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Some new properties of generalized (p, q)-numbers and generating functions for certain products

Nabiha Saba[a], Ali Boussayoud[a], Kasi Viswanadh V. Kanuri[b,*]

[a] *LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria.*

[b] *3669 Leatherwood Dr, Frisco, TX 75033, USA.*

Abstract

In this paper, we first give some new results and properties on the generalized (p, q)-numbers including explicit formula, negative extension, etc. After that, by using the complete homogeneous symmetric functions we obtain the new generating functions for the products of some bivariate polynomials with certain (p, q)-numbers at positive and negative indices.

Keywords: Generalized (p, q)-numbers, bivariate Mersenne polynomials, bivariate Mersenne Lucas polynomials, complete homogeneous symmetric functions, generating functions, explicit formula.

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1. Introduction

Special integer sequences especially Mersenne sequence are encountered in different branches of science. Also, the Mersenne Lucas sequence is one of the most famous and curious numerical sequence in mathematics. The Mersenne sequence $\{M_n\}_{n\in\mathbb{N}}$ (see [5]) is defined by:

$$M_n := \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 3M_{n-1} - 2M_{n-2}, & \text{if } n \geqslant 2, \end{cases}$$

and the Mersenne Lucas sequence $\{m_n\}_{n\in\mathbb{N}}$ (see [16]) is defined by:

$$m_n := \begin{cases} 2, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 3m_{n-1} - 2m_{n-2}, & \text{if } n \geqslant 2. \end{cases}$$

From the work [1], we know that the generalized Mersenne numbers called bivariate Mersenne poly-

*Corresponding author

Email address: vis.kanuri@gmail.com (Kasi Viswanadh V. Kanuri)

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nomials which are defined as follows:

$$M_n(x,y) := \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 3yM_{n-1}(x,y) - 2xM_{n-2}(x,y), & \text{if } n \geqslant 2, \end{cases} \tag{1.1}$$

and in [11], the authors introduced the generalized Mersenne Lucas numbers called bivariate Mersenne Lucas polynomials which are defined as follows:

$$m_n(x,y) := \begin{cases} 2, & \text{if } n = 0, \\ 3y, & \text{if } n = 1, \\ 3ym_{n-1}(x,y) - 2xm_{n-2}(x,y), & \text{if } n \geqslant 2. \end{cases} \tag{1.2}$$

From Eqs. (1.1) and (1.2) we thus have Table 1 for the bivariate Mersenne polynomials $\{M_n(x,y)\}_{n\in\mathbb{N}}$ and bivariate Mersenne Lucas polynomials $\{m_n(x,y)\}_{n\in\mathbb{N}}$.

Table 1: The first few terms of bivariate Mersenne and bivariate Mersenne Lucas polynomials.

n	$M_n(x,y)$	$m_n(x,y)$
0	0	2
1	1	$3y$
2	$3y$	$9y^2-4x$
3	$9y^2-2x$	$27y^3-18xy$
4	$27y^3-12xy$	$81y^4-72xy^2+8x^2$
5	$81y^4-54xy^2+4x^2$	$243y^5-270xy^3+60x^2y$
$\vdots$	$\vdots$	$\vdots$

In modern science, there are a huge interest in (p,q)-numbers and thier properties in [8, 10, 14, 18, 19]. There are many generalizations of these numbers, the generalized (p,q)-Fibonacci numbers $\{f_{p,q,n}(\alpha,\beta,\gamma)\}_{n\in\mathbb{N}}$, generalized (p,q)-Pell numbers $\{l_{p,q,n}(\alpha,\beta,\gamma)\}_{n\in\mathbb{N}}$, and generalized (p,q)-Jacobsthal numbers $\{C_{p,q,n}(\alpha,\beta,\gamma)\}_{n\in\mathbb{N}}$ [16] are one of them,

$$f_{p,q,0} = \alpha, \quad f_{p,q,1} = \beta + \gamma p, \quad \text{and} \quad f_{p,q,n} = pf_{p,q,n-1} + qf_{p,q,n-2}, \tag{1.3}$$
$$l_{p,q,0} = \alpha, \quad l_{p,q,1} = \beta + 2\gamma p, \quad \text{and} \quad l_{p,q,n} = 2pl_{p,q,n-1} + ql_{p,q,n-2}, \tag{1.4}$$

and

$$C_{p,q,0} = \alpha, \quad C_{p,q,1} = \beta + \gamma p, \quad \text{and} \quad C_{p,q,n} = pC_{p,q,n-1} + 2qC_{p,q,n-2}. \tag{1.5}$$

Note that for the case $\alpha = \gamma = 0$ and $\beta = 1$ the relationships (1.3), (1.4), and (1.5) are reduced to the well-known recurrence relations for (p,q)-Fibonacci, (p,q)-Pell, and (p,q)-Jacobsthal numbers (see [6, 19, 20]). And for the case $\alpha = 2, \gamma = 1$, and $\beta = 0$ in (1.3), (1.4), and (1.5) we get the recurrence relations for (p,q)-Lucas, (p,q)-Pell Lucas, and (p,q)-Jacobsthal Lucas numbers (see [6, 18, 20]). In addition, Saba et al. in [13] defined the generalized (p,q)-numbers $\{W_{p,q,n}(a,b,\alpha,\beta,\gamma)\}_{n\in\mathbb{N}}$ or shortly $\{W_{p,q,n}\}_{n\in\mathbb{N}}$ as follows:

$$W_{p,q,n} = apW_{p,q,n-1} + bqW_{p,q,n-2}, \ n \geqslant 2,$$

with $W_{p,q,0} = \alpha$, $W_{p,q,1} = \beta p + \gamma$, and $\{a,b,\alpha,\beta,\gamma\} \in \mathbb{C}$.

The members of this sequence can also be obtained different ways. It appears that this can be done in either of two ways: the Binet's formula or generating function. The Binet's formula of generalized (p,q)-numbers is given by:

$$W_{p,q,n} = \frac{Ax_1^n - Bx_2^n}{x_1 - x_2}, \tag{1.6}$$

with $A = \beta p + \gamma - \alpha x_2$ and $B = \beta p + \gamma - \alpha x_1$. As a second way, the generating function of generalized (p, q)-numbers is given by:

$$\sum_{n=0}^{\infty} W_{p,q,n} z^n = \frac{\alpha + (p(\beta - \alpha a) + \gamma) z}{1 - apz - bqz^2},$$

with

$$W_{p,q,n} = \alpha h_n \left(\frac{ap + \sqrt{a^2p^2 + 4bq}}{2}, \frac{ap - \sqrt{a^2p^2 + 4bq}}{2} \right) + (p(\beta - \alpha a) + \gamma) h_{n-1} \left(\frac{ap + \sqrt{a^2p^2 + 4bq}}{2}, \frac{ap - \sqrt{a^2p^2 + 4bq}}{2} \right).$$

Remark 1.1. From which, specifying a, b, α, β, and γ, we get the generating functions of some (p, q)-numbers given in the study [13] as follows.

- If we take $a = b = \gamma = 1$ and $\alpha = \beta = 0$, then we get the generating function of (p, q)-**Fibonacci** numbers as:
$$\sum_{n=0}^{\infty} F_{p,q,n} z^n = \frac{z}{1 - pz - qz^2},$$
with $F_{p,q,n} = h_{n-1} \left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2} \right)$.
- If we take $a = b = \beta = 1$, $\alpha = 2$, and $\gamma = 0$, then we get the generating function of (p, q)-**Lucas** numbers as:
$$\sum_{n=0}^{\infty} L_{p,q,n} z^n = \frac{2 - pz}{1 - pz - qz^2},$$
with $L_{p,q,n} = 2h_n \left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2} \right) - ph_{n-1} \left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2} \right)$.
- If we take $a = \gamma = 1$, $b = 2$, and $\alpha = \beta = 0$, then we get the generating function of (p, q)-**Jacobsthal** numbers as:
$$\sum_{n=0}^{\infty} J_{p,q,n} z^n = \frac{z}{1 - pz - 2qz^2},$$
with $J_{p,q,n} = h_{n-1} \left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2} \right)$.
- If we take $a = \beta = 1$, $b = \alpha = 2$, and $\gamma = 0$, then we get the generating function of (p, q)-**Jacobsthal Lucas** numbers as:
$$\sum_{n=0}^{\infty} j_{p,q,n} z^n = \frac{2 - pz}{1 - pz - 2qz^2},$$
with $j_{p,q,n} = 2h_n \left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2} \right) - ph_{n-1} \left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2} \right)$.
- If we take $a = 2$, $b = \gamma = 1$, and $\alpha = \beta = 0$, then we get the generating function of (p, q)-**Pell** numbers as:
$$\sum_{n=0}^{\infty} P_{p,q,n} z^n = \frac{z}{1 - 2pz - qz^2},$$
with $P_{p,q,n} = h_{n-1} \left(p + \sqrt{p^2 + q}, p - \sqrt{p^2 + q} \right)$.

