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Fuzzy set approach to ideal theory on Sheffer stroke BE-algebras

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Abstract

This paper aims to apply fuzzy set theory to Sheffer stroke BE-algebras. The concepts of fuzzy SBE-ideals and anti-fuzzy SBE-ideals of Sheffer stroke BE-algebras are introduced, and their important properties are investigated. Characterizations of SBE-ideals of SBE-algebras are given in terms of fuzzy SBE-ideals and anti-fuzzy SBE-ideals. Relationships between fuzzy SBE-ideals and anti-fuzzy SBE-ideals and their level subsets are discussed. We especially characterize fuzzy SBE-ideals and anti-fuzzy SBE-ideals by their level subsets.

Keywords: SBE-ideal, fuzzy SBE-ideal, anti-fuzzy SBE-ideal, level subset, Sheffer stroke BE-algebra.

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1. Introduction

Sheffer stroke (Sheffer operation), which is one of the two operators that can be used by itself without any other logical operators, was originally introduced by Sheffer to build a logical formal system [13]. Since it offers novel axiom systems that are straightforward and easily adaptable for a variety of algebraic structures owing to its commutative property, there are numerous uses for this operation in algebraic structures, such as Sheffer Stroke Hilbert algebras [8], Sheffer stroke BCK-algebras [9], Sheffer stroke BCH-algebras [5], strong Sheffer stroke nonassociative MV-algebras [6], Sheffer stroke BL-algebras [10], Sheffer stroke UP-algebras [7], and Sheffer stroke BE-algebras [1, 2, 11, 12].

Sheffer stroke BE-algebras, or SBE-algebras, were presented by Katican et al. [2] and links between SBE-algebras and BE-algebras were examined. It is demonstrated by presenting an SBE-filter and an SBE-subalgebra on an SBE-algebra that any SBE-filter of an SBE-algebra is an SBE-subalgebra, but the inverse is not true. Relationships between SBE-algebras and Sheffer stroke Hilbert algebras were studied. Recently, fuzzy Sheffer stroke BE-algebras were explored by Oner et al. [11]. They defined concepts of fuzzy SBE-subalgebras, fuzzy SBE-filters, and the Cartesian product of fuzzy points and fuzzy subsets

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on SBE-algebras. Additionally, they looked into how different SBE-filters on SBE-algebras related to one another.

As previously stated, it motivated us to study fuzzy set theory in SBE-algebras. We introduce the concepts of fuzzy SBE-ideals and anti-fuzzy SBE-ideals of SBE-algebras, and investigate their important properties. It is demonstrated that every fuzzy SBE-ideal of an SBE-algebra is a fuzzy SBE-subalgebra, but the inverse does not always hold in general. Also, characterizations of SBE-ideals of SBE-algebras are given in terms of fuzzy SBE-ideals and anti-fuzzy SBE-ideals. Finally, we will discuss the relationships between fuzzy SBE-ideals and anti-fuzzy SBE-ideals and their level subsets.

2. Preliminaries

In this section, definitions and notions about BE-algebras and Sheffer stroke BE-algebras are given.

Definition 2.1 ([4])**.** An algebra $\langle X, \rightsquigarrow, 1\rangle$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following conditions:

(BE-1) $a \rightsquigarrow a = 1$ for all $a \in X$;
(BE-2) $a \rightsquigarrow 1 = 1$ for all $a \in X$;
(BE-3) $1 \rightsquigarrow a = a$ for all $a \in X$;
(BE-4) $a \rightsquigarrow (b \rightsquigarrow c) = b \rightsquigarrow (a \rightsquigarrow c)$ for all $a, b, c \in X$.

Example 2.2 ([4])**.** Consider $X = \{1, a, b, c, d\}$ and a binary operation $\rightsquigarrow$ on X has the Cayley table:

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
1	1	1	1	1	1

Then $\langle X, \rightsquigarrow, 1\rangle$ is a BE-algebra.

Definition 2.3 ([2])**.** Let $\langle S, \uparrow\rangle$ be a groupoid. The operation $\uparrow$ on S is said to be a *Sheffer stroke* (or Sheffer operation) if it satisfies the following conditions:

(S1) $a \uparrow b = b \uparrow a$ for all $a, b \in S$;
(S2) $(a \uparrow a) \uparrow (a \uparrow b) = a$ for all $a, b \in S$;
(S3) $a \uparrow ((b \uparrow c) \uparrow (b \uparrow c)) = ((a \uparrow b) \uparrow (a \uparrow b)) \uparrow c$ for all $a, b, c \in S$;
(S4) $(a \uparrow ((a \uparrow a) \uparrow (b \uparrow b))) \uparrow (a \uparrow ((a \uparrow a) \uparrow (b \uparrow b))) = a$ for all $a, b \in S$.

Definition 2.4 ([2])**.** A *Sheffer stroke BE-algebra* (shortly, SBE-algebra) is a structure $\langle X, \uparrow, 1\rangle$ of type $(2, 0)$ such that 1 is the constant in X, $\uparrow$ is a Sheffer operation on X, and the following conditions are satisfied:

(SBE-1) $a \uparrow (a \uparrow a) = 1$ for all $a \in X$;
(SBE-2) $a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) = b \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))$ for all $a, b, c \in X$.

In what follows, let X denote an SBE-algebra $\langle X, \uparrow, 1\rangle$ unless otherwise specified.

