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ *with* $MK(\{k/u\}) \neq MK(\{m/v\})$, *there are* $G, K \in \tau$ *such that* $cl_m(\{k/u\}) \subseteq G$, $cl_m(\{m/v\}) \subseteq K$ *and* $G \cap K = \emptyset$.

Proof. It follows from Lemma 3.10. □

From Definition 3.1, Lemma 3.14, and above theorem, one can verify the next corollary.

Corollary 3.17. *For an* $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, *the following properties are equivalent.*

(i) $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_1.

(ii) *For every distinct* $\mathcal{M}$*-singletons* $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ *with* $\{k/u\} \nsubseteq cl_m(\{m/v\})$, *there are* $G,\ K \in \tau$ *such that* $\{k/u\} \subseteq G$, $\{m/v\} \subseteq K$, *and* $G \cap K = \emptyset$.

(iii) *For any two distinct* $\mathcal{M}$*-singletons* $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ *with* $cl_m(\{k/u\} \neq cl_m(\{m/v\})$, *there are* $G,\ K \in \tau$ *such that* $cl_m(\{k/u\}) \subseteq G, cl_m(\{m/v\}) \subseteq K$, *and* $G \cap K = \emptyset$.

Theorem 3.18. *Every* $\mathcal{M}$-R_1 *is* $\mathcal{M}$-R_0.

Proof. Assume that $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ are two distinct M-singletons with $\{k/u\} \nsubseteq cl_m(\{m/v\})$, then $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1, there is $F \in \tau$ such that $\{m/v\} \subseteq F$, $\{k/u\} \nsubseteq F$, and so, then $\{m/v\} \nsubseteq cl_m(\{k/u\})$. Therefore $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0. □

The next example demonstrates that the converse of Theorem 3.18 may not always hold.

Example 3.19. Consider an infinite set U and $\mathcal{M} \in [U]^w$, then the class $\tau_\infty = \{\emptyset\} \cup \{K \subseteq \mathcal{M} : K^c \text{ is finite}\}$ is an $\mathcal{M}$-topology on M, called a cofinite $\mathcal{M}$-topology. Now, to show that $(\mathcal{M}, \tau_\infty)$ is $\mathcal{M}$-R_0 but not $\mathcal{M}$-R_1, assume that $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ are two distinct $\mathcal{M}$-singletons with $\{k/u\} \subseteq (\{m/v\})^c$. Clearly, $(\{k/u\})^c, (\{m/v\})^c \in \tau_\infty$, we have $\{k/u\}$ is a closed m-set, this implies that $cl_m(\{k/u\}) = \{k/u\} \subseteq (\{m/v\})^c$. Hence $(\mathcal{M}, \tau_\infty)$ is $\mathcal{M}$-R_0. On other hand, suppose $(\mathcal{M}, \tau_\infty)$ is $\mathcal{M}$-R_1, we have for any two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with, $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$, there are $F,\ G \in \tau_\infty$ such that $\{k/u\} \subseteq F$, $\{m/v\} \subseteq G$, and $F \cap G = \emptyset$. Hence $F^c \cup G^c = U$. Since $F^c,\ G^c$ are two finite sub m-sets of $\mathcal{M}$, this contradicts that U is infinite. Hence $(\mathcal{M}, \tau_\infty)$ is not $\mathcal{M}$-R_1.

4. More characterizations and relations

In the following discussion, we will explore more properties and some related theorems of $\mathcal{M}$-$R_i, i = 0, 1$. The interconnections between these classes and other separation properties such as M-$T_i, i = 0, 1, 2$ are examined.

Theorem 4.1. *Every* $\mathcal{M}$*-subspace* (N, τ_N) *of* $\mathcal{M}$-R_i *space* $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_i, $i = 0, 1$.

Proof. For $i = 1$, assume $\{k/u\}, \{m/v\}$ are distinct M-singletons in (N, τ_N) with $cl_m((\{k/u\}) \neq cl_m(\{m/v\})$, we have $\{k/u\}, \{m/v\}$ are distinct $\mathcal{M}$-singletons in $(\mathcal{M}, \tau)$ with $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1, there are $F,\ G \in \tau$ such that $\{k/u\} \subseteq F$, $\{m/v\} \subseteq G$ with $F \cap G = \emptyset$. So that, there are open m-sets, $P_N = N \cap F \in \tau_N$ and $Q_N = N \cap G \in \tau_N$ which are containing $\{k/u\}$ and $\{m/v\}$, respectively such that $P_N \cap Q_N = \emptyset$. Therefore (N, τ_N) is $\mathcal{M}$-R_1. The proof for $i = 0$ is analogous. □

Theorem 4.2. *Every* $\mathcal{M}$-T_i *space* $(\mathcal{M}, \tau)$ *is* $\mathcal{M}$-R_{i-1}, $i = 1, 2$.

