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Qualitative Study on ψ−Caputo fractional differential inclusion with non-local conditions and feedback control

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Abstract

This article discuss the existence of solutions for certain classes of nonlinear ψ–Caputo fractional derivative differential inclusion via a nonlocal infinite-point or Riemann–Stieltjes integral boundary conditions and with a feedback control in Banach spaces. Our approach is based on Schauder's fixed point theorem. We establish appropriate conditions that guarantee unique solutions and demonstrate that the solution continuously depends on the set of selections and some other functions. Additionally, we include example to illustrate the key findings.

Keywords: ψ-Caputo fractional operator, Riemann-Stieltjes integral boundary conditions, infinite-point boundary conditions, feedback control.

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1. Introduction

Differential and integral equation models can be found in a wide range of applications (see [10, 13, 15, 16]). In some of these problems, subsidiary conditions are enforced locally. Other times, conditions outside the area are applied. Because the measurements needed to determine a non-local condition are perhaps even more accurate with data supplied by a local condition, non-local conditions are regularly used preferred to local conditions. The resulting fractional boundary value problems (abbreviated BVPs) with resonant requirements have produced a number of notable achievements. Bai [8] studied a class of fractional differential equations with m-point boundary conditions. Kosmatov [22] studied the BVP of three points of fractional order with the resonant case using the same methodology. Given that the investigation of fractional BVPs at resonance has produced useful results, It must be noted that there are many issues that include Riemann-Stieltjes integrals. As a result, research on fractional BVPs at resonance has been successful. It should be noted that these Riemann-Stieltjes integral problems are quite rare. Therefore, further research is needed. As can be seen in the pertinent publications attributed to

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Ahmad et al. [4], the Riemann-Stieltjes integral has been viewed as both a multipoint and an integral in one construction, the latter of which is more typical. There are many situations where boundary value concerns for nonlinear differential equations could come up, such as as those in physics, applied mathematics, and variations problems of control theory, we recommend to read works [16–18]. The sensors act linearly; one gives feedback to a controller at one endpoint from a portion of the interval, and the other provides feedback to a controller at the other extremity. Numerous positive solutions and a few nonexistent solutions were established by the proof of certain useful characteristics.

The importance of dealing with problems involving control variables is due to the unforeseen factors that continually upset ecosystems in the actual world, which can lead to changes in biological characteristics such as survival rates. Ecology has a practical interest in the question of whether an ecosystem can withstand those unpredictable disruptive events that continue for a short period of time. In the context of control variables, the disturbance functions are what we refer to as control variables. Chen derived certain criteria for the long-term stability and overall attractiveness of a nonautonomous Feedback-controlled Lotka-Volterra system in a study referenced as [11]. This was done through the development of an appropriate Lyapunov function.

Nasertayoob has shown the existence of a type of nonlinear functional-integral equations with feedback control that are asymptotically stable and globally attractive. This was achieved by utilizing the measure of noncompactness along with Darbo's fixed point theorem [25]. In addition, the research investigates the presence of a positive periodic solution in a nonlinear neutral delay population system with feedback control under specific conditions. The demonstration relies on the fixed-point theorem of strict-set-contraction operators [26]. El-Sayed et al. [5] address a functional integral equation with multi-valued feedback control that meets a constraint functional equation.

Many academics and researcherss have recently develope interest in studying boundary value problems of fractional order, with interest in this subject expanding across various fields of study (see [1–3, 20, 24]). The Liouville-Caputo fractional derivative was used to show the existence of continuous solutions to the non-local first-order BVP [27]

$$\frac{d\nu}{d\mathfrak{r}} = g(\mathfrak{r}, D^{\beta}\nu(\mathfrak{r})), \quad \mathfrak{r} \in (0,1), \ \ 0 < \beta < 1,$$

in addition to the infinite-point boundary conditions provided by

$$\sum_{j=1}^{\infty} c_j \ \nu(\tau_j) = \nu_\circ, \quad c_j \ > 0, \tau_j \in (0,1], \text{ and } \nu_\circ \in R^{+},$$

alternatively, the Riemann-Stieltjes functional integral boundary conditions

$$\int_0^T \nu(\sigma) \ d\mathfrak{h}(\sigma) \ = \ \nu_\circ, \ \ \mathfrak{h} : [0,T] \to \mathbb{R} \ \text{ is nondecreasing function.}$$

In light of the previously mentioned problems, in this study, we consider a modified version of the issue discussed in [27]. Specifically, we investigate the existence of solutions for a fractional differential inclusion of the ψ-Caputo type.

$$^{c}D^{\beta;\psi}\nu(\mathfrak{r}) \in \Theta_1(t, \mu(\mathfrak{r}), I^{\alpha;\psi}\theta_2(t, \nu(\varphi(\mathfrak{r}))), \ \ 0 < \alpha < 1, \ \mathfrak{r} \in (0,T] \tag{1.1}$$

equipped with the infinite-point boundary conditions

$$\nu(0) \ + \ \sum_{j=1}^{\infty} a_j \ \nu(\tau_j) = \nu_\circ, \quad a_j \ > 0, \tau_j \in (0,T] \tag{1.2}$$

or Riemann–Stieltjes integro boundary conditions

$$\nu(0) \ + \ \int_0^T \nu(\eta) \ d\mathfrak{h}(\eta) \ = \ \nu_\circ, \ \ \mathfrak{h} : [0,T] \to \mathbb{R} \ \text{ is nondecreasing function} \tag{1.3}$$

with feedback control provided by

$$\mu(\mathfrak{r}) = \varpi\big(\mathfrak{r}, \mu(\mathfrak{r}), \nu(\mathfrak{r})\big), \tag{1.4}$$

where $\psi(\mathfrak{r})$ is an increasing function with $\psi'(\mathfrak{r}) \neq 0, \ \forall\ \mathfrak{r} \in I = [0, T], 0 < \alpha \leqslant 1$, and ${}^{c}D^{\alpha,\psi}$ is the ψ-Caputo fractional derivative, and $\Theta_1 : [0, T] \times \mathbb{R}^+ \to P(\mathbb{R})$ is a multivalued map, where the family of all nonempty subsets of $\mathbb{R}$ is known as $P(\mathbb{R})$. Our inquiry is centered on the selections of the set-valued function Θ_1 by varying the inclusion of the functional integral into a coupled system. First, obtain the continuous solution of the problem (1.1) utilizing the n-point BCs provided by

$$\nu(0) + \sum_{j=1}^{n} a_j\ \nu(\tau_j) = \nu_\circ, \quad a_j > 0, \tau_j \in [0, 1], \tag{1.5}$$

and following that, by using the Riemann sum's characteristics for continuous functions, we show that the solutions of the BVP with the Riemann-Stieltjes integral provided in (1.1) and (1.3) and also the BVP with infinite points provided in (1.1) and (1.2). To achieve the main goal, the initial problem is transformed into a corresponding integral equation, and the Schauder fixed point theorem is applied to prove the existence of the solution.