Example 2.5 ([2])**.** Consider $X = \{0, a, b, c, 1\}$ and a binary operation $\uparrow$ on X has the Cayley table:

$\uparrow$	0	a	b	c	1
0	1	1	1	1	1
a	1	b	1	1	b
b	1	1	a	1	a
c	1	1	1	c	c
1	1	b	a	c	0

Then $\langle X, \uparrow, 1\rangle$ is an SBE-algebra. However, $\langle X, \uparrow, 1\rangle$ is not a BE-algebra since $a \uparrow a \neq 1$.

Example 2.6. Consider the BE-algebra $\langle X, \rightsquigarrow, 1\rangle$ in Example 2.2. We see that $\langle X, \rightsquigarrow, 1\rangle$ is not an SBE-algebra since $a \rightsquigarrow b \neq b \rightsquigarrow a$.

From Examples 2.2, 2.5, and 2.6, we have that every SBE-algebra may not be a BE-algebra and every BE-algebra may not be an SBE-algebra. In 2022, Katican et al. [2] gave relationships between BE-algebras and SBE-algebras in the following two theorems.

Theorem 2.7 ([2]). *Let $\langle X, \uparrow, 1\rangle$ be an SBE-algebra and a binary operation $\rightsquigarrow$ on X defined by $a \rightsquigarrow b = a \uparrow (b \uparrow b)$ for all $a, b \in X$. Then $\langle X, \rightsquigarrow, 1\rangle$ is a BE-algebra.*

Theorem 2.8 ([2]). *Let $\langle X, \rightsquigarrow, 1\rangle$ be a BE-algebra and 0 be a constant of X such that $0 \neq 1$. Define a unary operation $'$ and a binary operation $\uparrow$ on X by $a' = a \rightsquigarrow 0$ and $a \uparrow b = a \rightsquigarrow b'$ for all $a, b \in X$. Then $\langle X, \uparrow, 1\rangle$ is an SBE-algebra.*

Definition 2.9. A nonempty subset I of an SBE-algebra X is called an SBE-ideal of X if

(SBEI-1) $(\forall x \in X)(\forall a \in I)$, $x \uparrow (a \uparrow a) \in I$;
(SBEI-2) $(\forall x \in X)(\forall a, b \in I)$, $(a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x) \in I$.

Example 2.10. Consider $\langle X, \uparrow, 1\rangle$, where $X = \{0, a, b, 1\}$ and a binary operation $\uparrow$ has the Cayley table:

$\uparrow$	0	a	b	1
0	1	1	1	1
a	1	b	1	b
b	1	1	a	a
1	1	b	a	0

Then $\langle X, \uparrow, 1\rangle$ is an SBE-algebra [2]. We see that $\{1\}$, $\{1, a\}$, and X are SBE-ideals of X.

Definition 2.11 ([2]). An SBE-algebra X is called

(1) commutative if $(a \uparrow (b \uparrow b)) \uparrow (b \uparrow b) = (b \uparrow (a \uparrow a)) \uparrow (a \uparrow a)$ for all $a, b \in X$;
(2) self-distributive if $a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) = (a \uparrow (b \uparrow b)) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))$ for all $a, b, c \in X$.

Definition 2.12 ([2]). Define a relation $\precsim$ on an SBE-algebra X by for all $a, b \in X$,

$$a \precsim b \text{ if and if only if } a \uparrow (b \uparrow b) = 1.$$

Definition 2.13. An SBE-algebra X is called transitive if $b \uparrow (c \uparrow c) \precsim (a \uparrow (b \uparrow b)) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))$ for all $a, b, c \in X$.

For the study of Sheffer stroke BE-algebras, the following lemma is crucial.

Lemma 2.14 ([2]). *Let X be an SBE-algebra and $a, b, c \in X$. Then the following statements hold:*

(1) $a \uparrow (1 \uparrow 1) = 1$;
(2) $1 \uparrow (a \uparrow a) = a$;
(3) $a \uparrow ((b \uparrow (a \uparrow a)) \uparrow (b \uparrow (a \uparrow a))) = 1$;
(4) $a \uparrow (((a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) \uparrow (a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) = 1$;
(5) $(a \uparrow 1) \uparrow (a \uparrow 1) = a$;
(6) $((a \uparrow b) \uparrow (a \uparrow b)) \uparrow (a \uparrow a) = 1$ *and* $((a \uparrow b) \uparrow (a \uparrow b)) \uparrow (b \uparrow b) = 1$;
(7) $a \uparrow ((a \uparrow b) \uparrow (a \uparrow b)) = a \uparrow b = ((a \uparrow b) \uparrow (a \uparrow b)) \uparrow b$;
(8) *if* $a \precsim b$, *then* $b \uparrow b \precsim a \uparrow a$;
(9) $a \precsim b \uparrow (a \uparrow a)$;
(10) $b \precsim (b \uparrow (a \uparrow a)) \uparrow (a \uparrow a)$;
(11) *if X is self-distributive, then* $a \precsim b$ *implies* $b \uparrow c \precsim a \uparrow c$;
(12) *if X is self-distributive, then* $b \uparrow (c \uparrow c) \precsim (c \uparrow (a \uparrow a)) \uparrow ((b \uparrow (a \uparrow a) \uparrow (b \uparrow (a \uparrow a)))$.

3. Fuzzy SBE-ideals and anti-fuzzy SBE-ideals

In this section, fuzzy SBE-ideals and anti-fuzzy SBE-ideals of SBE-algebras are introduced and their properties are investigated. Moreover, we discuss the relationships among SBE-ideals, fuzzy SBE-ideals, and anti-fuzzy SBE-ideals of SBE-algebras.