Proof. For $i = 1$, let $H \in \tau$ with $\{k/u\} \subseteq H$. We want to show that $cl_m(\{k/u\}) \subseteq H$. Assume that $\{m/v\} \nsubseteq H$, we have $\{k/u\} \nsubseteq cl_m(\{m/v\})$, where $\{k/u\}, \{m/v\}$ are distinct $\mathcal{M}$-singletons in $(\mathcal{M}, \tau)$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-T_1, there is $G \in \tau$ such that $\{m/v\} \subseteq G$, $\{k/u\} \nsubseteq G$ and so, $\{m/v\} \subseteq cl_m\{k/u\}$. Thus, $cl_m\{k/u\} \subseteq H$. Hence $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0. The proof for $i = 0$ is obvious. □

The next example explains that the converse of Theorem 4.2 does not necessarily hold.

Example 4.3. Consider the m-set $\mathcal{M} = \{2/u, 3/v, 1/t\}$ with an $\mathcal{M}$-topology $\tau = \{\emptyset, \mathcal{M}, \{3/v\}, \{2/u, 1/t\}\}$ on $\mathcal{M}$. One can verify that $(\mathcal{M}, \tau)$ is an $\mathcal{M}$-R_0 space but doesn't $\mathcal{M}$-T_1. Indeed, for two distinct $\mathcal{M}$-singletons $\{1/u\}, \{1/t\} \subseteq \mathcal{M}$ there are no open sub m-sets G, H of $\mathcal{M}$ such that $\{1/u\} \subseteq G$, $\{1/t\} \nsubseteq G$, and $\{1/u\} \subseteq H$, $\{1/t\} \nsubseteq H$.

Theorem 4.4. *For an $\mathcal{M}$-TS $(\mathcal{M}, \tau)$, we have:*

(1) *$(\mathcal{M}, \tau)$ is $\mathcal{M}$-T_1 $\Longleftrightarrow$ it is both $\mathcal{M}$-R_0 and $\mathcal{M}$-T_0;*
(2) *$(\mathcal{M}, \tau)$ is $\mathcal{M}$-T_2 $\Longleftrightarrow$ it is both $\mathcal{M}$-R_1 and $\mathcal{M}$-T_1.*

Proof.

(1) Necessity. It can be inferred from Theorem 4.2 and Remark 2.17.

Converse. Assume that $\{k/u\}, \{m/v\}$ are distinct $\mathcal{M}$-singletons in $(\mathcal{M}, \tau)$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-T_0 and $\mathcal{M}$-R_0, we have $cl_m\{k/u\} \neq cl_m\{m/v\}$. From Theorem 3.5, $cl_m\{k/u\} \cap cl_m\{m/v\} = \emptyset$. Thus, $[cl_m\{m/v\}]^c$ is an open m-set containing $\{k/u\}$ not $\{m/v\}$ and $[cl_m\{k/u\}]^c$ is an open m-set containing $\{m/v\}$ not $\{k/u\}$.

Thus $(\mathcal{M}, \tau)$ is $\mathcal{M}$- T_1.

(2) Necessity. It can be deduced from Theorem 4.2 and Remark 2.17.

Converse. Consider two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $\{k/u\} \nsubseteq cl_m\{m/v\}$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_0, then $\{m/v\} \nsubseteq cl_m(\{k/u\})$. So that, $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$. Also, $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1. So, there is disjoint open m-sets G, K such that $\{k/u\} \subseteq G$ and $\{m/v\} \subseteq K$. Hence $(\mathcal{M}, \tau)$ is $\mathcal{M}$-T_2. □

Corollary 4.5. *An $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ is M-T_2 $\Longleftrightarrow$ it is both $\mathcal{M}$-R_1 and M-T_0.*

Proof. It is consequence of that of Theorem 4.4. □

Theorem 4.6. *Every $\mathcal{M}$-regular($\mathcal{M}$-R_2) space is $\mathcal{M}$-R_i, $i = 0, 1$.*

Proof. For $i = 1$, consider two distinct $\mathcal{M}$-singletons $\{k/u\}, \{m/v\} \subseteq \mathcal{M}$ with $cl_m\{k/u\} \neq cl_m(\{m/v\})$, then either $\{k/u\} \nsubseteq cl_m(\{m/v\})$ or $\{m/v\} \nsubseteq cl_m(\{k/u\})$. Without loss of generality, suppose that $\{k/u\} \nsubseteq cl_m(\{m/v\})$, where $cl_m(\{m/v\})$ is a closed m-set with $\{k/u\} \nsubseteq cl_m(\{m/v\})$. Since $(\mathcal{M}, \tau)$ is $\mathcal{M}$-regular, there are disjoint open m-sets G, K with $\{k/u\} \subseteq G$ and $\{m/v\} \subseteq cl_m(\{m/v\}) \subseteq K$. Hence $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1. The proof for $i = 0$ is analogous. □

Based on Theorems 3.18 and 4.6, we conclude the next result.