- If we take $a = \alpha = \beta = 2$, $b = 1$, and $\gamma = 0$, then we get the generating function of (p, q)-**Pell Lucas** numbers as: $\sum_{n=0}^{\infty} Q_{p,q,n}z^n = \frac{2-2pz}{1-2pz-qz^2}$, with $Q_{p,q,n} = 2h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) - 2ph_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right)$.

Here we are going to recall some elementary facts on the symmetric functions and to state our main results on this paper. For more details on the symmetric functions, we let the reader to the papers [4, 7, 12, 15].

Definition 1.2 ([7]). Let k and n be two positive integers and $\{a_1, a_2, \ldots, a_n\}$ are set of given variables the k^{th} elementary symmetric function $e_k(a_1, a_2, \cdots, a_n)$ is defined by:

$$e_k^{(n)} = e_k(a_1, a_2, \cdots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n} \qquad (0 \leqslant k \leqslant n),$$

with $i_1, i_2, \ldots, i_n = 0$ or 1.

Remark 1.3. Set $e_0(a_1, a_2, \cdots, a_n) = 1$, by usual convention. For $k < 0$ or $k > n$, we set $e_k(a_1, a_2, \ldots, a_n) = 0$.

Example 1.4. The following lists the n elementary symmetric function for the first three positive values of n.

For $n = 1$: $e_1(a_1) = a_1$.
For $n = 2$: $e_1(a_1, a_2) = a_1 + a_2$, $e_2(a_1, a_2) = a_1a_2$.
For $n = 3$: $e_1(a_1, a_2, a_3) = a_1 + a_2 + a_3$, $e_2(a_1, a_2, a_3) = a_1a_2 + a_1a_3 + a_2a_3$, $e_3(a_1, a_2, a_3) = a_1a_2a_3$.

Proposition 1.5 ([16]). *Given an alphabet $A = \{a_1, a_2, \ldots, a_n\}$, the elementary symmetric function is characterized by the following identity of formal power series in z:*

$$\sum_{k=0}^{\infty} e_k(a_1, a_2, \ldots, a_n)z^k = \prod_{a\in A}(1 + az).$$

Definition 1.6 ([7]). For any natural numbers k and n, the complete homogeneous symmetric function of degree k in n variables $a_1, a_2, \ldots, a_n$ is defined by:

$$h_k^{(n)} = h_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}, \quad (k \geqslant 0),$$

with $i_1, i_2, \ldots, i_n \geqslant 0$.

Remark 1.7. Set $h_0(a_1, a_2, \ldots, a_n) = 1$, by usual convention. For $k < 0$, we set $h_k(a_1, a_2, \ldots, a_n) = 0$.

Definition 1.8 ([9]). Let k be positive integer and $A = \{a_1, a_2\}$ an alphabet. Then, the k^{th} complete homogeneous symmetric function $h_k(a_1, a_2)$ is defined by:

$$h_k^{(2)} = h_k(a_1, a_2) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}, \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}),$$

with

$$h_0(a_1, a_2) = 1, \ h_1(a_1, a_2) = a_1 + a_2, \ h_2(a_1, a_2) = a_1^2 + a_1a_2 + a_2^2, \ \ldots.$$

Proposition 1.9 ([16]). *Given an alphabet $A = \{a_1, a_2, \ldots, a_n\}$, the complete homogeneous symmetric function is characterized by the following identity of formal power series in z:*

$$\sum_{k=0}^{\infty} h_k(a_1, a_2, \ldots, a_n)z^k = \frac{1}{\prod\limits_{a\in A}(1 - az)}.$$

There is a fundamental relation between the elementary symmetric functions and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1,a_2,\dots,a_n)h_{k-j}(a_1,a_2,\dots,a_n)=0,$$

which is valid for all $k>0$.

The following theorem is one of key tools of the proof of our main result. It has been proved in [14].

Theorem 1.10. *Let A and B be two alphabets, respectively, $\{a_1,a_2,\dots,a_k\}$ and $\{b_1,b_2\}$, then we have*

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(a_1,a_2,\dots,a_k)\,h_{n+1-l}(b_1,b_2)\,z^n &= \frac{h_{1-l}(b_1,b_2)-e_1(a_1,a_2,\dots,a_k)\,b_1b_2h_{-l}(b_1,b_2)z}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)}\\ &\quad-\frac{b_1^{2-l}b_2^{2-l}z^{3-l}\sum\limits_{n=0}^{\infty}(-1)^{n-l+3}e_{n-l+3}(a_1,a_2,\dots,a_k)\,h_n(b_1,b_2)z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)},\end{aligned}$$

for all $n\in\mathbb{N}_0$ and $l\in\{0,1,2\}$.

2. Main results of generalized (p,q)-numbers

In this section, we introduce the negative extension, generating function at negative indice and explicit formula of generalized (p,q)-numbers and we give some special cases of them.

2.1. *The negative extension of generalized (p,q)-numbers*

Now, we present the negative extension of the generalized (p,q)-numbers.

Theorem 2.1. *Let n be any positive integer. Then we have:*

$$W_{p,q,-n}=\frac{(-1)^n}{(bq)^n}\left(\frac{Ax_2^n-Bx_1^n}{x_1-x_2}\right), \tag{2.1}$$

with $A=\beta p+\gamma-\alpha x_2$ and $B=\beta p+\gamma-\alpha x_1$.

Proof. Replacing (n) by $(-n)$ in the Binet's formula (1.6), we can write:

$$W_{p,q,-n}=\frac{Ax_1^{-n}-Bx_2^{-n}}{x_1-x_2}=\frac{\frac{A}{x_1^n}-\frac{B}{x_2^n}}{x_1-x_2}=\frac{Ax_2^n-Bx_1^n}{(x_1x_2)^n(x_1-x_2)}=\frac{(-1)^n}{(bq)^n}\left(\frac{Ax_2^n-Bx_1^n}{x_1-x_2}\right).$$

Thus, this completes the proof. □

The special cases of Eq. (2.1) are listed in the Table 2.