A fuzzy set [16] of a nonempty set X is defined to be a mapping $\mu : X \longrightarrow [0,1]$. Oner et al. [11] studied fuzzy Sheffer stroke BE-algebras and defined a fuzzy SBE-subalgebra on an SBE-algebra as follows.

Definition 3.1 ([11])**.** A fuzzy set μ in an SBE-algebra X is called a fuzzy SBE-subalgebra of X if $\mu(a \uparrow (b \uparrow b)) \geqslant \min\{\mu(a), \mu(b)\}$ for all $a, b \in X$.

Example 3.2. Consider the SBE-algebra X in Example 2.5. Define a fuzzy set μ in X by

$$\mu \colon X \to \{0,1\}, \quad x \mapsto \begin{cases} 0.3, & \text{if } x = 0, \\ 0.5, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy SBE-subalgebra of X.

Now, we define a fuzzy SBE-ideal of an SBE-algebra as the following.

Definition 3.3. A fuzzy set μ in an SBE-algebra X is called a fuzzy SBE-ideal of X if

(FSI-1) $\mu(a \uparrow (b \uparrow b)) \geqslant \mu(b)$ for all $a, b \in X$;
(FSI-2) $\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\mu(a), \mu(b)\}$ for all $a, b, c \in X$.

Example 3.4. Consider an SBE-algebra $\langle X, \uparrow, 1\rangle$ defined in Example 2.10. Define a fuzzy set μ in X by $\mu(1) = 0.9$, $\mu(a) = 0.7$ and $\mu(b) = \mu(0) = 0.2$. Then μ is a fuzzy SBE-ideal of X.

Proposition 3.5. *Every fuzzy SBE-ideal of an SBE-algebra is a fuzzy SBE-subalgebra.*

Proof. It follows from (FSI-1). □

The following example illustrates why the inverse of Proposition 3.5 is not true in general.

Example 3.6. The fuzzy SBE-subalgebra μ of an SBE-algebra X in Example 3.2 is not a fuzzy SBE-ideal of X because

$$\mu((a \uparrow ((b \uparrow (0 \uparrow 0)) \uparrow (b \uparrow (0 \uparrow 0)))) \uparrow (0 \uparrow 0)) = \mu(0) = 0.3 < 0.5 = \min\{\mu(a), \mu(b)\}.$$

Proposition 3.7. *If μ is a fuzzy SBE-ideal of an SBE-algebra X and $a, b \in X$, then*

$$\mu((a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) \geqslant \mu(a).$$

Proof. Assume that μ is a fuzzy SBE-ideal of X and $a, b \in X$. Using Lemma 2.14 (2) and (FSI-2), we get

$$\begin{aligned} \mu((a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) &= \mu((a \uparrow ((1 \uparrow (b \uparrow b)) \uparrow (1 \uparrow (b \uparrow b)))) \uparrow (b \uparrow b)) \\ &\geqslant \min\{\mu(a), \mu(1)\} = \min\{\mu(a), \mu(a \uparrow (a \uparrow a))\} \geqslant \mu(a). \end{aligned}$$

□

Proposition 3.8. *Every fuzzy SBE-ideal μ of an SBE-algebra X is order-preserving.*

Proof. Let $a, b \in X$ be such that $a \precsim b$. Then $a \uparrow (b \uparrow b) = 1$. From Lemma 2.14 (2) and Proposition 3.7, we have

$$\mu(b) = \mu(1 \uparrow (b \uparrow b)) = \mu((a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) \geqslant \mu(a).$$

Hence, μ is order-preserving. □

Proposition 3.9. *If μ is a fuzzy SBE-ideal of an SBE-algebra X and $a \in X$, then*

$$\mu(1) \geqslant \mu(a). \tag{3.1}$$

Proof. Let $a \in X$. By using (SBE-1) and (FSI-1), we have $\mu(1) = \mu(a \uparrow (a \uparrow a)) \geqslant \mu(a)$. □

Proposition 3.10. *Let μ be a fuzzy set in an SBE-algebra X, which satisfies $\mu(1) \geqslant \mu(a)$ and*

$$\mu(a \uparrow (c \uparrow c)) \geqslant \min\{\mu(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))), \mu(b)\} \tag{3.2}$$

for all $a, b, c \in X$. Then μ is order-preserving.

Proof. Let $a, b \in X$ be such that $a \precsim b$. Then

$$\begin{aligned} \mu(b) = \mu(1 \uparrow (b \uparrow b)) &\geqslant \min\{\mu(1 \uparrow ((a \uparrow (b \uparrow b)) \uparrow (a \uparrow (b \uparrow b)))), \mu(a)\} \\ &= \min\{\mu(1 \uparrow (1 \uparrow 1)), \mu(a)\} = \min\{\mu(1), \mu(a)\} = \mu(a). \end{aligned}$$

Hence, μ is order-preserving. □

Theorem 3.11. *A fuzzy set μ in a transitive SBE-algebra X is a fuzzy SBE-ideal of X if and only if it satisfies (3.1) and (3.2).*

Proof. Assume that μ is a fuzzy SBE-ideal of X. By Proposition 3.9, we have $\mu(1) \geqslant \mu(a)$ for all $a \in X$. Since X is transitive, we get

$$(b \uparrow (c \uparrow c)) \uparrow (c \uparrow c) \precsim (a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))$$

for all $a, b, c \in X$. Thus,

$$\begin{aligned} ((b \uparrow (c \uparrow c)) \uparrow (c \uparrow c)) \uparrow ((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \uparrow ((a \uparrow (c \uparrow c)) \\ \uparrow (a \uparrow (c \uparrow c))) \uparrow (a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \\ \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))) = 1 \end{aligned}$$

for all $a, b, c \in X$. We consider

$$\begin{aligned} \mu(a \uparrow (c \uparrow c)) &= \mu(1 \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))) \\ &= \mu(((b \uparrow (c \uparrow c)) \uparrow (c \uparrow c) \uparrow ((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \\ &\quad \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c))) \uparrow (a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \\ &\quad \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c))))) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c))))) \\ &\geqslant \min\{\mu((b \uparrow (c \uparrow c)) \uparrow (c \uparrow c)), \mu(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))))\} \\ &\geqslant \min\{\mu(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))), \mu(b)\} \end{aligned}$$

for all $a, b, c \in X$. Hence, the conditions (3.1) and (3.2) are true.