The remaining parts of the paper are structured as follows: Our main finding in relation to issues (1.1)–(1.5) is presented in Section 2. Considering the conclusion of the development, we look into the BVP given in (1.1)-(1.3) with a feedback control (1.4) and by (1.1)-(1.2) with a feedback control (1.4). We show that both the infinite-point BC (1.2) and the Riemann-Stieltjes functional integral BC (1.3) satisfy sufficient criteria for the problem (1.1) in each case, while Section 3 addresses dependency and the uniqueness of continuous solutions. Section 4 includes an example that will demonstrate our findings.

2. Existence of solution

Consider the assumptions listed below

(i) The multivalued map $\Theta_1 : I \times \mathbb{R} \to 2^{\mathbb{R}}$ is Lipschitzian, and has a nonempty convex compact subset of $2^{\mathbb{R}}$, with Lipschitz constant $k > 0$

$$\|\Theta_1(\mathfrak{r}, \mu) - \Theta_1(\mathfrak{r}, \nu)\| \leqslant k\ |\mu - \nu|.$$

Here the set of Lipschitz selections for Θ_1 is not empty and there exists $\theta_1 \in \Theta_1$ (see [7]), with

$$|\theta_1(\mathfrak{r}, \mu) - \theta_1(\mathfrak{r}, \nu)| \leqslant k\ |\mu - \nu|.$$

(ii) The function $\varphi : I \to I$ is a continuous.

(iii) The function $\theta_2 : I \times \mathbb{R} \to \mathbb{R}$, satisfied Caratheodory requirement. There exist a measurable bounded function $a(\mathfrak{r})$ and a constant $b > 0$, with

$$|\theta_2(\mathfrak{r}, \mu)| \leqslant\ a(\mathfrak{r}) +\ b|\mu|, \quad \forall\ \mathfrak{r}\ \in I \text{ and } \mu \in \mathbb{R}.$$

(iv) $\omega \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exists a measurable and bounded function $\delta : I \to \mathbb{R}$, have norm $\|\delta\|$, with

$$|\varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| \leqslant \delta(\mathfrak{r}), \quad \mathfrak{r} \in I.$$

(v) $[a(|\nu_\circ| + \sum_{j=1}^{n} |a_j|)\ + 1]k(\psi(T) - \psi(0)) < 1$, $\frac{b\ (\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)} < 1$, and $I_c^{\gamma;\psi} a(.) \leqslant M, \ \ \forall\, \gamma\ \leqslant \alpha,\ c \geqslant 0.$

Lemma 2.1. *For any $\nu \in C(I, \mathbb{R})$, the solution of the linear fractional boundary value problem*

$$D^{\beta;\psi}\nu(\mathfrak{r}) = \theta_1(\mathfrak{r}, \mu(\mathfrak{r}), I^{\alpha;\psi}\theta_2(\mathfrak{r}, \nu(\varphi(\mathfrak{r})))), \quad \alpha \in (0, 1),\ \mathfrak{r} \in I, \tag{2.1}$$

additionally with feedback control (1.4) and condition (1.5), are equivalent to equation

$$\begin{aligned}\nu(\mathfrak{r}) =& a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta\right) \\ &+ \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta,\end{aligned} \tag{2.2}$$

here $a = (1+\sum_{j=1}^{n} a_j)^{-1}$.

Proof. We begin with investigating the issue (2.1) with the m-point BCs in (1.5). Integrating the two sides of (2.1), we get

$$\nu(\mathfrak{r}) = \nu(0) + I^{\beta;\psi}\theta_1(\mathfrak{r},\mu(\mathfrak{r}),I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r})))).$$

Utilize condition (1.5) to obtain

$$\nu(\mathfrak{r}) = \nu_\circ - \sum_{j=1}^{n} a_j\, \nu(\tau_j) + \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta. \tag{2.3}$$

In actuality, when set $\mathfrak{r} = \tau_j \in [0,T]$ in Eq. (2.3), we obtain

$$\nu(\tau_j) = \nu_\circ - \sum_{j=1}^{n} a_j\, \nu(\tau_j) + \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta. \tag{2.4}$$

So, we have

$$\begin{aligned}\nu(\tau_j) = \nu(\mathfrak{r}) &+ \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta \\ &- \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta.\end{aligned} \tag{2.5}$$

From (2.5) and (2.3)

$$\begin{aligned}\nu(\mathfrak{r}) = \nu_\circ - \sum_{j=1}^{n} a_j\, (\nu(\mathfrak{r}) &+ \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta \\ &- \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta) \\ &+ \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta.\end{aligned}$$

Consequently, we achieve

$$\begin{aligned}\left(1+\sum_{j=1}^{n} a_j\right)\nu(\mathfrak{r}) =& \nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(s)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta \\ &+ (1+\sum_{j=1}^{n} a_j)\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\, d\eta.\end{aligned}$$

Letting $a = (1+\sum_{j=1}^{n} a_j)^{-1}$,

$$\nu(\mathfrak{r}) = a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta\right)$$

$$+\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta.$$

From (2.2), we have

$$\begin{aligned}\nu(\tau_j) =& a\left(\nu_\circ-\sum_{j=1}^{n}a_j\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu(\eta),\nu(\varphi(\eta))))d\eta\right)\\ &+\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta,\end{aligned}\tag{2.6}$$

then

$$\begin{aligned}\left(1+\sum_{j=1}^{n}a_j\right)\nu(\tau_j) =& \nu_\circ-\sum_{j=1}^{n}a_j\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta\\ &+\left(1+\sum_{j=1}^{n}a_j\right)\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,x(\varphi(\eta))))d\eta,\end{aligned}$$

so,

$$\nu(\tau_j)+\sum_{j=1}^{n}a_j\ \nu(\tau_j)=\nu_\circ+\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu(\varphi(\eta))))d\eta,$$

hence

$$\sum_{j=1}^{n}a_j\ \nu(\tau_j)=\nu_\circ-\nu(\tau_j)+\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta.\tag{2.7}$$

From (2.2) we have

$$\nu(0)=a\left(\nu_\circ-\sum_{j=1}^{n}a_j\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta\right).$$

Substitute the value of $\nu(0)$, we obtain

$$\nu(\tau_j)=\nu(0)+\int_0^{\tau_j}\frac{\psi'(s)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta,$$

and

$$\nu(0)=\nu(\tau_j)-\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta.\tag{2.8}$$

By combining (2.7) and (2.8), we arrive at n-point BC (1.5)

$$\nu(0)+\sum_{j=1}^{\infty}a_j\ \nu(\tau_j)=\nu_\circ.$$

□

Remark 2.2. From assumption (i), it follows that the set of Lipschitz selection for Θ_1 is not empty. Furthermore, there is $\theta_1\in S_{\Theta_1}$, where

$$|\theta_1(\eta,\mu)-\theta_1(\eta,\nu)|\leqslant k|\mu-\nu|.$$

Hence, clearly, we have

$$|\theta_1(\eta,\mu)|\leqslant k|\mu|+\theta_1^*,\quad \text{where}\quad \theta_1^*=\sup_{\eta\in[0,T]}|\theta_1(\eta,0)|.$$

The next step shall be taken

$$w(\eta) = I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))), \quad \eta \in I. \tag{2.9}$$

Thus, we can express the equation (2.2) as

$$\nu(\mathfrak{r}) = a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\,d\eta\right) + \int_0^{\eta} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\,d\eta, \quad \mathfrak{r} \in I. \tag{2.10}$$

Then, the equation (2.2) is equivalent to the coupled system (2.9) and (2.10).