Remark 2.2. If we put $p=k$ and $q=1$ in $F_{p,q,-n}$, $L_{p,q,-n}$, $J_{p,q,-n}$ and $j_{p,q,-n}$ in Table 2, we get the negative extension of k-Fibonacci, k-Lucas, k-Jacobsthal, and k-Jacobsthal Lucas numbers as follows:

$$F_{k,-n}=(-1)^{n+1}F_{k,n},\quad L_{k,-n}=(-1)^nL_{k,n},\quad J_{k,-n}=\frac{(-1)^{n+1}}{2^n}J_{k,n},\quad j_{k,-n}=\frac{(-1)^n}{2^n}j_{k,n}.$$

Table 2: The (p,q)-numbers at negative indices.

a	b	α	β	γ	$W_{p,q,-n}$
1	1	0	0	1	$F_{p,q,-n} = \frac{(-1)^{n+1}}{q^n} F_{p,q,n}$
1	1	2	1	0	$L_{p,q,-n} = \frac{(-1)^{n}}{q^n} L_{p,q,n}$
1	2	0	0	1	$J_{p,q,-n} = \frac{(-1)^{n+1}}{(2q)^n} J_{p,q,n}$
1	2	2	1	0	$j_{p,q,-n} = \frac{(-1)^{n}}{(2q)^n} j_{p,q,n}$
2	1	0	0	1	$P_{p,q,-n} = \frac{(-1)^{n+1}}{q^n} P_{p,q,n}$
2	1	2	2	0	$Q_{p,q,-n} = \frac{(-1)^{n}}{q^n} Q_{p,q,n}$

Remark 2.3. If we take $p = 1$ and $q = k$ in $P_{p,q,-n}$ and $Q_{p,q,-n}$ in Table 2, we get the negative extension of k-Pell and k-Pell Lucas numbers as follows:

$$P_{k,-n} = \frac{(-1)^{n+1}}{k^n} P_{k,n}, \qquad Q_{k,-n} = \frac{(-1)^{n}}{k^n} Q_{k,n}.$$

Remark 2.4. If we put $p = q = 1$ in Table 2, we get the negative extension of Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell and Pell Lucas numbers as $F_{-n} = (-1)^{n+1} F_n$, $L_{-n} = (-1)^n L_n$, $J_{-n} = \frac{(-1)^{n+1}}{2^n} J_n$, $j_{-n} = \frac{(-1)^{n}}{2^n} j_n$, $P_{-n} = (-1)^{n+1} P_n$, and $Q_{-n} = (-1)^n Q_n$.

2.2. *The generating functions of (p,q)-numbers at negative indices*

Now, we give the generating functions of the generalized (p,q)-numbers at negative indices.

Theorem 2.5. *For $n \in \mathbb{N}$, the new generating functions of (p,q)-Fibonacci, (p,q)-Lucas, (p,q)-Jacobsthal, (p,q)-Jacobsthal Lucas (p,q)-Pell, and (p,q)-Pell Lucas numbers at negative indices are respectively given by:*

$$1)\ \sum_{n=0}^{\infty} F_{p,q,-n} z^n = \frac{z}{q + pz - z^2}, \qquad 2)\ \sum_{n=0}^{\infty} L_{p,q,-n} z^n = \frac{2q + pz}{q + pz - z^2},$$

$$3)\ \sum_{n=0}^{\infty} J_{p,q,-n} z^n = \frac{z}{2q + pz - z^2}, \qquad 4)\ \sum_{n=0}^{\infty} j_{p,q,-n} z^n = \frac{4q + pz}{2q + pz - z^2},$$

$$5)\ \sum_{n=0}^{\infty} P_{p,q,-n} z^n = \frac{z}{q + 2pz - z^2}, \qquad 6)\ \sum_{n=0}^{\infty} Q_{p,q,-n} z^n = \frac{2q + 2pz}{q + 2pz - z^2}.$$

Proof. By [13], we have:

$$\sum_{n=0}^{\infty} F_{p,q,n} z^n = \frac{z}{1 - pz - qz^2}.$$

Using Table 2, we get:

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,-n} z^n &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{q^n} F_{p,q,n} z^n \\ &= -\sum_{n=0}^{\infty} F_{p,q,n} \left(\frac{-z}{q}\right)^n = -\left(\frac{\left(\frac{-z}{q}\right)}{1 - p\left(\frac{-z}{q}\right) - q\left(\frac{-z}{q}\right)^2}\right) = \frac{z}{q + pz - z^2},\end{aligned}$$

which is the first equation. Other equations can be proved similarly. □

Remark 2.6. If we put $p = k$ and $q = 1$ in 1, 2, 3, and 4 in Theorem 2.5, we get the generating functions of k-Fibonacci, k-Lucas, k-Jacobsthal, and k-Jacobsthal Lucas numbers at negative indices as follows:

$$\sum_{n=0}^{\infty} F_{k,-n} z^n = \frac{z}{1+kz-z^2}, \qquad \sum_{n=0}^{\infty} L_{k,-n} z^n = \frac{2+kz}{1+kz-z^2},$$
$$\sum_{n=0}^{\infty} J_{k,-n} z^n = \frac{z}{2+kz-z^2}, \qquad \sum_{n=0}^{\infty} j_{k,-n} z^n = \frac{4+kz}{2+kz-z^2}.$$

Remark 2.7. If we take $p = 1$ and $q = k$ in 5 and 6 in Theorem 2.5, we get the generating functions of k-Pell and k-Pell Lucas numbers at negative indices as

$$\sum_{n=0}^{\infty} P_{k,-n} z^n = \frac{z}{k+2z-z^2} \quad \text{and} \quad \sum_{n=0}^{\infty} Q_{k,-n} z^n = \frac{2k+2z}{k+2z-z^2}.$$

Remark 2.8. If we put $p = q = 1$ in Theorem 2.5, we get the generating functions of Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas Pell, and Pell Lucas numbers at negative indices as

$$\sum_{n=0}^{\infty} F_{-n} z^n = \frac{z}{1+z-z^2}, \qquad \sum_{n=0}^{\infty} L_{-n} z^n = \frac{2+z}{1+z-z^2}, \qquad \sum_{n=0}^{\infty} J_{-n} z^n = \frac{z}{2+z-z^2},$$
$$\sum_{n=0}^{\infty} j_{-n} z^n = \frac{4+z}{2+z-z^2}, \qquad \sum_{n=0}^{\infty} P_{-n} z^n = \frac{z}{1+2z-z^2}, \qquad \sum_{n=0}^{\infty} Q_{-n} z^n = \frac{2+2z}{1+2z-z^2}.$$

2.3. *Explicit formula of generalized (p, q)-numbers*

Now, we will give the explicit formula of the generalized (p, q)-numbers. We start by the following theorem.

Theorem 2.9. *The complete homogeneous symmetric function $h_n(a_1, a_2)$ can be written explicitly as follows:*

$$h_n(a_1, a_2) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (a_1+a_2)^{n-2j} (a_1 a_2)^j. \tag{2.2}$$

Proof. By [4], we have:

$$g(z) = \sum_{n=0}^{\infty} h_n(a_1, a_2) z^n = \frac{1}{1-(a_1+a_2)z + a_1 a_2 z^2}.$$

Then, we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(a_1, a_2) z^n &= \frac{1}{1-((a_1+a_2)z - a_1 a_2 z^2)} \\
&= \sum_{n=0}^{\infty} ((a_1+a_2)z - a_1 a_2 z^2)^n \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} ((a_1+a_2)z)^{n-j} (a_1 a_2 z^2)^j \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (a_1+a_2)^{n-j} (a_1 a_2)^j z^{n+j}.
\end{aligned}$$

Writing (n) instead of $(n+j)$, we obtain:

$$\sum_{n=0}^{\infty} h_n(a_1,a_2) z^n = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (a_1+a_2)^{n-2j} (a_1a_2)^j \right) z^n.$$

Comparing of the coefficients of z^n, we obtain the desired result. □

From the Eq. (2.2), we get:

$$h_{n-1}(a_1,a_2) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-j-1}{j} (a_1+a_2)^{n-2j-1} (a_1a_2)^j. \tag{2.3}$$

Now, by using Eqs. (2.2) and (2.3) we give the following lemma.

Lemma 2.10. *For $n \geqslant 1$, we have*

$$\begin{aligned} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (a_1+a_2)^{n-2j} (a_1a_2)^j \\ = 2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (a_1+a_2)^{n-2j} (a_1a_2)^j \\ - \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-j-1}{j} (a_1+a_2)^{n-2j} (a_1a_2)^j. \end{aligned}$$

And we have the following theorem.