Conversely, using (3.1), (3.2), (SBE-1), and Lemma 2.14 (2), we have

$$\begin{aligned} \mu(a \uparrow (b \uparrow b)) &\geqslant \min\{\mu(a \uparrow ((b \uparrow (b \uparrow b)) \uparrow (b \uparrow (b \uparrow b)))), \mu(b)\} \\ &= \min\{\mu(a \uparrow (a \uparrow 1)), \mu(b)\} = \min\{\mu(1), \mu(b)\} = \mu(b) \end{aligned}$$

and

$$\begin{aligned} \mu((a \uparrow (b \uparrow b)) \uparrow (b \uparrow b)) &\geqslant \min\{\mu((a \uparrow (b \uparrow b)) \uparrow ((a \uparrow (b \uparrow b)) \uparrow (a \uparrow (b \uparrow b)))), \mu(a)\} \\ &= \min\{\mu(1), \mu(a)\} = \mu(a) \end{aligned}$$

for all $a, b \in X$. Since μ satisfies (3.1) and (3.2) and by Proposition 3.10, we have μ is order-preserving. Since X is transitive, we see that

$$\mu((b \uparrow (c \uparrow c)) \uparrow (c \uparrow c)) \leqslant \mu(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))) \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))),$$

for all $a, b, c \in X$. Hence, for all $a, b, c \in X$, we have

$$\begin{aligned}\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) &\geqslant \min\{\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c))))\\ &\quad \uparrow ((a \uparrow (c \uparrow c)) \uparrow (a \uparrow (c \uparrow c)))), \mu(a)\}\\ &\geqslant \min\{\mu((b \uparrow (c \uparrow c)) \uparrow (c \uparrow c)), \mu(a)\} \geqslant \min\{\mu(a), \mu(b)\}.\end{aligned}$$

Therefore, μ is a fuzzy SBE-ideal of X. □

Corollary 3.12. *Let μ be a fuzzy set in a self-distributive SBE-algebra X. Then μ is a fuzzy SBE-ideal of X if and only if it satisfies conditions* (3.1) *and* (3.2).

Proof. Straightforward. □

Definition 3.13. A fuzzy set μ in an SBE-algebra X is called an anti-fuzzy SBE-ideal of X if it satisfies the two conditions:

(AFSI-1) $\mu(a \uparrow (b \uparrow b)) \leqslant \mu(b)$ for all $a, b \in X$; and
(AFSI-2) $\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \leqslant \max\{\mu(a), \mu(b)\}$ for all $a, b, c \in X$.

Example 3.14. Consider an SBE-algebra $\langle X, \uparrow, 1\rangle$ defined in Example 2.10. Define a fuzzy set μ in X by $\mu(1) = 0$, $\mu(a) = 0.2$, and $\mu(b) = \mu(0) = 0.6$. Then μ is an anti-fuzzy SBE-ideal of X.

Let μ be a fuzzy set in a nonempty set X. The fuzzy set $\bar{\mu}$ in X, defined by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$, is called the complement [14] of μ in X. Note that $\mu = \bar{\bar{\mu}}$. Now, we show the relationships between fuzzy SBE-ideals and anti-fuzzy SBE-ideals and their complement.

Lemma 3.15 ([14])**.** *Let μ be a fuzzy set in a nonempty set X and $x, y \in X$. Then*

(1) $1 - \max\{\mu(x), \mu(y)\} = \min\{1 - \mu(x), 1 - \mu(y)\}$; *and*
(2) $1 - \min\{\mu(x), \mu(y)\} = \max\{1 - \mu(x), 1 - \mu(y)\}$.

Theorem 3.16. *A fuzzy set μ in an SBE-algebra X is a fuzzy SBE-ideal of X if and only if $\bar{\mu}$ is an anti-fuzzy SBE-ideal of X.*

Proof.

($\Rightarrow$) Assume that μ is a fuzzy SBE-ideal of X. Let $a, b, c \in X$. Then $\mu(a \uparrow (b \uparrow b)) \geqslant \mu(b)$ and

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\mu(a), \mu(b)\}.$$

Thus,

$$\bar{\mu}(a \uparrow (b \uparrow b)) = 1 - \mu(a \uparrow (b \uparrow b)) \leqslant 1 - \mu(b) = \bar{\mu}(b),$$

and by Lemma 3.15, we have

$$\begin{aligned}\bar{\mu}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) &= 1 - \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c))\\ &\leqslant 1 - \min\{\mu(a), \mu(b)\}\\ &= \max\{1 - \mu(a), 1 - \mu(b)\} = \max\{\bar{\mu}(a), \bar{\mu}(b)\}.\end{aligned}$$

Hence, $\bar{\mu}$ satisfies the conditions (AFSI-1) and (AFSI-2), that is, $\bar{\mu}$ is an anti-fuzzy SBE-ideal of X.