Theorem 2.3. *Let assumptions* $(\mathrm{i})-(\mathrm{v})$ *meet. Then the problem* (2.9), (2.10) *has at least one continuous solution* $u = (\nu, w),\ \nu, w \in C(I, \mathbb{R})$.

Proof. Let the set Q_r be specified as

$$Q_r = \{u = (\nu, w) \in \mathbb{R}^2, \|u\| \leqslant r\}$$

where

$$r = r_1 + r_2 = \frac{a|x_\circ| + [a\sum_{j=1}^{n}|a_j| + 1]\theta_1^*(\psi(T)-\psi(0))}{1-[a\sum_{j=1}^{n}|a_j|+1]kT} + (1 - \frac{b\,(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})^{-1}\frac{M(\psi(T)-\psi(0))^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.$$

The set Q_r is nonempty, closed, bounded and convex.

Next, set A to signify the operator defined on the space $C(I, \mathbb{R})$ by

$$Au(\mathfrak{r}) = A(\nu, w)(\mathfrak{r}) = (A_1 w(\mathfrak{r}), A_2\nu(\mathfrak{r})),$$

$$A_1 w(\mathfrak{r}) = a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\,d\eta\right) + \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta)))\,d\eta, \quad \mathfrak{r} \in I,$$

and

$$A_2\nu(\eta) = \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\mu(\varphi(\eta)))\,d\eta, \quad \eta \in I.$$

As a result, as indicated to $u = (w, \nu) \in Q_r$,

$$\begin{aligned}
|A_1 w(\mathfrak{r})| &= \Bigg| a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\,d\eta\right) \\
&\quad + \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta)))\,d\eta\Bigg| \\
&\leqslant a\,|\nu_\circ| + a\sum_{j=1}^{n} a_j \frac{(\psi(\tau_j)-\psi(0))^{\beta}}{\Gamma(\beta+1)}(k(\|\delta\|+|w|)+\theta_1^*) + \frac{(\psi(\mathfrak{r})-\psi(0))^{\beta}}{\Gamma(\beta+1)}(k(\|\delta\|+|w|)+\theta_1^*) \\
&\leqslant a\,|\nu_\circ| + [a\sum_{j=1}^{n}|a_j| + 1]\frac{(k(\|\delta\|+|w|)+\theta_1^*)(\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)},
\end{aligned}$$

then

$$\|A_1 w\| \leqslant a\, |\nu_\circ| + [a \sum_{j=1}^{n} |a_j| + 1]\frac{(k(\|\delta\| + |w|) + \theta_1^*)(\psi(T) - \psi(0))^\beta}{\Gamma(\beta+1)} = r_1,$$

$$r_1 = \frac{a|\nu_\circ| + [a \sum_{j=1}^{n} |a_j| + 1]\frac{\theta_1^*\ (\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}}{1 - [a \sum_{j=1}^{n} |a_j| + 1]\frac{j\ (\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}}.$$

Also

$$\begin{aligned}|A_2\nu(\mathfrak{r})| &= \left|\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\alpha-1}}{\Gamma(\alpha)} \theta_2(\eta, \nu(\varphi(\eta)))d\eta\right| \\ &\leqslant \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\alpha-1}}{\Gamma(\alpha)} [a(\eta) + b\ |\nu(\varphi(\eta))|]d\eta.\end{aligned}$$

Taking supremum over $\mathfrak{r} \in I$, we have

$$\begin{aligned}\|A_2\nu\| &\leqslant I^{\alpha-\gamma;\psi} I^{\gamma;\psi} a(\mathfrak{r}) + b r_2 I^{\alpha;\psi}(\mathfrak{r}) \\ &\leqslant M \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} d\eta + b r_2 \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\alpha-1}}{\Gamma(\alpha)} d\eta \\ &\leqslant \frac{M\ (\psi(\mathfrak{r}) - \psi(0))^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + b r_2\ \frac{(\psi(\mathfrak{r}) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)}, \\ r_2 &= (1 - \frac{b\ (\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)})^{-1} \frac{M\ (\psi(T) - \psi(0))^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.\end{aligned}$$

Now

$$\begin{aligned}\|Au\|_X &= \|A_1 w\|_C + \|A_2\nu\|_C \\ &\leqslant r_1 + r_2 \\ &\leqslant \frac{a|\nu_\circ| + [a \sum_{j=1}^{n} |a_j| + 1]\frac{\theta_1^*\ (\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}}{1 - [a \sum_{j=1}^{n} |a_j| + 1]\frac{k\ (\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}} \\ &\quad + (1 - \frac{b\ (\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)})^{-1} \frac{M\ (\psi(T) - \psi(0))^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\ &= r.\end{aligned}$$

Then $AQ_r \subset Q_r$ and the class $\{Au\}$, $u \in Q_r$ is uniformly bounded.

Now, for $u = (\nu, \mu) \in Q_r$, for all $\epsilon > 0$, $\delta > 0$ and for each $\mathfrak{r}_1, \mathfrak{r}_2 \in [0, T]$, $\mathfrak{r}_1 < \mathfrak{r}_2$ such that $|\mathfrak{r}_2 - \mathfrak{r}_1| < \delta$, we have

$$\begin{aligned}|A_1 w(\mathfrak{r}_2) - A_1 w(\mathfrak{r}_1)| &\leqslant \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{\psi'(\eta)(\psi(\mathfrak{r}_2) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} |\theta_1(\eta, \mu(\eta), w(\eta))| d\eta \\ &\quad + \int_0^{\mathfrak{r}_1} \left[\frac{\psi'(\eta)(\psi(\mathfrak{r}_2) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} - \frac{\psi'(\eta)(\psi(\mathfrak{r}_1) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)}\right] |\theta_1(\eta, \mu(\eta))| d\eta \\ &\leqslant (k(\|\delta\| + |w|) + \theta_1^*) \frac{(\psi(\mathfrak{r}_2) - \psi(\mathfrak{r}_1))^\beta}{\Gamma(\beta+1)} \\ &\quad + (k(\|\delta\| + |w|) + \theta_1^*) \left(\frac{-(\psi(\mathfrak{r}_2) - \psi(\mathfrak{r}_1))^\beta}{\Gamma(\beta+1)} + \frac{\psi(\mathfrak{r}_2)^\beta}{\Gamma(\beta+1)} - \frac{\psi(\mathfrak{r}_1)^\beta}{\Gamma(\beta+1)}\right) \\ &\leqslant (k(\|\delta\| + |w|) + \theta_1^*) (\frac{\psi(\mathfrak{r}_2)^\beta - \psi(\mathfrak{r}_1)^\beta}{\Gamma(\beta+1)}),\end{aligned}$$