Theorem 2.11. *The explicit formula of the generalized (p,q)-numbers is given by:*

$$W_{p,q,n} = \alpha \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (ap)^{n-2j} (bq)^j + (p(\beta - a\alpha) + \gamma) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (ap)^{n-2j-1} (bq)^j. \tag{2.4}$$

Proof. Setting $\begin{cases} a_1 = \frac{ap+\sqrt{a^2p^2+4bq}}{2}, \\ a_2 = \frac{ap-\sqrt{a^2p^2+4bq}}{2}, \end{cases}$ in Eqs. (2.2) and (2.3), we get

$$h_n\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (ap)^{n-2j} (bq)^j, \tag{2.5}$$

$$h_{n-1}\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (ap)^{n-2j-1} (bq)^j. \tag{2.6}$$

On the other hand, we have:

$$\begin{aligned} W_{p,q,n} = \alpha h_n\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) \\ + (p(\beta - a\alpha) + \gamma) h_{n-1}\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right). \end{aligned}$$

Multiplying the equation (2.5) by (α) and adding it to the equation obtained by (2.6) multiplying by $(p(\beta - a\alpha) + \gamma)$, then we get

$$W_{p,q,n} = \alpha \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (ap)^{n-2j} (bq)^j + (p(\beta - a\alpha) + \gamma) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (ap)^{n-2j-1} (bq)^j .$$

Thus, this completes the proof. □

The special cases of Eq. (2.4) are listed in Table 3.

Table 3: The explicit formulas of some (p, q)-numbers.

a	b	α	β	γ	Explicit formula
1	1	0	0	1	$F_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} p^{n-2j-1} q^j$
1	1	2	1	0	$L_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} p^{n-2j} q^j$
1	2	0	0	1	$J_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} p^{n-2j-1} (2q)^j$
1	2	2	1	0	$j_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} p^{n-2j} (2q)^j$
2	1	0	0	1	$P_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (2p)^{n-2j-1} q^j$
2	1	2	2	0	$Q_{p,q,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} (2p)^{n-2j} q^j$

Remark 2.12. If we put $p = k$ and $q = 1$ in $F_{p,q,n}$, $L_{p,q,n}$, $J_{p,q,n}$ and $j_{p,q,n}$ in Table 3, we get the explicit formula of k-Fibonacci, k-Lucas, k-Jacobsthal, and k-Jacobsthal Lucas numbers as follows:

$$F_{k,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} k^{n-2j-1}, \qquad L_{k,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} k^{n-2j},$$

$$J_{k,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} k^{n-2j-1} 2^j, \qquad j_{k,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} k^{n-2j} 2^j.$$

Remark 2.13. If we take $p = 1$ and $q = k$ in $P_{p,q,n}$ and $Q_{p,q,n}$ in Table 3, we get the negative extension of k-Pell and k-Pell Lucas numbers as follows:

$$P_{k,n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} 2^{n-2j-1} k^j, \qquad Q_{k,n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} 2^{n-2j} k^j.$$

Remark 2.14. If we put $p = q = 1$ in Table 3, we get the explicit formula of Fibonacci, Lucas, Jacobsthal Jacobsthal Lucas, Pell, and Pell Lucas numbers as follows:

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}, \qquad L_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j}, \qquad J_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} 2^j,$$

$$j_n=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}2^j,\quad P_n=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}2^{n-2j-1},\quad Q_n=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}2^{n-2j}.$$

3. Ordinary generating functions of binary products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with some (p,q)-numbers

In this part, we are now in a position to provide theorems. Also we derive the new generating functions for the products of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q)-Pell numbers, (p,q)-Pell Lucas numbers, (p,q)-Jacobsthal numbers, and (p,q)-Jacobsthal Lucas numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

We consider $A=\{a_1,a_2\}$, $B=\{b_1,b_1\}$, and $l\in\{0,1,2\}$ in Theorem 1.10, we deduce the following lemmas.

Lemma 3.1. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$, *then we have:*

$$\sum_{n=0}^{\infty}h_n(a_1,a_2)h_{n+1}(b_1,b_2)z^n=\frac{(b_1+b_2)z^2-b_1b_2(a_1+a_2)z^3}{(1-a_1b_1z)(1-a_2b_1z)(1-a_1b_2z)(1-a_2b_2z)}.$$

The previous lemma gives the following result as particular example:

$$\sum_{n=0}^{\infty}h_{n-1}(a_1,a_2)h_n(b_1,b_2)z^n=\frac{(b_1+b_2)z-b_1b_2(a_1+a_2)z^2}{(1-a_1b_1z)(1-a_2b_1z)(1-a_1b_2z)(1-a_2b_2z)}.\tag{3.1}$$

Lemma 3.2. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$, *then we have:*

$$\sum_{n=0}^{\infty}h_n(a_1,a_2)h_n(b_1,b_2)z^n=\frac{1-a_1a_2b_1b_2z^2}{(1-a_1b_1z)(1-a_2b_1z)(1-a_1b_2z)(1-a_2b_2z)}.\tag{3.2}$$

Note that, based on relationship (3.2), we get:

$$\sum_{n=0}^{\infty}h_{n-1}(a_1,a_2)h_{n-1}(b_1,b_2)z^n=\frac{z-a_1a_2b_1b_2z^3}{(1-a_1b_1z)(1-a_2b_1z)(1-a_1b_2z)(1-a_2b_2z)}.\tag{3.3}$$

Lemma 3.3. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $B=\{b_1,b_2\}$, *then we have:*

$$\sum_{n=0}^{\infty}h_n(a_1,a_2)h_{n-1}(b_1,b_2)z^n=\frac{(a_1+a_2)z-a_1a_2(b_1+b_2)z^2}{(1-a_1b_1z)(1-a_2b_1z)(1-a_1b_2z)(1-a_2b_2z)}.\tag{3.4}$$

3.1. Generating functions of the products of $(\mathbf{p},\mathbf{q})$-Fibonacci and $(\mathbf{p},\mathbf{q})$-Lucas numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials

Now, we give the new generating functions for the products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with (p,q)-Fibonacci and (p,q)-Lucas numbers, k-Fibonacci and k-Lucas numbers, Fibonacci and Lucas numbers at positive and negative indices.

The substitutions of $\begin{cases} a_1=\frac{p+\sqrt{p^2+4q}}{2}\\ a_2=\frac{p-\sqrt{p^2+4q}}{2}\end{cases}$ and $\begin{cases} b_1=\frac{3y+\sqrt{9y^2-8x}}{2}\\ b_2=\frac{3y-\sqrt{9y^2-8x}}{2}\end{cases}$ in Eqs. (3.1), (3.2), (3.3), and (3.4), gives

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right)\\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix}z^n=\frac{N_1}{D_1},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n=\frac{L_1}{D_1},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n=\frac{K_1}{D_1},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n=\frac{R_1}{D_1},$$

with

$$D_1=1-3pyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2+6pqxyz^3+4q^2x^2z^4,$$
$$L_1=1+2qxz^2,\quad K_1=z+2qxz^3,\quad N_1=3yz-2pxz^2,\quad R_1=pz+3qyz^2,$$

and we deduce the following proposition and theorems.

Proposition 3.4. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with bivariate Mersenne polynomials is given by:*

$$\sum_{n=0}^{\infty}F_{p,q,n}M_n(x,y)z^n=\frac{K_1}{D_1}=\frac{z+2qxz^3}{1-3pyz-(9qy^2-2x(p^2+2q))z^2+6pqxyz^3+4q^2x^2z^4},\tag{3.5}$$

with $F_{p,q,n}M_n(x,y)=h_{n-1}\left(\frac{p+\sqrt{p^2+q}}{2},\frac{p-\sqrt{p^2+q}}{2}\right)h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)$.