($\Leftarrow$) Assume that $\bar{\mu}$ is an anti-fuzzy SBE-ideal of X. Let $a, b \in X$. By using the condition (AFSI-1), we have $\bar{\mu}(a \uparrow (b \uparrow b)) \leqslant \bar{\mu}(b)$. Thus, $1 - \mu(a \uparrow (b \uparrow b)) \leqslant 1 - \mu(b)$, and so $\mu(a \uparrow (b \uparrow b)) \geqslant \mu(b)$. Hence, μ satisfies the condition (FSI-1).

Now, let $a, b, c \in X$. By using (AFSI-2) and Lemma 3.15, we have

$$\begin{aligned} 1 - \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) &= \bar{\mu}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \\ &\leqslant \max\{\bar{\mu}(a), \bar{\mu}(b)\} \\ &= \max\{1 - \mu(a), 1 - \mu(b)\} = 1 - \min\{\mu(a), \mu(b)\}. \end{aligned}$$

Thus,

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\mu(a), \mu(b)\}.$$

Hence, μ satisfies the condition (FSI-2). Since μ satisfies the conditions (FSI-1) and (FSI-2), we see that μ is a fuzzy SBE-ideal of X. □

Theorem 3.17. *A fuzzy set μ in an SBE-algebra X is an anti-fuzzy SBE-ideal of X if and only if $\bar{\mu}$ is a fuzzy SBE-ideal of X.*

Proof. By using $\mu = \bar{\bar{\mu}}$ and Theorem 3.16, we have Theorem 3.17. □

For a subset Y of a nonempty set X, the characteristic function χ_Y of Y is defined by

$$\chi_Y \colon X \to \{0, 1\}, \quad y \mapsto \begin{cases} 1, & \text{if } y \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

Note that χ_Y is a fuzzy set in X. Next, we show the relationships among SBE-ideals, fuzzy SBE-ideals, and anti-fuzzy SBE-ideals of SBE-algebras.

Theorem 3.18. *Let Y be a nonempty subset of an SBE-algebra X. The following statements are true.*

(1) *Y is an SBE-ideal of X;*
(2) *χ_Y is a fuzzy SBE-ideal of X;*
(3) *$\overline{\chi_Y}$ is an anti-fuzzy SBE-ideal of X.*

Proof.

(1)$\Rightarrow$(2) Assume that Y is an SBE-ideal of X and $a, b \in X$. If $b \notin Y$, then $\chi_Y(a \uparrow (b \uparrow b)) \geqslant 0 = \chi_Y(b)$. Suppose that $b \in Y$. Then $a \uparrow (b \uparrow b) \in Y$. Thus, $\chi_Y(a \uparrow (b \uparrow b)) = 1 = \chi_Y(b)$. Hence, χ_Y satisfies the condition (FSI-1).

Now, let $a, b, c \in X$. If $a \notin Y$ or $b \notin Y$, then

$$\chi_Y((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant 0 = \min\{\chi_Y(a), \chi_Y(b)\}.$$

Suppose that $a, b \in Y$. Since Y is an SBE-ideal of X, we have $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \in Y$. Thus,

$$\chi_Y((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) = 1 = \min\{\chi_Y(a), \chi_Y(b)\}.$$

Hence, χ_Y satisfies the condition (FSI-2). Since χ_Y satisfies the two conditions (FSI-1) and (FSI-2), we have χ_Y is a fuzzy SBE-ideal of X.

(2)$\Leftrightarrow$(3) It follows from Theorem 3.16.

(3)$\Rightarrow$(1) Assume that $\overline{\chi_Y}$ is an anti-fuzzy SBE-ideal of X. Let $a \in X$ and $b \in Y$. Then $\chi_Y(b) = 1$. By the assumption, we have

$$\overline{\chi_Y}(a \uparrow (b \uparrow b)) \leqslant \overline{\chi_Y}(b) = 1 - \chi_Y(b) = 0.$$

Thus, $\overline{\chi_Y}(a \uparrow (b \uparrow b)) = 0$, which implies that $a \uparrow (b \uparrow b) \in Y$. Hence, Y satisfies the condition (SBEI-1).

To show that Y satisfies the condition (SBEI-2), let $a, b \in Y$ and $c \in X$. Then $\min\{\chi_Y(a), \chi_Y(b)\} = 1$. By the assumption and Lemma 3.15, we have

$$\overline{\chi_Y}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \leqslant \max\{\overline{\chi_Y}(a), \overline{\chi_Y}(b)\} = 1 - \min\{\chi_Y(a), \chi_Y(b)\} = 0.$$

Thus,

$$\overline{\chi_Y}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) = 0,$$

and then $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \in Y$. Hence, Y satisfies the condition (SBEI-2). Since Y satisfies the conditions (SBEI-1) and (SBEI-2), we see that Y is a SBE-ideal of X. □

4. Level subsets of fuzzy SBE-ideals and anti-fuzzy SBE-ideals

In this section, we discuss level subsets of fuzzy sets in SBE-algebras. Characterizations of fuzzy SBE-ideals and anti-fuzzy SBE-ideals of SBE-algebras are given in terms of their level subsets.

Definition 4.1 ([14])**.** Let μ be a fuzzy set in an SBE-algebra X and $t \in [0, 1]$. The sets

$$U(\mu; t) = \{a \in X \mid \mu(a) \geqslant t\} \quad \text{and} \quad U^+(\mu; t) = \{a \in X \mid \mu(a) > t\}$$

are called an upper t-level subset and an upper t-strong level subset of μ, respectively. The sets

$$L(\mu; t) = \{a \in X \mid \mu(a) \leqslant t\} \quad \text{and} \quad L^-(\mu; t) = \{a \in X \mid \mu(a) < t\}$$

are called a lower t-level subset and a lower t-strong level subset of μ, respectively.