and

$$\begin{aligned}
&|A_2\nu(\mathfrak{r}_2)-A_2\nu(\mathfrak{r}_1)|\\
&\quad\leqslant\left|\int_0^{\mathfrak{r}_2}\frac{\psi'(s)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta-\int_0^{\mathfrak{r}_1}\frac{\psi'(s)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\right|\\
&\quad\quad+\left|\int_0^{\mathfrak{r}_1}\frac{\psi'(s)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta-\int_0^{\mathfrak{r}_1}\frac{\psi'(\eta)(\psi(\mathfrak{r}_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\right|\\
&\quad\leqslant\left|\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{\psi'(\eta)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\right|+\left|\int_0^{\mathfrak{r}_1}\frac{\psi'(\eta)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\right|\\
&\quad\quad-\left|\int_0^{\mathfrak{r}_1}\frac{\psi'(\eta)(\psi(\mathfrak{r}_1)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\right|\\
&\quad\leqslant\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{\psi'(s)(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\eta,\nu(\varphi(\eta)))|d\eta\\
&\quad\quad+\int_0^{\mathfrak{r}_1}\frac{(\psi(\mathfrak{r}_2)-\psi(\eta))^{\alpha-1}-(\psi(\mathfrak{r}_1)-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\eta,\nu(\varphi(\eta)))|\,\psi'(\eta)\,d\eta\\
&\quad\leqslant(a+br_2)\frac{(\psi(\mathfrak{r}_2)-\psi(\mathfrak{r}_1))^{\alpha}}{\Gamma(\alpha+1)}+(a+br_2)\left(\frac{-(\psi(\mathfrak{r}_2)-\psi(\mathfrak{r}_1))^{\alpha}}{\Gamma(\alpha+1)}+\frac{\psi(\mathfrak{r}_2)^{\alpha}}{\Gamma(\alpha+1)}-\frac{\psi(\mathfrak{r}_1)^{\alpha}}{\Gamma(\alpha+1)}\right)\\
&\quad\leqslant(a+br_2)\frac{\psi(\mathfrak{r}_2)^{\alpha}-\psi(\mathfrak{r}_1)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
Au(\mathfrak{r}_2)-Au(\mathfrak{r}_1)&=A(\nu,w)(\mathfrak{r}_2)-A(\nu,w)(\mathfrak{r}_1)\\
&=(A_2\nu(\mathfrak{r}_2),A_1w(\mathfrak{r}_2))-(A_2\nu(\mathfrak{r}_1),A_1w(\mathfrak{r}_1))\\
&=(A_2\nu(\mathfrak{r}_2)-A_2\nu(\mathfrak{r}_1),A_1w(\mathfrak{r}_2)-A_1w(\mathfrak{r}_1)),
\end{aligned}$$

so,

$$\begin{aligned}
|Au(\mathfrak{r}_2)-Au(\mathfrak{r}_1)|_X&=|A(\nu,\mu)(\mathfrak{r}_2)-A(\nu,\mu)(\mathfrak{r}_1)|_X,\\
&=|A_1w(\mathfrak{r}_2)-A_1w(\mathfrak{r}_2)|_C+|A_2\nu(\mathfrak{r}_2)-A_2\nu(\mathfrak{r}_1)|_C\\
&=(k(\|\delta\|+|w|)+\theta_1^*)\frac{\psi(\mathfrak{r}_2)^{\beta}-\psi(\mathfrak{r}_1)^{\beta}}{\Gamma(\beta+1)}\;+\;(a+br_2)\;\frac{\psi(\mathfrak{r}_2)^{\alpha}-\psi(\mathfrak{r}_1)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

The class of functions $\{Au\}$ is hence equi-continuous on Q_r. The Arzela-Ascoli Theorem [12] proves that the operator A is compact. There is still evidence to establish the continuity of $A : Q_r \to Q_r$. Let $u_n = (w_n, \nu_n)$ be a sequence in Qr with $w_n \to w$,and $\nu_n \to \nu$ and since $\theta_2(\mathfrak{r}, w(\mathfrak{r}))$ is continuous in $C(I,\mathbb{R})$, then $\theta_2(\mathfrak{r}, \mu_n(\mathfrak{r}))$ converges to $\theta_2(\mathfrak{r}, \mu(\mathfrak{r}))$, thus $\theta_2(\mathfrak{r}, \mu_n(\varphi(\mathfrak{r})))$ converges to $\theta_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))$, By using the Lebesgue Dominated Convergence Theorem and making use of the assumptions (iii)–(iv), we arrive to

$$\lim_{n\to\infty}\int_0^{\mathfrak{r}}\frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\;\theta_2(\eta,\nu_n(\varphi(\eta)))\;d\eta\;=\;\int_0^{\mathfrak{r}}\frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\;\theta_2(\eta,\nu(\varphi(\eta)))\;d\eta,$$

then

$$\begin{aligned}
\lim_{n\to\infty}A_2\nu_n(\mathfrak{r})&=\int_0^{\mathfrak{r}}\frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\;\lim_{n\to\infty}\theta_2(\eta,\nu_n(\varphi(s)))ds\\
&=\int_0^{\mathfrak{r}}\frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\alpha-1}}{\Gamma(\alpha)}\theta_2(\eta,\nu(\varphi(\eta)))d\eta\\
&=A_2\nu(\mathfrak{r}),
\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} A_1 w_n(\mathfrak{r}) =& a(\nu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{\psi'(\eta)(\psi(\tau_k)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \lim_{n\to\infty} \theta_1(\eta,\mu_n(\eta),w_n(\eta))d\eta)\\ &+\int_0^{\mathfrak{r}} \frac{\psi'(s)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \lim_{n\to\infty} \theta_1(\eta,\mu_n(\eta),w_n(\eta)d\eta\\ =& a(\nu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{\psi'(\eta(\psi(\tau_k)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \mathfrak{r}_1(\eta,\mu(\eta),w(\eta))\ d\eta)\\ &+\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta,\mu(\eta),w(\eta)d\eta\\ =& A_1 w(\mathfrak{r}).\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} Au_n(\mathfrak{r}) &= \lim_{n\to\infty}(A_1 w_n(\mathfrak{r}), A_2\nu_n(\mathfrak{r}))\\ &=(\lim_{n\to\infty} A_1 w_n(\mathfrak{r}) \lim_{n\to\infty} A_2\nu_n(\mathfrak{r}))\\ &=(A_1 w(\mathfrak{r}), A_2\nu(\mathfrak{r}))\\ &= Au(\mathfrak{r}).\end{aligned}$$

After that, $Au_n \to Au$ as $n \to \infty$. As a result, the operator A is continuous. If the Schauder fixed-point theorem [13] 's requirements are met, then A has a fixed point $u \in Q_r$, and the problem (2.9)-(2.10) has at least one continuous solutions $u = (\mu, \nu) \in Q_r$, $\mu; \nu \in C(I, \mathbb{R})$. Therefore, there is at least one solution $\mu \in C(I, \mathbb{R})$ to the functional integral equation (1.1).