Corollary 4.7. $\mathcal{M}$-$R_2 \Longrightarrow \mathcal{M}$-$R_1 \Longrightarrow \mathcal{M}$-$R_0$.

The next example explains that the reverse of Theorem 4.6 does not necessarily hold.

Example 4.8. From Example 3.19, we demonstrated that $(\mathcal{M}, \tau_\infty)$ is $\mathcal{M}$-R_0 but it is not $\mathcal{M}$-R_1 and so, it is not $\mathcal{M}$-regular (or $\mathcal{M}$-R_2).

Definition 4.9. An $\mathcal{M}$-TS $(\mathcal{M}, \tau)$ is named whole $\mathcal{M}$-regular (or $W\mathcal{M}$-regular) iff for any $\{k/u\} \subseteq \mathcal{M}$ and any closed whole sub m-set F of $\mathcal{M}$ with $\{k/u\} \nsubseteq F$, there are disjoint open m-sets G, H such that $\{k/u\} \subseteq G$, $F \subseteq H$.

Note. Evidently, every $\mathcal{M}$-regular space is $W\mathcal{M}$-regular. However, the converse is not necessarily hold. In fact, not every sub m-set of $\mathcal{M}$ is a whole sub m-set.

Theorem 4.10. *A compact $\mathcal{M}$-space $(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1 $\Longleftrightarrow$ it is $W\mathcal{M}$-regular.*

Proof.

Necessity. Assume that $(\mathcal{M}, \tau)$ is a compact $\mathcal{M}$-R_1 space. To show that $(\mathcal{M}, \tau)$ is $W\mathcal{M}$-regular, suppose that H is a closed whole sub m-set in $(\mathcal{M}, \tau)$ and $\{k/u\} \subseteq \mathcal{M}$ with $\{k/u\} \nsubseteq H$. Now for all $\{m/v\} \subseteq H$, $cl_m(\{m/v\}) \subseteq H$. Since $\{k/u\} \nsubseteq H$, then $\{k/u\} \nsubseteq cl_m(\{m/v\})$ and so, $cl_m(\{k/u\}) \neq cl_m(\{m/v\})$. Since

$(\mathcal{M}, \tau)$ is $\mathcal{M}$-R_1, then for any $\{m/\nu\} \subseteq H$, there are disjoint open m-sets $K^{\{m/\nu\}}$, $G^{\{m/\nu\}}$ such that $\{k/u\} \subseteq K^{\{m/\nu\}}$ and $\{m/\nu\} \subseteq G^{\{m/\nu\}}$. So that $\mathcal{A} = \{G_i^{\{m/\nu\}} : \{m/\nu\} \subseteq H, i \in J\}$ is an open cover of H. By Theorem 2.19, H is a compact m-set, so there is a finite subclass $\{G_1^{\{m/\nu\}}, G_2^{\{m/\nu\}}, \ldots, G_n^{\{m/\nu\}}\}$ of $\mathcal{A}$ that covers H. Consider corresponding sub-class $\{K_1^{\{m/\nu\}}, K_2^{\{m/\nu\}}, \ldots, K_n^{\{m/\nu\}}\}$ of $\{K_i^{\{m/\nu\}} : \{m/\nu\} \subseteq H\}$. It is clear that, $P = \cap_{i=1}^n K_i$ and $Q = \cup_{i=1}^n G_i$ are open m-sets and P is disjoint from Q because $P \subseteq K_i$ for all i, which is disjoint from the corresponding G_i, with $\{k/u\} \subseteq P$, $H \subseteq Q$. Therefore, $(\mathcal{M}, \tau)$ is $W\mathcal{M}$-regular. The proof of the converse follows from Theorem 4.6. □

5. Conclusion

$\mathcal{M}$-topology represents the extension of an ordinary topology into the context of m-sets. This article provides and analyzes some novel separation properties termed $\mathcal{M}$-R_0 and $\mathcal{M}$-R_1 in M-topological spaces, investigating certain properties and characterizing them with some illustrative examples. The interconnections between these classes and other M-separation properties are explored. It is demonstrated that $\mathcal{M}$-R_0 and $\mathcal{M}$-R_1 are special instances of $\mathcal{M}$-regularity. Additionally, in the context of compact $\mathcal{M}$-spaces, it is shown that $\mathcal{M}$-R_1 is equivalent to $W\mathcal{M}$-regularity. Finally, the hereditary property of these classes is explored.

As a future work, we intend to discuss the current separation axioms using the other generalizations of open m-sets. Additionally, we plan to extend the characterizations of these classes to the contexts of fuzzy (soft) m-sets settings and look at the possible applications of them.

Competing interests

The authors state that they have no conflicts of interest regarding the publication of this article.

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