Theorem 4.2. *Let μ be a fuzzy set in an SBE-algebra X. Then μ is a fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0, 1]$, $U(\mu; t) \neq \emptyset$ implies $U(\mu; t)$ is an SBE-ideal of X.*

Proof.

($\Rightarrow$) Assume that μ is a fuzzy SBE-ideal of X, $t \in [0, 1]$ and $U(\mu; t) \neq \emptyset$. Let $x \in X$ and $a \in U(\mu; t)$. Then $\mu(a) \geqslant t$, so $\mu(x \uparrow (a \uparrow a)) \geqslant \mu(a) \geqslant t$ by the condition (FSI-1). Thus, $x \uparrow (a \uparrow a) \in U(\mu; t)$. Hence, the condition (SBEI-1) holds.

Now, let $x \in X$ and $a, b \in U(\mu; t)$. Then $\mu(a) \geqslant t$ and $\mu(b) \geqslant t$. It follows from (FSI-2) that

$$\mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \geqslant \min\{\mu(a), \mu(b)\} \geqslant t,$$

which implies that $(a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x) \in U(\mu; t)$. Hence, the condition (SBEI-2) holds. Therefore, $U(\mu; t)$ is an SBE-ideal of X.

($\Leftarrow$) Assume that $U(\mu; t)$ is an SBE-ideal of X for all $t \in [0, 1]$ such that $U(\mu; t) \neq \emptyset$. If $\mu(a \uparrow (b \uparrow b)) < \mu(b)$ for some $a, b \in X$, then there exists $t_0 \in [0, 1]$ such that $\mu(a \uparrow (b \uparrow b)) < t_0 < \mu(b)$ by taking

$$t_0 = \frac{\mu(a \uparrow (b \uparrow b)) + \mu(b)}{2}.$$

Thus, $a \uparrow (b \uparrow b) \notin U(\mu; t_0)$ and $b \in U(\mu; t_0)$, which is a contradiction with the assumption. Hence, $\mu(a \uparrow (b \uparrow b)) \geqslant \mu(b)$ for all $a, b \in X$, that is, the condition (FSI-1) holds.

Now, let $a, b, c \in X$ be such that

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) < \min\{\mu(a), \mu(b)\}.$$

Taking $k_0 = \dfrac{\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) + \min\{\mu(a), \mu(b)\}}{2}$, we get that $k_0 \in [0, 1]$ and

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) < k_0 < \min\{\mu(a), \mu(b)\}.$$

It follows that $a, b \in U(\mu; k_0)$ and $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \notin U(\mu; k_0)$. Thus, $U(\mu; k_0)$ is not an SBE-ideal of X, which is a contradiction with the assumption. Hence, $\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\mu(a), \mu(b)\}$ for all $a, b, c \in X$, that is, the condition (FSI-2) holds. Therefore, μ is a fuzzy SBE-ideal of X. □

Theorem 4.3. *Let μ be a fuzzy set in an SBE-algebra X. Then $\overline{\mu}$ is an anti-fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0,1]$, $U(\mu;t) \neq \emptyset$ implies $U(\mu;t)$ is an SBE-ideal of X.*

Proof. It follows from Theorems 3.16 and 4.2. □

Theorem 4.4. *Let μ be a fuzzy set in an SBE-algebra X. Then μ is a fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0,1]$, $U^+(\mu;t) \neq \emptyset$ implies $U^+(\mu;t)$ is an SBE-ideal of X.*

Proof.

($\Rightarrow$) Assume that μ is a fuzzy SBE-ideal of X, $t \in [0,1]$ and $U^+(\mu;t) \neq \emptyset$. Let $x \in X$ and $a \in U^+(\mu;t)$. Then $\mu(a) > t$. Since μ is a fuzzy SBE-ideal of X, we have $\mu(x \uparrow (a \uparrow a)) > t$. Thus, $x \uparrow (a \uparrow a) \in U^+(\mu;t)$. Hence, the set $U^+(\mu;t)$ satisfies the condition (SBEI-1).

Now, let $x \in X$ and $a, b \in U^+(\mu;t)$. Then $\mu(a) > t$ and $\mu(b) > t$. Since μ is an SBE-ideal of X, we get that

$$\mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) > t,$$

which implies that $(a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x) \in U^+(\mu;t)$. Hence, the set $U^+(\mu;t)$ satisfies the condition (SBEI-2). Since the set $U^+(\mu;t)$ satisfies the conditions (SBEI-1) and (SBEI-2), we have the set $U^+(\mu;t)$ is an SBE-ideal of X.

($\Leftarrow$) Assume that $U^+(\mu;t)$ is an SBE-ideal of X for all $t \in [0,1]$ such that $U^+(\mu;t) \neq \emptyset$. Suppose that $\mu(a \uparrow (b \uparrow b)) < \mu(b)$ for some $a, b \in X$. Choose $t_0 = \mu(a \uparrow (b \uparrow b))$, we have $\mu(b) > t_0$. Thus, $b \in U^+(\mu;t_0)$. By the assumption, we get that $U^+(\mu;t_0)$ is an SBE-ideal of X. Hence, $a \uparrow (b \uparrow b) \in U^+(\mu;t_0)$, that is,

$$\mu(a \uparrow (b \uparrow b)) > t_0 = \mu(a \uparrow (b \uparrow b)).$$

This is a contradiction. Then $\mu(a \uparrow (b \uparrow b)) \geqslant \mu(b)$ for all $a, b \in X$, that is, μ satisfies the condition (FSI-1).