Theorem 3.5. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with bivariate Mersenne Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty}F_{p,q,n}m_n(x,y)z^n=\frac{3yz-4pxz^2-6qxyz^3}{1-3pyz-(9qy^2-2x(p^2+2q))z^2+6pqxyz^3+4q^2x^2z^4}.\tag{3.6}$$

Proof. We have

$$\sum_{n=0}^{\infty}F_{p,q,n}m_n(x,y)z^n=\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times\begin{pmatrix} 2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right) \\ -3yh_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix}\end{pmatrix}z^n$$
$$=2\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix}z^n$$
$$-3y\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix}z^n$$

$$
\begin{aligned}
&= \frac{2N_1 - 3yK_1}{D_1} \\
&= \frac{2\left(3yz - 2pxz^2\right) - 3y\left(z + 2qxz^3\right)}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4} \\
&= \frac{3yz - 4pxz^2 - 6qxyz^3}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Lucas numbers with bivariate Mersenne polynomials is given by:*

$$
\sum_{n=0}^{\infty} L_{p,q,n} M_n(x, y) z^n = \frac{pz + 6qyz^2 - 2pqxz^3}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4}. \tag{3.7}
$$

Proof. We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n} M_n(x, y) z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{c} \left(\begin{array}{c} 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \end{array} \right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{c} h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&\quad - p\sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= \frac{2R_1 - pK_1}{D_1} \\
&= \frac{2\left(pz + 3qyz^2\right) - p\left(z + 2qxz^3\right)}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4} \\
&= \frac{pz + 6qyz^2 - 2pqxz^3}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Lucas numbers with bivariate Mersenne Lucas polynomials is given by:*

$$
\sum_{n=0}^{\infty} L_{p,q,n} m_n(x, y) z^n = \frac{4 - 9pyz + \left(4x\left(p^2 + 2q\right) - 18qy^2\right)z^2 + 6pqxyz^3}{1 - 3pyz - \left(9qy^2 - 2x\left(p^2 + 2q\right)\right)z^2 + 6pqxyz^3 + 4q^2x^2z^4}. \tag{3.8}
$$

Proof. We have

$$\sum_{n=0}^{\infty} L_{p,q,n} m_n(x,y) z^n = \sum_{n=0}^{\infty} \left(\begin{pmatrix} 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \end{pmatrix} \times \begin{pmatrix} 2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \\ -3yh_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} \right) z^n$$

$$= 4\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n$$

$$- 6y\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n$$

$$- 2p\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n$$

$$+ 3py\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n$$

$$= \frac{4L_1 - 6yR_1 - 2pN_1 + 3pyK_1}{D_1}$$

$$= \frac{4\left(1+2qxz^2\right) - 6y\left(pz+3qyz^2\right) - 2p\left(3yz-2pxz^2\right) + 3py\left(z+2qxz^3\right)}{1-3pyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2+6pqxyz^3+4q^2x^2z^4}$$

$$= \frac{4-9pyz+\left(4x\left(p^2+2q\right)-18qy^2\right)z^2+6pqxyz^3}{1-3pyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2+6pqxyz^3+4q^2x^2z^4}.$$

This completes the proof. □

By using Table 2 and according to the Eqs. (3.5), (3.6), (3.7), and (3.8), we get the following theorem.

Theorem 3.8. *For* $n \in \mathbb{N}$, *the new generating functions of* $(F_{p,q,-n}M_n(x,y))$, $(F_{p,q,-n}m_n(x,y))$, $(L_{p,q,-n}M_n(x,y))$, *and* $(L_{p,q,-n}m_n(x,y))$ *are respectively given by*

$$\sum_{n=0}^{\infty} F_{p,q,-n}M_n(x,y) z^n = \frac{qz+2xz^3}{q^2+3pqyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2-6pxyz^3+4x^2z^4}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} F_{p,q,-n}m_n(x,y) z^n = \frac{3qyz+4pxz^2-6xyz^3}{q^2+3pqyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2-6pxyz^3+4x^2z^4}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} L_{p,q,-n}M_n(x,y) z^n = \frac{-pqz+6qyz^2+2pxz^3}{q^2+3pqyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2-6pxyz^3+4x^2z^4}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} L_{p,q,-n}m_n(x,y) z^n = \frac{4q^2+9pqyz+\left(4x\left(p^2+2q\right)-18qy^2\right)z^2-6pxyz^3}{q^2+3pqyz-\left(9qy^2-2x\left(p^2+2q\right)\right)z^2-6pxyz^3+4x^2z^4}. \tag{3.12}$$

Proof. We have:

$$\sum_{n=0}^{\infty} F_{p,q,n} M_n (x,y) z^n = \frac{z+2qxz^3}{1-3pyz-(9qy^2-2x(p^2+2q))z^2+6pqxyz^3+4q^2x^2z^4}.$$

Writing $\left(\frac{-z}{q}\right)$ instead of (z), we obtain:

$$\sum_{n=0}^{\infty} F_{p,q,n} M_n (x,y) \left(\frac{-z}{q}\right)^n = \frac{\left(\frac{-z}{q}\right)+2qx\left(\frac{-z}{q}\right)^3}{1-3py\left(\frac{-z}{q}\right)-(9qy^2-2x(p^2+2q))\left(\frac{-z}{q}\right)^2+6pqxy\left(\frac{-z}{q}\right)^3+4q^2x^2\left(\frac{-z}{q}\right)^4}.$$

Since

$$F_{p,q,-n} = \frac{(-1)^{n+1}}{q^n} F_{p,q,n}.$$

Then, we get

$$\sum_{n=0}^{\infty} F_{p,q,-n} M_n (x,y) z^n = \frac{qz+2xz^3}{q^2+3pqyz-(9qy^2-2x(p^2+2q))z^2-6pxyz^3+4x^2z^4}.$$

So, the desired result is achieved. Applying the same method, the other equations can be proved. □

Corollary 3.9. *Putting* $p = k$ *and* $q = 1$ *in Eqs.* (3.5)-(3.12) *we get the new generating functions of the products of* k*-Fibonacci and* k*-Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials. The calculation and results are listed as follows:*

$$\sum_{n=0}^{\infty} F_{k,n} M_n (x,y) z^n = \frac{z+2xz^3}{1-3kyz-(9y^2-2x(k^2+2))z^2+6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} F_{k,n} m_n (x,y) z^n = \frac{3yz-4kxz^2-6xyz^3}{1-3kyz-(9y^2-2x(k^2+2))z^2+6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} L_{k,n} M_n (x,y) z^n = \frac{kz+6yz^2-2kxz^3}{1-3kyz-(9y^2-2x(k^2+2))z^2+6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} L_{k,n} m_n (x,y) z^n = \frac{4-9kyz+\left(4x\left(k^2+2\right)-18y^2\right)z^2+6kxyz^3}{1-3kyz-(9y^2-2x(k^2+2))z^2+6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} F_{k,-n} M_n (x,y) z^n = \frac{z+2xz^3}{1+3kyz-(9y^2-2x(k^2+2))z^2-6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} F_{k,-n} m_n (x,y) z^n = \frac{3yz+4kxz^2-6xyz^3}{1+3kyz-(9y^2-2x(k^2+2))z^2-6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} L_{k,-n} M_n (x,y) z^n = \frac{-kz+6yz^2+2kxz^3}{1+3kyz-(9y^2-2x(k^2+2))z^2-6kxyz^3+4x^2z^4},$$

$$\sum_{n=0}^{\infty} L_{k,-n} m_n (x,y) z^n = \frac{4+9kyz+\left(4x\left(k^2+2\right)-18y^2\right)z^2-6kxyz^3}{1+3kyz-(9y^2-2x(k^2+2))z^2-6kxyz^3+4x^2z^4}.$$

Putting $k = 1$ in Corollary 3.9, we obtain Table 4.