Conversely, when we differentiate (2.2), we obtain

$$\begin{aligned}{}^cD^{\beta;\psi}\nu(\mathfrak{r}) = {}^cD^{\beta;\psi}\Bigg\{ a\Bigg(\nu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{\psi'(\eta)(\psi(\tau_k)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta\Bigg)\\ +\int_0^{\tau_k} \frac{\psi'(s)(\psi(\tau_k)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta\Bigg\},\end{aligned}$$

$$w(\mathfrak{r}) = I^{\alpha;\phi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r}))).$$

In addition, we find from the equation (2.9)–(2.10)

$$\begin{aligned}\nu(\tau_j) =\ & a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\ d\eta\right)\\ &+\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta))\ d\eta,\\ \nu(0) =\ & a\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))\ d\eta\right),\\ w(\mathfrak{r}) =\ & I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r}))),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sum_{j=1}^{n} a_j\nu(\tau_j) =\ & a\sum_{j=1}^{n} a_j \left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta\right)\\ &+\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta,\\ w(\mathfrak{r}) =\ & I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r}))).\end{aligned} \tag{2.12}$$

So, (2.11) and (2.12) provide the following

$$\nu(0)+\sum_{j=1}^{n} a_j\ \nu(\tau_j) = a(1+\sum_{j=1}^{n} a_j)\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta\right)$$
$$+\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),w(\eta))d\eta.$$

Then

$$\nu(0)+\sum_{j=1}^{n} a_j\ \nu(\tau_j) = \nu_\circ.$$

Thus, the problem (1.1)-(1.5) with feedback control (1.4) has $\nu \in C(I,\mathbb{R})$ as at least one solution. □

2.1. Infinite-point boundary condition (1.2)

Take consideration as well $\nu \in C(I,\mathbb{R})$ be the solution to the non-local problem provided by (1.1) and (1.2).

Theorem 2.4. *Let assumptions* (i)–(v) *of Theorem 2.3 meet and* $S_n^{-1} = 1+\sum_{j=1}^{n} a_j$ *be convergent sequence. Then the non-local problem of* (1.1)-(1.2) *with a feedback control* (1.4) *given by the following equation*

$$\nu(t) = S_n\nu_\circ - S_n\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta$$
$$+\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta, \tag{2.13}$$

has at least one solution $\nu \in C(I,\mathbb{R})$.

Proof. Assume that the infinite point BVP (1.1) and (1.2) with a feedback control (1.4) given by (2.2) has $\nu \in C(I,\mathbb{R})$ as a solution.

$$\nu_n(t) = \frac{1}{(1+\sum_{j=1}^{n} a_j)}\left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))d\eta\right)$$
$$+\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_m(\varphi(\eta))))d\eta. \tag{2.14}$$

Taking into account the limit to (2.14), as $n\to\infty$, we have

$$\lim_{n\to\infty}\nu_n(t)$$
$$=\lim_{n\to\infty}\frac{1}{(1+\sum_{j=1}^{n} a_j)}\left[\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\ d\eta\right]$$
$$+\lim_{n\to\infty}\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_m(\varphi(\eta))))d\eta. \tag{2.15}$$

Now $|a_j\nu(\tau_j)| \leqslant |a_j|\|\nu\|$, therefore by comparison test $\sum_{j=1}^{n} a_j\nu(\tau_j)$is convergent. Also,

$$\left|\int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\ d\eta\right|$$
$$\leqslant \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}(k\ (\|\delta\|+|I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))|)\ +\theta_1^*)d\eta$$

$$\leqslant k \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} (\|\delta\| + [I^{\alpha-\gamma;\psi} I^{\gamma;\psi} a(\eta) + br_2 \int_0^{\eta} \frac{\psi'(\theta)(\psi(\eta)-\psi(\theta))^{\alpha-1}}{\Gamma(\alpha)} d\theta\,] d\eta)$$
$$+ k\, \theta_1^* \frac{\psi(\tau_k)^{\beta}}{\Gamma(\beta+1)}$$
$$\leqslant \frac{(\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)} \left(k\|\delta\| + \left[\frac{k\, M\, (\psi(T)-\psi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+1)} + \frac{k\, br_2\, (\psi(T)-\psi(0))^{\alpha+1}}{\Gamma(\alpha+1)} + k\, \theta_1^* \right] \right)$$
$$\leqslant N,$$

then

$$|a_j|\, | \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))\, d\eta| \leqslant |a_j|\, N.$$

The sequence $\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))\, d\eta$ is convergent according to the comparison test.

By applying the Lebesgue-Dominated convergence Theorem [21] and making assumptions $(i)-(iii)$, we can derive (2.13) from (2.15). In addition, from (2.13), we have

$$(1+\sum_{j=1}^{n} a_j)\nu(\tau_j) = S_n^{-1} S_n \nu_\circ - S_n^{-1} S_n \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(s)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta$$
$$+ (1+\sum_{j=1}^{n} a_j) \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta,$$

$$\nu(\tau_j) + \sum_{j=1}^{n} a_j\, \nu(\tau_j) = \nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta$$
$$+ \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta$$
$$+ \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta,$$

$$\sum_{k=1}^{m} a_k\, \nu(\tau_k) = \nu_\circ - \nu(\tau_k) + \int_0^{\tau_k} \frac{\psi'(\eta)(\psi(\tau_k)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta. \tag{2.16}$$

From (2.2), we have

$$\nu(0) = a \left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta \right),$$

and

$$\nu(\tau_j) = a \left(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \mu(\varphi(\eta))))d\eta \right)$$
$$+ \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\phi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta.$$

So

$$\nu(0) = \nu(\tau_j) - \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta.$$

Return to (2.16) we receive infinite-point (1.2)

$$\nu(0) + \sum_{j=1}^{\infty} a_j \, \nu(\tau_j) = \nu_\circ.$$

As a result, there is at least one solution $\nu \in C(I, \mathbb{R})$ to the nonlocal problem of functional differential inclusion (1.1)-(1.2). □

2.2. Riemann-Stieltjes Integral BCs (1.3)

Assume that $\nu \in C(I, \mathbb{R})$ be the solution of the non-local problem of (1.1)-(1.5). Let $a_j = h(\mathfrak{r}_j) - h(\mathfrak{r}_{j-1})$, h is non-decreasing function, $\tau_j \in (\mathfrak{r}_{j-1}, \mathfrak{r}_j)$, $0 = \mathfrak{r}_0 < \mathfrak{r}_1 < \mathfrak{r}_2 \cdots < T$. Then, the nonlocal condition (1.5) will be in the form

$$\nu(0) + \sum_{j=1}^{n} \nu(\tau_j) \, (\mathfrak{h}(\mathfrak{r}_j) - \mathfrak{h}(\mathfrak{r}_{j-1})) = \nu_\circ.$$

From [18], we deduce as $n \to \infty$ the solution of the nonlocal problem (1.1)-(1.5) is continues.