Now, we will show that μ satisfies the condition (FSI-2). Let $a, b, c \in X$ be such that

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) < \min\{\mu(a), \mu(b)\}.$$

Choose $k_0 = \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c))$, we have that $\mu(a) > k_0$ and $\mu(b) > k_0$. Thus, $a, b \in U^+(\mu;k_0)$. By the assumption, we get that $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \in U^+(\mu;k_0)$, that is,

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) > k_0 = \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)).$$

It is a contradiction. Hence, for all $a, b, c \in X$, we have

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\mu(a), \mu(b)\}.$$

We have that μ satisfies the condition (FSI-2). Since μ satisfies the two conditions (FSI-1) and (FSI-2), we see that μ is a fuzzy SBE-ideal of X. □

Theorem 4.5. *Let μ be a fuzzy set in an SBE-algebra X. Then $\overline{\mu}$ is an anti-fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0,1]$, $U^+(\mu;t) \neq \emptyset$ implies $U^+(\mu;t)$ is an SBE-ideal of X.*

Proof. It follows from Theorems 3.16 and 4.4. □

Theorem 4.6. *Let μ be a fuzzy set in an SBE-algebra X. Then $\bar{\mu}$ is a fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0,1]$, $L(\mu;t) \neq \emptyset$ implies $L(\mu;t)$ is an SBE-ideal of X.*

Proof.

($\Rightarrow$) Assume that $\bar{\mu}$ is a fuzzy SBE-ideal of X, $t \in [0,1]$ and $L(\mu;t) \neq \emptyset$. Let $a \in X$ and $b \in L(\mu;t)$. Then $\mu(b) \leqslant t$. By the assumption, we have

$$1 - t \leqslant 1 - \mu(b) = \bar{\mu}(b) \leqslant \bar{\mu}(a \uparrow (b \uparrow b)) = 1 - \mu(a \uparrow (b \uparrow b)).$$

Thus, $\mu(a \uparrow (b \uparrow b)) \leqslant t$, which implies that $a \uparrow (b \uparrow b) \in L(\mu;t)$. Hence, the set $L(\mu;t)$ satisfies the condition (SBEI-1).

Now, let $a, b \in L(\mu; t)$ and $x \in X$. Then $\mu(a) \leqslant t$ and $\mu(b) \leqslant t$. By using the assumption and Lemma 3.15, we get that

$$\begin{aligned}1-&\mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \\ &= \bar{\mu}((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \\ &\geqslant \min\{\bar{\mu}(a), \bar{\mu}(b)\} = \min\{1-\mu(a), 1-\mu(b)\} = 1-\max\{\mu(a), \mu(b)\} \geqslant 1-t.\end{aligned}$$

Thus,

$$\mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \leqslant t,$$

which implies that

$$(a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x) \in L(\mu; t).$$

Hence, the set $L(\mu; t)$ satisfies the condition (SBEI-2). Since the set $L(\mu; t)$ satisfies the conditions (SBEI-1) and (SBEI-2), we have that the set $L(\mu; t)$ is an SBE-ideal of X.

($\Leftarrow$) Assume that $L(\mu; t)$ is an SBE-ideal of X for all $t \in [0, 1]$ such that $L(\mu; t) \neq \emptyset$. Let $a, b \in X$. Choose $t_0 = \mu(b)$, we get $\mu(b) \leqslant t_0$. Thus, $b \in L(\mu; t_0)$. By the assumption, we get that $L(\mu; t_0)$ is an SBE-ideal of X. Hence, $a \uparrow (b \uparrow b) \in L(\mu; t_0)$, that is,

$$\mu(a \uparrow (b \uparrow b)) \leqslant t_0 = \mu(b).$$

Then

$$\bar{\mu}(a \uparrow (b \uparrow b)) = 1 - \mu(a \uparrow (b \uparrow b)) \geqslant 1 - \mu(b) = \bar{\mu}(b).$$

Hence, $\bar{\mu}$ satisfies the condition (FSI-1).

To show that $\bar{\mu}$ satisfies the condition (FSI-2), let $a, b, c \in X$. Choose $k_0 = \max\{\mu(a), \mu(b)\}$, we have $a, b \in L(\mu; k_0)$. By the assumption, we get that $L(\mu; k_0)$ is an SBE-ideal of X. Hence, $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \in L(\mu; k_0)$, that is,

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \leqslant k_0 = \max\{\mu(a), \mu(b)\}.$$

Hence,

$$\begin{aligned}\bar{\mu}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (x \uparrow c)) &= 1 - \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \\ &\geqslant 1 - \max\{\mu(a), \mu(b)\} \\ &= \min\{1-\mu(a), 1-\mu(b)\} = \min\{\bar{\mu}(a), \bar{\mu}(b)\}.\end{aligned}$$

We have that $\bar{\mu}$ satisfies the condition (FSI-2). Since $\bar{\mu}$ satisfies the two conditions (FSI-1) and (FSI-2), we get that $\bar{\mu}$ is a fuzzy SBE-ideal of X. □

Theorem 4.7. *Let μ be a fuzzy set in an SBE-algebra X. Then μ is an anti-fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0, 1]$, $L(\mu; t) \neq \emptyset$ implies $L(\mu; t)$ is an SBE-ideal of X.*

Proof. It follows from Theorems 3.17 and 4.6. □

Theorem 4.8. *Let μ be a fuzzy set in an SBE-algebra X. Then $\bar{\mu}$ is a fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0, 1]$, $L^-(\mu; t) \neq \emptyset$ implies $L^-(\mu; t)$ is an SBE-ideal of X.*

Proof.