Table 4: New generating functions of the products of Fibonacci and Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

Coefficient of z^n	Generating function	Coefficient of z^n	Generating function
$F_n M_n(x,y)$	$\frac{z+2xz^3}{1-3yz-(9y^2-6x)z^2+6xyz^3+4x^2z^4}$	$F_{-n} M_n(x,y)$	$\frac{z+2xz^3}{1+3yz-(9y^2-6x)z^2-6xyz^3+4x^2z^4}$
$F_n m_n(x,y)$	$\frac{3yz-4xz^2-6xyz^3}{1-3yz-(9y^2-6x)z^2+6xyz^3+4x^2z^4}$	$F_{-n} m_n(x,y)$	$\frac{3yz+4xz^2-6xyz^3}{1+3yz-(9y^2-6x)z^2-6xyz^3+4x^2z^4}$
$L_n M_n(x,y)$	$\frac{z+6yz^2-2xz^3}{1-3yz-(9y^2-6x)z^2+6xyz^3+4x^2z^4}$	$L_{-n} M_n(x,y)$	$\frac{-z+6yz^2+2xz^3}{1+3yz-(9y^2-6x)z^2-6xyz^3+4x^2z^4}$
$L_n m_n(x,y)$	$\frac{4-9yz+(12x-18y^2)z^2+6xyz^3}{1-3yz-(9y^2-6x)z^2+6xyz^3+4x^2z^4}$	$L_{-n} m_n(x,y)$	$\frac{4+9yz+(12x-18y^2)z^2-6xyz^3}{1+3yz-(9y^2-6x)z^2-6xyz^3+4x^2z^4}$

3.2. *Generating functions of the products of* $(\mathbf{p},\mathbf{q})$*-Pell and* $(\mathbf{p},\mathbf{q})$*-Pell Lucas numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials*

Now, we calcule the new generating functions for the products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with (p,q)-Pell and (p,q)-Pell Lucas numbers, k-Pell and k-Pell Lucas numbers, Pell and Pell Lucas numbers at positive and negative indices.

The substitutions of $\begin{cases} a_1 = p+\sqrt{p^2+q} \\ a_2 = p-\sqrt{p^2+q} \end{cases}$ and $\begin{cases} b_1 = \frac{3y+\sqrt{9y^2-8x}}{2} \\ b_2 = \frac{3y-\sqrt{9y^2-8x}}{2} \end{cases}$ in Eqs. (3.1), (3.2), (3.3), and (3.4), we get

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n = \frac{N_2}{D_2},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n}\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n = \frac{L_2}{D_2},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n = \frac{K_2}{D_2},$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n}\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{pmatrix} z^n = \frac{R_2}{D_2},$$

with

$$D_2 = 1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4,$$
$$L_2 = 1+2qxz^2,\quad K_2 = z+2qxz^3,\quad N_2 = 3yz-4pxz^2,\quad R_2 = 2pz+3qyz^2,$$

and we deduce the following proposition and theorems.

Proposition 3.10. *For* $n\in\mathbb{N}$*, the new generating function of the product of* (p,q)*-Pell numbers with bivariate Mersenne polynomials is given by:*

$$\sum_{n=0}^{\infty} P_{p,q,n}M_n(x,y)z^n = \frac{K_2}{D_2} = \frac{z+2qxz^3}{1-6pyz-(9qy^2-4x(2p^2+q))z^2+12pqxyz^3+4q^2x^2z^4}, \tag{3.13}$$

with $P_{p,q,n}M_n(x,y) = h_{n-1}\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right)h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)$.

Theorem 3.11. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p, q)*-Pell numbers with bivariate Mersenne Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} P_{p,q,n} m_n(x,y) z^n = \frac{3yz - 8pxz^2 - 6qxyz^3}{1 - 6pyz - (9qy^2 - 4x(2p^2+q)) z^2 + 12pqxyz^3 + 4q^2x^2z^4}. \quad (3.14)$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{p,q,n} m_n(x,y) z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times \left(\begin{array}{c} 2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \\ -3yh_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&\quad - 3y\sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= \frac{2N_2 - 3yK_2}{D_2} \\
&= \frac{2(3yz - 4pxz^2) - 3y(z + 2qxz^3)}{1 - 6pyz - (9qy^2 - 4x(2p^2+q)) z^2 + 12pqxyz^3 + 4q^2x^2z^4} \\
&= \frac{3yz - 8pxz^2 - 6qxyz^3}{1 - 6pyz - (9qy^2 - 4x(2p^2+q)) z^2 + 12pqxyz^3 + 4q^2x^2z^4}.
\end{aligned}$$

This completes the proof. □

Theorem 3.12. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p, q)*-Pell Lucas numbers with bivariate Mersenne polynomials is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q,n} M_n(x,y) z^n = \frac{2pz + 6qyz^2 - 4pqxz^3}{1 - 6pyz - (9qy^2 - 4x(2p^2+q)) z^2 + 12pqxyz^3 + 4q^2x^2z^4}. \quad (3.15)$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n} M_n(x,y) z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{c} \left(\begin{array}{c} 2h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ -2ph_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \end{array} \right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{c} h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&\quad - 2p\sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= \frac{2R_2 - 2pK_2}{D_2}
\end{aligned}$$

$$= \frac{2\left(2pz+3qyz^2\right)-2p\left(z+2qxz^3\right)}{1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4}$$
$$= \frac{2pz+6qyz^2-4pqxz^3}{1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4}.$$

This completes the proof. □

Theorem 3.13. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with bivariate Mersenne Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q,n} m_n(x,y) z^n = \frac{4-18pyz+\left(8x\left(2p^2+q\right)-18qy^2\right)z^2+12pqxyz^3}{1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4}. \tag{3.16}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n} m_n(x,y) z^n &= \sum_{n=0}^{\infty} \left(\begin{pmatrix} 2h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ -2ph_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \end{pmatrix} \times \begin{pmatrix} 2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \\ -3yh_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} \right) z^n \\
&= 4\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n \\
&\quad - 6y\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n \\
&\quad - 4p\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n \\
&\quad + 6py\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q}, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n \\
&= \frac{4L_2-6yR_2-4pN_2+6pyK_2}{D_2} \\
&= \frac{4\left(1+2qxz^2\right)-6y\left(2pz+3qyz^2\right)-4p\left(3yz-4pxz^2\right)+6py\left(z+2qxz^3\right)}{1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4} \\
&= \frac{4-18pyz+\left(8x\left(2p^2+q\right)-18qy^2\right)z^2+12pqxyz^3}{1-6pyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2+12pqxyz^3+4q^2x^2z^4}.
\end{aligned}$$

This completes the proof. □

By using Table 2 and according to the Eqs. (3.13), (3.14), (3.15), and (3.16), we get the following theorem.

Theorem 3.14. *For $n \in \mathbb{N}$, the new generating functions of $\left(P_{p,q,-n} M_n(x,y)\right)$, $\left(P_{p,q,-n} m_n(x,y)\right)$, $\left(Q_{p,q,-n} M_n(x,y)\right)$, and $\left(Q_{p,q,-n} m_n(x,y)\right)$ are respectively given by*

$$\sum_{n=0}^{\infty} P_{p,q,-n} M_n(x,y) z^n = \frac{qz+2xz^3}{q^2+6pqyz-\left(9qy^2-4x\left(2p^2+q\right)\right)z^2-12pxyz^3+4x^2z^4}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} P_{p,q,-n} m_n(x,y) z^n = \frac{3qyz + 8pxz^2 - 6xyz^3}{q^2 + 6pqyz - (9qy^2 - 4x(2p^2+q))z^2 - 12pxyz^3 + 4x^2z^4}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} Q_{p,q,-n} M_n(x,y) z^n = \frac{-2pqz + 6qyz^2 + 4pxz^3}{q^2 + 6pqyz - (9qy^2 - 4x(2p^2+q))z^2 - 12pxyz^3 + 4x^2z^4}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} Q_{p,q,-n} m_n(x,y) z^n = \frac{4q^2 + 18pqyz + \left(8x\left(2p^2+q\right) - 18qy^2\right)z^2 - 12pxyz^3}{q^2 + 6pqyz - (9qy^2 - 4x(2p^2+q))z^2 - 12pxyz^3 + 4x^2z^4}. \tag{3.20}$$

Proof. We have

$$\sum_{n=0}^{\infty} P_{p,q,n} M_n(x,y) z^n = \frac{z + 2qxz^3}{1 - 6pyz - (9qy^2 - 4x(2p^2+q))z^2 + 12pqxyz^3 + 4q^2x^2z^4}.$$

Writing $\left(\frac{-z}{q}\right)$ instead of (z), we obtain

$$\sum_{n=0}^{\infty} P_{p,q,n} M_n(x,y) \left(\frac{-z}{q}\right)^n$$

$$= \frac{\left(\frac{-z}{q}\right) + 2qx\left(\frac{-z}{q}\right)^3}{1 - 6py\left(\frac{-z}{q}\right) - (9qy^2 - 4x(2p^2+q))\left(\frac{-z}{q}\right)^2 + 12pqxy\left(\frac{-z}{q}\right)^3 + 4q^2x^2\left(\frac{-z}{q}\right)^4}.$$