$$\lim_{n\to\infty} \sum_{j=1}^{n} \nu(\tau_j) \, (\mathfrak{h}(\mathfrak{r}_j) - \mathfrak{h}(\mathfrak{r}_{j-1})) = \int_0^T \nu(\eta) \, d\mathfrak{h}(\eta).$$

This means that the Riemann-Steltjes integral condition as $n \to \infty$ is modified by the non-local constraints in (1.5) .

$$\nu(0) + \lim_{n\to\infty} \sum_{j=1}^{n} \nu(\tau_j) \, (\mathfrak{h}(\mathfrak{r}_j) - \mathfrak{h}(\mathfrak{r}_{j-1})) = \nu(0) + \int_0^T \nu(\eta) \, d\mathfrak{h}(\eta) = \nu_\circ.$$

Theorem 2.5. *Let assumptions* (i)–(v) *of Theorem 2.3 meet and the function* $\mathfrak{h} : I \to I$ *is an increasing, then the non-local issue* (1.1) *and the Riemann-Stieltjes functional integral condition* (1.3) *with a feedback control* (1.4) *have a solution* $\nu \in C(I, \mathbb{R})$ *denoted by*

$$\begin{aligned}\nu(\mathfrak{r}) =& (1 + \mathfrak{h}(T) - \mathfrak{h}(0))^{-1}\nu_\circ - (1 + \mathfrak{h}(T) \\ &- \mathfrak{h}(0))^{-1} \int_0^T \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta \, d\mathfrak{h}(\eta) \\ &+ \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta.\end{aligned} \tag{2.17}$$

Proof. The following will be the answer to the non-local problem (1.1)-(1.5) as $n \to \infty$:

$$\begin{aligned}\nu(\mathfrak{r}) =& \lim_{n\to\infty} \frac{1}{(1 + \sum_{j=1}^{n} a_j)} \Big(\nu_\circ - \sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta\Big) \\ &+ \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r}) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta \\ =& \frac{1}{(1 + \mathfrak{h}(T) - \mathfrak{h}(0))} \Big(\nu_\circ - \lim_{n\to\infty} \sum_{j=1}^{n} (\mathfrak{h}(\mathfrak{r}_j) - h(\mathfrak{r}_{j-1})) \\ &\times \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j) - \psi(\eta))^{\beta-1}}{\Gamma(\beta)} \theta_1(\eta, \mu(\eta), I^{\alpha;\psi}\theta_2(\eta, \nu(\varphi(\eta))))d\eta\Big)\end{aligned}$$

$$+\int_0^{\tau}\frac{(\tau-\eta)^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\,d\eta$$
$$=\frac{1}{(1+\mathfrak{h}(T)-\mathfrak{h}(0))}\Big(\nu_\circ-\int_0^T\int_0^{\tau}\frac{\psi'(\eta)(\psi(\tau)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,x(\varphi(\eta))))d\eta\,d\mathfrak{h}\Big)$$
$$+\int_0^{\tau}\frac{\psi'(\eta)(\psi(\tau)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta))))\,d\eta.$$

□

Consequently, the first-order nonlinear differential Equation (1.1)'s solution $\nu \in C(I,\mathbb{R})$ via the Riemann-Stieltjes integral condition (1.3) is denoted by (2.17). Thus, the inclusion problem (1.1)-(1.3) has at least one solution, $\nu \in C(I,\mathbb{R})$.

3. Existence of unique solutions

This section provides the conditions needed for the uniqueness solution for the non-local problems (1.1)-(1.5) . The next assumption should be taken

$(iii)^*$ Let $\theta_2 : I\times R \to R$, be a continuous function satisfying the Lipschitz condition, such that

$$|\theta_2(\tau,\nu)-\theta_2(\tau,\mu)| \leqslant c\,|\nu-\mu|.$$

Theorem 3.1. *Let the assumptions of Theorem 2.3 be satisfied with replace condition (iii) by* $(iii)^*$, *if*

$$\frac{(\,a\,\sum_{j=1}^{n}\,a_j+1)\,(\psi(T)-\psi(0))^{\alpha+\beta}\,k\,c}{\Gamma(\beta+1)\Gamma(\alpha+1)}<1.$$

Then the non-local problem (1.1)-(1.5) *with a feedback control* (1.4) *has a unique solution* $\nu \in C(I,\mathbb{R})$.

Proof. Assuming there are two solutions $\nu_1(\tau)$ and $\nu_2(\tau)$ to equation (2.2), then

$$\begin{aligned}
&|\nu_1(\tau)-\nu_2(\tau)|\\
&\leqslant a\Big|\sum_{j=1}^{n}a_j\,\Big(\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_2(\varphi(\eta))))d\eta\\
&\quad-\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\phi(\eta))^{\beta-1}}{\Gamma(\beta)}\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_1(\varphi(\eta))))\,d\eta\Big|\\
&\quad+\Big|\int_0^{t}\frac{\psi'(\eta)(\phi(\tau)-\phi(\eta))^{\beta-1}}{\Gamma(\beta)}\big[\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_1(\varphi(\eta))))-\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_2(\varphi(\eta))))\,\big]\,d\eta\Big|\\
&\leqslant a\sum_{j=1}^{n}a_j\Big(\int_0^{\tau_j}\frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_2(\varphi(\eta)))\\
&\quad-\theta_1(\eta,\mu(\eta),I^{\alpha}\theta_2(\eta,\nu_1(\varphi(\eta))))|d\eta\Big)\\
&\quad+\int_0^{\tau}\frac{\psi'(\eta)(\psi(\tau)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_1(\varphi(\eta))))-\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu_2(\varphi(\eta))))|d\eta.
\end{aligned}$$

Using Lipschitz condition for θ_1, we obtain

$$
\begin{aligned}
&|v_1(t)-v_2(t)|\\
&\leqslant a\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} k|I^{\alpha;\psi}\theta_2(\eta,v_1(\varphi(\eta)))-I^{\alpha}\theta_2(\eta,v_2(\varphi(\eta)))|\,d\eta\\
&\quad+\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} k|I^{\alpha;\psi}\theta_2(\eta,v_1(\varphi(\eta)))-I^{\alpha}\theta_2(\eta,v_2(\varphi(\eta)))|\,d\eta\\
&\leqslant a\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\\
&\quad\times\ k\int_0^{\eta} \frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\tau,v_1(\varphi(\tau)))-\theta_2(\tau,v_2(\varphi(\tau)))|d\tau d\eta\\
&\quad+\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} k\int_0^{\eta} \frac{\psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\tau,v_1(\varphi(\tau)))-\theta_2(\tau,v_2(\varphi(\tau)))|d\tau d\eta.
\end{aligned}
$$