($\Rightarrow$) Assume that $\bar{\mu}$ is a fuzzy SBE-ideal of X, $t \in [0, 1]$ and $L^-(\mu; t) \neq \emptyset$. Let $a \in X$ and $b \in L^-(\mu; t)$. Since $\bar{\mu}$ is a fuzzy SBE-ideal of X, we get $\bar{\mu}(b) \leqslant \bar{\mu}(a \uparrow (b \uparrow b))$. Then $1 - t < 1 - \mu(a \uparrow (b \uparrow b))$, so $\mu(a \uparrow (b \uparrow b)) < t$. Thus, $a \uparrow (b \uparrow b) \in L^-(\mu; t)$. Hence, the set $L^-(\mu; t)$ satisfies the condition (SBEI-1).

Now, let $a, b \in L^-(\mu; t)$ and $x \in X$. Then $\max\{\mu(a), \mu(b)\} < t$. Since $\bar{\mu}$ is a fuzzy SBE-ideal of X, we have

$$\bar{\mu}((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \geqslant \min\{\bar{\mu}(a), \bar{\mu}(b)\}.$$

By using Lemma 3.15, we get that

$$1 - \mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) \geqslant 1 - \max\{\mu(a), \mu(b)\} > 1 - t.$$

Thus,

$$\mu((a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x)) < t,$$

which implies that

$$(a \uparrow ((b \uparrow (x \uparrow x)) \uparrow (b \uparrow (x \uparrow x)))) \uparrow (x \uparrow x) \in L^-(\mu; t).$$

Hence, the set $L^-(\mu; t)$ satisfies the condition (SBEI-2). Since the set $L^-(\mu; t)$ satisfies the two conditions (SBEI-1) and (SBEI-2), we have that the set $L^-(\mu; t)$ is an SBE-ideal of X.

($\Leftarrow$) Assume that for all $t \in [0, 1]$, $L^-(\mu; t)$ is an SBE-ideal of X if $L^-(\mu; t)$ is nonempty. Suppose that there exist $a, b \in X$ such that $\bar{\mu}(a \uparrow (b \uparrow b)) < \bar{\mu}(b)$. Then $\mu(a \uparrow (b \uparrow b)) > \mu(b)$. Choose $t_0 = \mu(a \uparrow (b \uparrow b))$, we have $\mu(b) < t_0$. Thus, $b \in L^-(\mu; t_0)$ but $a \uparrow (b \uparrow b) \notin L^-(\mu; t_0)$. By the assumption, we have that $L^-(\mu; t_0)$ is an SBE-ideal of X. Hence, $a \uparrow (b \uparrow b) \in L^-(\mu; t_0)$, a contradiction. Then $\bar{\mu}(a \uparrow (b \uparrow b)) \geqslant \bar{\mu}(b)$ for all $a, b \in X$, that is, $\bar{\mu}$ satisfies the condition (FSI-1).

Now, we will show that $\bar{\mu}$ satisfies the condition (FSI-2). Let $a, b, c \in X$ be such that

$$\bar{\mu}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) < \min\{\bar{\mu}(a), \bar{\mu}(b)\}.$$

By using Lemma 3.15, we get $\min\{\bar{\mu}(a), \bar{\mu}(b)\} = 1 - \max\{\mu(a), \mu(b)\}$, and so

$$\mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) > \max\{\mu(a), \mu(b)\}.$$

Choose

$$k_0 = \mu((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)),$$

we have $a, b \in L^-(\mu; k_0)$ but $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \notin L^-(\mu; k_0)$. By the assumption, the set $L^-(\mu; k_0)$ is an SBE-ideal of X. Thus, $(a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c) \in L^-(\mu; k_0)$, a contradiction. Hence,

$$\bar{\mu}((a \uparrow ((b \uparrow (c \uparrow c)) \uparrow (b \uparrow (c \uparrow c)))) \uparrow (c \uparrow c)) \geqslant \min\{\bar{\mu}(a), \bar{\mu}(b)\},$$

for all $a, b, c \in X$. We have that $\bar{\mu}$ satisfies the condition (FSI-2). Since $\bar{\mu}$ satisfies the two conditions (FSI-1) and (FSI-2), we have that $\bar{\mu}$ is a fuzzy SBE-ideal of X. □

Theorem 4.9. *Let μ be a fuzzy set in an SBE-algebra X. Then μ is an anti-fuzzy SBE-ideal of X if and only if it satisfies the condition: for all $t \in [0, 1]$, $L^-(\mu; t) \neq \emptyset$ implies $L^-(\mu; t)$ is an SBE-ideal of X.*

Proof. It follows from Theorems 3.17 and 4.8. □

5. Conclusions

In the present paper, we have introduced the concepts of fuzzy SBE-ideals and anti-fuzzy SBE-ideals of SBE-algebras and given their important properties. We have discussed the relationships between fuzzy SBE-ideals and anti-fuzzy SBE-ideals and their level subsets. Further, characterizations of SBE-ideals of SBE-algebras in terms of fuzzy SBE-ideals and anti-fuzzy SBE-ideals.

Our research group plans to extend the study of this article to Q-fuzzy sets and intuitionistic fuzzy sets in the near future, building on the work of [15] and [3], respectively, and to additional kinds of SBE-ideals in SBE-algebras.

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