Since

$$P_{p,q,-n} = \frac{(-1)^{n+1}}{q^n} P_{p,q,n}.$$

Then, we get

$$\sum_{n=0}^{\infty} P_{p,q,-n} M_n(x,y) z^n = \frac{qz + 2xz^3}{q^2 + 6pqyz - (9qy^2 - 4x(2p^2+q))z^2 - 12pxyz^3 + 4x^2z^4}.$$

So, the desired result is achieved. Applying the same method, the other equations can be proved. □

Corollary 3.15. *Putting* $p = 1$ *and* $q = k$ *in Eqs.* (3.13)-(3.20) *we get the new generating functions of the products of* k*-Pell and* k*-Pell Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials. The calculation and results are listed as follows:*

$$\sum_{n=0}^{\infty} P_{k,n} M_n(x,y) z^n = \frac{z + 2kxz^3}{1 - 6yz - (9ky^2 - 4x(2+k))z^2 + 12kxyz^3 + 4k^2x^2z^4},$$

$$\sum_{n=0}^{\infty} P_{k,n} m_n(x,y) z^n = \frac{3yz - 8xz^2 - 6kxyz^3}{1 - 6yz - (9ky^2 - 4x(2+k))z^2 + 12kxyz^3 + 4k^2x^2z^4},$$

$$\sum_{n=0}^{\infty} Q_{k,n} M_n(x,y) z^n = \frac{2z + 6kyz^2 - 4kxz^3}{1 - 6yz - (9ky^2 - 4x(2+k))z^2 + 12kxyz^3 + 4k^2x^2z^4},$$

$$\sum_{n=0}^{\infty} Q_{k,n} m_n(x,y) z^n = \frac{4 - 18yz + \left(8x(2+k) - 18ky^2\right)z^2 + 12kxyz^3}{1 - 6yz - (9ky^2 - 4x(2+k))z^2 + 12kxyz^3 + 4k^2x^2z^4},$$

$$\sum_{n=0}^{\infty} P_{k,-n} M_n(x,y) z^n = \frac{kz + 2xz^3}{k^2 + 6kyz - (9ky^2 - 4x(2+k))z^2 - 12xyz^3 + 4x^2z^4},$$

$$\sum_{n=0}^{\infty} P_{k,-n} m_n(x,y) z^n = \frac{3kyz + 8xz^2 - 6xyz^3}{k^2 + 6kyz - (9ky^2 - 4x(2+k)) z^2 - 12xyz^3 + 4x^2z^4},$$

$$\sum_{n=0}^{\infty} Q_{k,-n} M_n(x,y) z^n = \frac{-2kz + 6kyz^2 + 4xz^3}{k^2 + 6kyz - (9ky^2 - 4x(2+k)) z^2 - 12xyz^3 + 4x^2z^4},$$

$$\sum_{n=0}^{\infty} Q_{k,-n} m_n(x,y) z^n = \frac{4k^2 + 18kyz + \left(8x(2+k) - 18ky^2\right) z^2 - 12xyz^3}{k^2 + 6kyz - (9ky^2 - 4x(2+k)) z^2 - 12xyz^3 + 4x^2z^4}.$$

Putting $k = 1$ in Corollary 3.15, we obtain Table 5.

Table 5: New generating functions of the products of Pell and Pell Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

Coefficient of z^n	Generating function	Coefficient of z^n	Generating function
$P_n M_n(x,y)$	$\frac{z+2xz^3}{1-6yz-(9y^2-12x)z^2+12xyz^3+4x^2z^4}$	$P_{-n} M_n(x,y)$	$\frac{z+2xz^3}{1+6yz-(9y^2-12x)z^2-12xyz^3+4x^2z^4}$
$P_n m_n(x,y)$	$\frac{3yz-8xz^2-6xyz^3}{1-6yz-(9y^2-12x)z^2+12xyz^3+4x^2z^4}$	$P_{-n} m_n(x,y)$	$\frac{3yz+8xz^2-6xyz^3}{1+6yz-(9y^2-12x)z^2-12xyz^3+4x^2z^4}$
$Q_n M_n(x,y)$	$\frac{2z+6yz^2-4xz^3}{1-6yz-(9y^2-12x)z^2+12xyz^3+4x^2z^4}$	$Q_{-n} M_n(x,y)$	$\frac{-2z+6yz^2+4xz^3}{1+6yz-(9y^2-12x)z^2-12xyz^3+4x^2z^4}$
$Q_n m_n(x,y)$	$\frac{4-18yz+\left(24x-18y^2\right)z^2+12xyz^3}{1-6yz-(9y^2-12x)z^2+12xyz^3+4x^2z^4}$	$Q_{-n} m_n(x,y)$	$\frac{4+18yz+\left(24x-18y^2\right)z^2-12xyz^3}{1+6yz-(9y^2-12x)z^2-12xyz^3+4x^2z^4}$

3.3. *Generating functions of the products of $(\mathbf{p},\mathbf{q})$-Jacobsthal and $(\mathbf{p},\mathbf{q})$-Jacobsthal Lucas numbers with bivariate Mersenne and bivariate Mersenne Lucas polynomials*

Now, we introduce the new generating functions for the products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with (p,q)-Jacobsthal and (p,q)-Jacobsthal Lucas numbers, k-Jacobsthal and k-Jacobsthal Lucas numbers, Jacobsthal and Jacobsthal Lucas numbers at positive and negative indices.

The substitutions of $\begin{cases} a_1 = \frac{p+\sqrt{p^2+8q}}{2} \\ a_2 = \frac{p-\sqrt{p^2+8q}}{2} \end{cases}$ and $\begin{cases} b_1 = \frac{3y+\sqrt{9y^2-8x}}{2} \\ b_2 = \frac{3y-\sqrt{9y^2-8x}}{2} \end{cases}$ in Eqs. (3.1), (3.2), (3.3), and (3.4), gives

$$\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n = \frac{N_3}{D_3},$$

$$\sum_{n=0}^{\infty} \begin{pmatrix} h_{n}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n = \frac{L_3}{D_3},$$

$$\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n = \frac{K_3}{D_3},$$

$$\sum_{n=0}^{\infty} \begin{pmatrix} h_{n}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{pmatrix} z^n = \frac{R_3}{D_3},$$

with

$$D_3 = 1 - 3pyz - \left(18qy^2 - 2x\left(p^2+4q\right)\right) z^2 + 12pqxyz^3 + 16q^2x^2z^4,$$

$$L_3 = 1 + 4qxz^2, \quad K_3 = z + 4qxz^3, \quad N_3 = 3yz - 2pxz^2, \quad R_3 = pz + 6qyz^2,$$

and we deduce the following proposition and theorems.

Proposition 3.16. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal numbers with bivariate Mersenne polynomials is given by:*

$$\sum_{n=0}^{\infty} J_{p,q,n} M_n(x,y) z^n = \frac{K_3}{D_3} = \frac{z + 4qxz^3}{1 - 3pyz - (18qy^2 - 2x(p^2+4q)) z^2 + 12pqxyz^3 + 16q^2x^2z^4}, \tag{3.21}$$

with $J_{p,q,n} M_n(x,y) = h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right)$.