Using Lipschitz condition for θ_2, we obtain

$$
\begin{aligned}
&|v_1(t)-v_2(t)|\\
&\leqslant a\sum_{j=1}^{n} a_j\, kc\int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|v_1(\varphi(\tau))-v_2(\varphi(\tau))|d\tau)d\eta\\
&\quad+kc\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|v_1(\varphi(\tau))-\sigma_2(\varphi(\tau))\ |d\tau d\eta\\
&\leqslant a\sum_{j=1}^{n} a_j k\ c\|v_1-v_2\|\int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\ \ d\eta\\
&\quad+k\ c\|v_1-v_2\|\int_0^{t} \frac{\psi'(\eta)(\psi(t)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\ \ d\eta,
\end{aligned}
$$

$$
\|v_1-v_2\|\leqslant\frac{(\ a\ \sum_{j=1}^{n}\ a_j+1)\ (\psi(T)-\psi(0))^{\alpha+\beta}\ k\ c}{\Gamma(\beta+1)\Gamma(\alpha+1)}\|v_1-v_2\|.
$$

Hence

$$
\left(1-\frac{(\ a\ \sum_{j=1}^{n}\ a_j+1)\ (\psi(T)-\psi(0))^{\alpha+\beta}\ k\ c}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right)\|v_1-v_2\|\leqslant 0.
$$

Since $\frac{(\ a\ \sum_{j=1}^{n}\ a_j+1)\ (\psi(T)-\psi(0))^{\alpha+\beta}\ k\ c}{\Gamma(\beta+1)\Gamma(\alpha+1)}<1$, then $v_1(t)=v_2(t)$, therefore the solution of the integral equation (2.2) is unique, and consequence the integral equation (2.2) has a unique solution, and as a result, this establishes the existence of unique solutions to the non-local problem (1.1), (1.5) with a feedback control (1.4). □

3.1. Continuous dependence

Theorem 3.2. *Count on the assumptions of Theorem 3.1 being valid. Therefore, the solution of the problem (1.1)-(1.5) is continuously dependent on S_{Φ_1}.*

Proof. Assuming that $\theta_1(t,v(t))$ and $\theta_1^*(t,v(t))$ are two different Lipschitzian selections of $\Theta_1(t,v(t))$, then

$$
|\theta_1(t,v(t))-\theta_1^*(t,v(t)|<\epsilon,\quad \epsilon>0,\quad t\in I,
$$

then

$$
\begin{aligned}
&|\nu(\mathfrak{r})-\nu^*(\mathfrak{r})| \\
&\leqslant a\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times |\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta)))-\theta_1^*(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|d\eta \\
&\quad +\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))-\theta_1^*(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|d\eta \\
&\leqslant a\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times |\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta)))-\theta_1(\eta,\mu(\eta),I^{\alpha}\theta_2(\eta,\nu^*(\varphi(\eta)))|d\eta \\
&\quad + a\sum_{j=1}^{n} a_j \int_0^{\tau_k} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times |\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))-\theta_1^*(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|d\eta \\
&\quad +\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu(\varphi(\eta)))-\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|\, d\eta \\
&\quad +\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu^*(\varphi(\eta)))-\theta_1^*(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|d\eta \\
&\leqslant a\sum_{j=1}^{n} a_j \int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times (|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu(\varphi(\eta)))-\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu^*(\varphi(\eta)))|+\delta)d\eta \\
&\quad +\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\mu(\varphi(\eta)))-\theta_1(\eta,\mu(\eta),I^{\alpha;\psi}\theta_2(\eta,\nu^*(\varphi(\eta)))|\, d\eta \\
&\quad +\int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\delta\, d\eta \\
&\leqslant a\sum_{j=1}^{n} a_j k\Bigg(\int_0^{\mathfrak{r}_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(t)))-I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu^*(\varphi(\mathfrak{r}))|d\eta \\
&\quad +\frac{\delta\,(\psi(\tau_j)-\psi(0))^{\beta}}{\Gamma(\beta+1)}\Bigg) \\
&\quad +k\int_0^{\mathfrak{r}} \frac{\psi'(\mathfrak{r})(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}|I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r})))-I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu^*(\varphi(\mathfrak{r}))|\, d\mathfrak{r}+\frac{\delta\,(\psi(\mathfrak{r})-\psi(0))^{\beta}}{\Gamma(\beta+1)} \\
&\leqslant a\sum_{j=1}^{n} a_j k\Bigg(\int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{(\eta-\tau)^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\tau,\nu(\varphi(\tau)))-\theta_2(\tau,\nu^*(\varphi(\tau)))|d\tau\, d\eta \\
&\quad +\frac{\delta\,(\psi(\tau_j)-\psi(0))^{\beta}}{\Gamma(\beta+1)}\Bigg) \\
&\quad +k\int_0^{\mathfrak{r}} \frac{\psi'(s)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}|\theta_2(\tau,\nu(\varphi(\tau)))-\theta_2(\tau,\nu^*(\varphi(\tau)))|\, d\tau\, d\rho \\
&\quad +\frac{\delta\,(\psi(\mathfrak{r})-\psi(0))^{\beta}}{\Gamma(\beta+1)} \\
&\leqslant \|\nu-\nu^*\|\Bigg(a\sum_{j=1}^{n} a_j\, k\, c\int_0^{\tau_j} \frac{\psi'(\eta)(\psi(\tau_j)-\psi(\eta))^{\beta-1}}{\Gamma(\beta)}\int_0^{\eta}\frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\, d\tau\, d\eta
\end{aligned}
$$

$$+ \ k\, c \int_0^{\mathfrak{r}} \frac{\psi'(\eta)(\psi(\mathfrak{r})-\psi(\eta))^{\beta-1}}{\Gamma(\beta)} \int_0^{\eta} \frac{\psi'(\tau)(\psi(\eta)-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\, d\tau\, d\eta \Big)$$
$$+ \Big(a \sum_{j=1}^{n} a_j\, k\, c\, + 1\Big) \frac{\delta\, (\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)}$$
$$\leqslant \frac{(a\sum_{j=1}^{n}\ a_j+1)kc(\psi(T)-\psi(0))^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\|\nu-\nu^*\| + (a\sum_{j=1}^{n}\ a_j kc+1)\frac{\delta\,(\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)}.$$

Then

$$\|\nu-\nu^*\| \leqslant \left(1 - \frac{(\ a\ \sum_{j=1}^{n} a_j+1)kc(\psi(T)-\psi(0))^{\alpha+\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\right)^{-1} (a\sum_{j=1}^{n} a_j\ kc+1)\ \frac{\delta\,(\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)}$$
$$=\epsilon.$$

Hence,

$$\|\nu-\nu^*\| \leqslant \epsilon.$$ □

4. Examples

In this section, we give an illustration to support our findings.

Example 1. Take into account the nonlinear integro-differential inclusion that follows:

$$^{c}D^{\alpha;\psi}\nu(\mathfrak{r}) \in \Theta_1(\mathfrak{r}, I^{\alpha;\psi}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r})))),\quad \mathfrak{r}\in[0,1],\ \alpha\in(0,1) \tag{4.1}$$

with infinite point boundary condition

$$\nu(0)+\sum_{j=1}^{\infty}\ \frac{1}{j^2}\ \nu(\frac{j-1}{j}) = \nu_\circ, \tag{4.2}$$

and feedback

$$\mu(\mathfrak{r}) = 0.1\ \mu(\mathfrak{r}) + \frac{1}{200}\cos(\mathfrak{r}) +\ e^{\frac{-3}{2}\mathfrak{r}}\ \nu(\mathfrak{r}). \tag{4.3}$$

To demonstrate Theorem 2.3, we select $\alpha = \frac{1}{4}$, $\psi(\mathfrak{r}) = \mathfrak{r}$, and $\Theta_1 : [0,1]\times\mathbb{R}\to 2^{\mathbb{R}^+}$ in (4.1) as follows:

$$\Theta_1(\mathfrak{r}, I^{\frac{1}{4};\mathfrak{r}}\theta_2(\mathfrak{r},\nu(\mathfrak{r}))) = \left[0, \mathfrak{r}^3+\mathfrak{r}+1+\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\eta)^{-\frac{3}{4}}}{2\ \Gamma(\frac{3}{4})}\big(\cos(\nu(\eta)+1)+\frac{\nu(\eta)}{e^{\eta}}\big)\ d\eta\right],$$

set

$$\theta_2(\mathfrak{r},\nu(\mathfrak{r})) = \frac{1}{2}\big(\cos(\nu(\mathfrak{r})+1)+\frac{\nu(\mathfrak{r})}{e^{\mathfrak{r}}}\big).$$

The function $\theta_1 : [0,1]\times\mathbb{R}\to\mathbb{R}$ is Define the continuous, observe that for $\theta_1 \in S_{\Theta_1}$, then we get

$$|\theta_1(\mathfrak{r}, I^{\frac{1}{4};\mathfrak{r}}\theta_2(\mathfrak{r},\nu(\varphi(\mathfrak{r})))) - \theta_1(\mathfrak{r}, I^{\frac{1}{4};\mathfrak{r}}\theta_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))))| \leqslant \frac{1+e}{2\ e\ \Gamma(\frac{1}{4})}|\nu-\mu|,$$

and

$$\big|\theta_2(\mathfrak{r},\nu(\mathfrak{r}))\big| \leqslant \frac{1}{2}|\cos(\nu(\mathfrak{r})+1)| + \frac{|\nu(\mathfrak{r})|}{2e}.$$

As a result, requirements of conditions (i) and (iii) are meet with $k = \frac{1+e}{2\ e\ \Gamma(\frac{1}{4})} \approx 0.1889 < 1$, $a(\mathfrak{r}) = \frac{1}{2}\ \cos(\nu(\mathfrak{r})+1)\ \in L^1[0,1]$, $b = \frac{1}{2e}$ and the series $\sum_{j=1}^{\infty}\ \frac{1}{j^4}$ is convergent. Furthermore, $[a(|\nu_\circ|+\sum_{j=1}^{n}|a_j|)+1]k(\psi(T)-\psi(0)) \approx 0.6136 < 1$ and $\frac{b\ (\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \approx 0.203 < 1$.

The above nonlocal issue (4.1)-(4.2) with feedback (4.3) has at least one continuous solution, according to Theorem 2.3, which is a logical consequence.

Example 2. Consider the following nonlinear integro-differential inclusion:

$$^{c}D^{\alpha,\psi}x(\mathfrak{r}) \in \Theta_1(\mathfrak{r},\mu(\mathfrak{r}), I^{\alpha;\psi}\theta_2(\mathfrak{r},x(\varphi(\mathfrak{r})))), \quad \mathfrak{r} \in [0,1], \ \alpha \in (0,1) \tag{4.4}$$

with infinite point boundary condition

$$x(0) + \sum_{k=1}^{\infty} \frac{1}{k^4}\, x\Big(\frac{k^2+k-1}{k^2+k}\Big) = x_\circ, \tag{4.5}$$

and feedback

$$\mu(\mathfrak{r}) = \frac{1}{20}\,\mu(\mathfrak{r}) + \frac{1}{300}\sin(\mathfrak{r}) + \, e^{\frac{-4}{3}\mathfrak{r}}\,\nu(\mathfrak{r}). \tag{4.6}$$

For illustrating Theorem 3.1, we choose $\alpha = \frac{1}{2}$, $\psi(t) = \sqrt{\mathfrak{r}+1}$, and $\Theta_1 : [0,1]\times\mathbb{R} \to 2^{\mathbb{R}^+}$ in (4.4) as follows

$$\Theta_1(t, I^{\frac{1}{2},\sqrt{\mathfrak{r}+1}}\theta_2(\mathfrak{r},x(\mathfrak{r})) = \left[0, \frac{e^{-t}}{e^t+5} + \int_0^{\mathfrak{r}} \frac{\sqrt{(\mathfrak{r}-s)}}{2\sqrt{\pi}e^{s+1}}\left(\frac{2+|\sin x(s)|}{1+|\sin x(s)|}\right) ds\right].$$

Set

$$\theta_2(\mathfrak{r},x(\mathfrak{r})) = \frac{1}{2e^{\mathfrak{r}+1}}\left(\frac{2+|\sin x(\mathfrak{r})|}{1+|\sin x(s)|}\right).$$

Define the continuous map $\theta_1 : [0,1]\times\mathbb{R} \to \mathbb{R}$, notice that for $\theta_1 \in S_{F_1}$, then we have

$$|\theta_1(\mathfrak{r}, I^{\frac{1}{2},\sqrt{\mathfrak{r}+1}}\theta_2(\mathfrak{r},x(\varphi(\mathfrak{r})))) - \theta_1(\mathfrak{r}, I^{\frac{1}{2},\sqrt{\mathfrak{r}+1}}\theta_2(\mathfrak{r},y(\varphi(\mathfrak{r}))))| \leqslant \frac{1}{2e^2\sqrt{\pi}}|x-y|.$$

and

$$\left|\theta_2(\mathfrak{r},x(\mathfrak{r})) - \theta_2(\mathfrak{r},y(\mathfrak{r}))\right| \leqslant \frac{1}{2e^2}|x-y|.$$

Thus conditions (i)-(iii)* are satisfied with $k = \frac{1}{2e^2\sqrt{\pi}}$, and $c = \frac{1}{2e^2}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent. Also $\frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ (\phi(T)-\phi(0)^{\alpha}\ k\ c}{\Gamma(\alpha+1)} = \frac{1}{4e^4\pi} < 1$. It follows from Theorem 3.1 that the nonlocal problem (4.4)-(4.5) has a unique continuous solution.

5. Conclusion

Considering the nonlocal infinite-point, Riemann-Stieltjes integral boundary conditions (BCs), we described the existence criterion for solutions to ψ-Caputo fractional fractional differential equations and inclusions. First, we converted the nonlinear fractional boundary value ψ-Caputo type problem into a fixed point issue. We have shown that if we can solve the boundary value problems with integral BCs or infinite-point BCs, we can also solve the boundary value problems with continuous solutions with m-point BCs. the uniqueness solution, the continuous dependence of the functional differential inclusion on the set of selections, and certain data were analyzed, and we showed the existence of a continuous solution for the single-valued and set-valued case. Appropriate examples were presented to ensure the validity of all the acquired theoretical results.

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