Theorem 3.17. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal numbers with bivariate Mersenne Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} J_{p,q,n} m_n(x,y) z^n = \frac{3yz - 4pxz^2 - 12qxyz^3}{1 - 3pyz - (18qy^2 - 2x(p^2+4q)) z^2 + 12pqxyz^3 + 16q^2x^2z^4}. \tag{3.22}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{p,q,n} m_n(x,y) z^n &= \sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times \left(\begin{array}{c} 2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \\ -3y h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&\quad - 3y \sum_{n=0}^{\infty} \left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2}, \frac{3y-\sqrt{9y^2-8x}}{2}\right) \end{array} \right) z^n \\
&= \frac{2N_3 - 3yK_3}{D_3} \\
&= \frac{2\left(3yz - 2pxz^2\right) - 3y\left(z + 4qxz^3\right)}{1 - 3pyz - (18qy^2 - 2x(p^2+4q)) z^2 + 12pqxyz^3 + 16q^2x^2z^4} \\
&= \frac{3yz - 4pxz^2 - 12qxyz^3}{1 - 3pyz - (18qy^2 - 2x(p^2+4q)) z^2 + 12pqxyz^3 + 16q^2x^2z^4}.
\end{aligned}$$

This completes the proof. □

Theorem 3.18. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal Lucas numbers with bivariate Mersenne polynomials is given by:*

$$\sum_{n=0}^{\infty} j_{p,q,n} M_n(x,y) z^n = \frac{pz + 12qyz^2 - 4pqxz^3}{1 - 3pyz - (18qy^2 - 2x(p^2+4q)) z^2 + 12pqxyz^3 + 16q^2x^2z^4}. \tag{3.23}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n} M_n(x,y) z^n &= \sum_{n=0}^{\infty}\left(\begin{array}{c}\left(\begin{array}{c}2h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\end{array}\right)\\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n\\
&= 2\sum_{n=0}^{\infty}\left(\begin{array}{c}h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n\\
&\quad - p\sum_{n=0}^{\infty}\left(\begin{array}{c}h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n\\
&= \frac{2R_3 - pK_3}{D_3}\\
&= \frac{2\left(pz+6qyz^2\right) - p\left(z+4qxz^3\right)}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}\\
&= \frac{pz+12qyz^2-4pqxz^3}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 3.19. *For* $n \in \mathbb{N}$, *the new generating function of the product of* (p,q)*-Jacobsthal Lucas numbers with bivariate Mersenne Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} j_{p,q,n} m_n(x,y) z^n = \frac{4-9pyz+\left(4x\left(p^2+4q\right)-36qy^2\right)z^2+12pqxyz^3}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}. \tag{3.24}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n} m_n(x,y) z^n &= \sum_{n=0}^{\infty}\left(\begin{array}{c}\left(\begin{array}{c}2h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\end{array}\right)\\ \times\left(\begin{array}{c}2h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\\ -3yh_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)\end{array}\right)z^n\\
&= 4\sum_{n=0}^{\infty}\left(\begin{array}{c}h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n\\
&\quad - 6y\sum_{n=0}^{\infty}\left(\begin{array}{c}h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n\end{aligned}$$

$$-2p\sum_{n=0}^{\infty}\left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n$$
$$+3py\sum_{n=0}^{\infty}\left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{3y+\sqrt{9y^2-8x}}{2},\frac{3y-\sqrt{9y^2-8x}}{2}\right)\end{array}\right)z^n$$
$$=\frac{4L_3-6yR_3-2pN_3+3pyK_3}{D_3}$$
$$=\frac{4\left(1+4qxz^2\right)-6y\left(pz+6qyz^2\right)-2p\left(3yz-2pxz^2\right)+3py\left(z+4qxz^3\right)}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}$$
$$=\frac{4-9pyz+\left(4x\left(p^2+4q\right)-36qy^2\right)z^2+12pqxyz^3}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}.$$

This completes the proof. □

By using Table 1 and according to the Eqs. (3.21), (3.22), (3.23), and (3.24), we get the following theorem.

Theorem 3.20. *For $n \in \mathbb{N}$, the new generating functions of $\left(J_{p,q,-n}M_n(x,y)\right)$, $\left(J_{p,q,-n}m_n(x,y)\right)$, $\left(j_{p,q,-n}M_n(x,y)\right)$, and $\left(j_{p,q,-n}m_n(x,y)\right)$ are respectively given by*

$$\sum_{n=0}^{\infty}J_{p,q,-n}M_n(x,y)z^n=\frac{qz+xz^3}{2q^2+3pqyz-\left(9qy^2-x\left(p^2+4q\right)\right)z^2-3pxyz^3+2x^2z^4},\tag{3.25}$$
$$\sum_{n=0}^{\infty}J_{p,q,-n}m_n(x,y)z^n=\frac{3qyz+2pxz^2-3xyz^3}{2q^2+3pqyz-\left(9qy^2-x\left(p^2+4q\right)\right)z^2-3pxyz^3+2x^2z^4},\tag{3.26}$$
$$\sum_{n=0}^{\infty}j_{p,q,-n}M_n(x,y)z^n=\frac{-pqz+6qyz^2+pxz^3}{2q^2+3pqyz-\left(9qy^2-x\left(p^2+4q\right)\right)z^2-3pxyz^3+2x^2z^4},\tag{3.27}$$
$$\sum_{n=0}^{\infty}j_{p,q,-n}m_n(x,y)z^n=\frac{8q^2+9pqyz+\left(2x\left(p^2+4q\right)-18qy^2\right)z^2-3pxyz^3}{2q^2+3pqyz-\left(9qy^2-x\left(p^2+4q\right)\right)z^2-3pxyz^3+2x^2z^4}.\tag{3.28}$$

Proof. We have

$$\sum_{n=0}^{\infty}J_{p,q,n}M_n(x,y)z^n=\frac{z+4qxz^3}{1-3pyz-\left(18qy^2-2x\left(p^2+4q\right)\right)z^2+12pqxyz^3+16q^2x^2z^4}.$$

Writing $\left(\frac{-z}{2q}\right)$ instead of (z), we obtain

$$\sum_{n=0}^{\infty}J_{p,q,n}M_n(x,y)\left(\frac{-z}{2q}\right)^n$$
$$=\frac{\left(\frac{-z}{2q}\right)+4qx\left(\frac{-z}{2q}\right)^3}{1-3py\left(\frac{-z}{2q}\right)-\left(18qy^2-2x\left(p^2+4q\right)\right)\left(\frac{-z}{2q}\right)^2+12pqxy\left(\frac{-z}{2q}\right)^3+16q^2x^2\left(\frac{-z}{2q}\right)^4}.$$

Since

$$J_{p,q,-n}=\frac{(-1)^{n+1}}{(2q)^n}J_{p,q,n}.$$

Then, we get

$$\sum_{n=0}^{\infty} J_{p,q,-n} M_n(x,y) z^n = \frac{qz + xz^3}{2q^2 + 3pqyz - (9qy^2 - x(p^2+4q)) z^2 - 3pxyz^3 + 2x^2z^4}.$$

So, the desired result is achieved. Applying the same method, the other equations can be proved. □

Corollary 3.21. *Putting* $p = k$ *and* $q = 1$ *in Eqs.* (3.21)-(3.28) *we get the new generating functions of the products of* k*-Jacobsthal and* k*-Jacobsthal Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials. The calculation and results are listed as follows:*

$$\sum_{n=0}^{\infty} J_{k,n} M_n(x,y) z^n = \frac{z + 4xz^3}{1 - 3kyz - (18y^2 - 2x(k^2+4)) z^2 + 12kxyz^3 + 16x^2z^4},$$

$$\sum_{n=0}^{\infty} J_{k,n} m_n(x,y) z^n = \frac{3yz - 4kxz^2 - 12xyz^3}{1 - 3kyz - (18y^2 - 2x(k^2+4)) z^2 + 12kxyz^3 + 16x^2z^4},$$

$$\sum_{n=0}^{\infty} j_{k,n} M_n(x,y) z^n = \frac{kz + 12yz^2 - 4kxz^3}{1 - 3kyz - (18y^2 - 2x(k^2+4)) z^2 + 12kxyz^3 + 16x^2z^4},$$

$$\sum_{n=0}^{\infty} j_{k,n} m_n(x,y) z^n = \frac{4 - 9kyz + \left(4x\left(k^2+4\right) - 36y^2\right) z^2 + 12kxyz^3}{1 - 3kyz - (18y^2 - 2x(k^2+4)) z^2 + 12kxyz^3 + 16x^2z^4},$$

$$\sum_{n=0}^{\infty} J_{k,-n} M_n(x,y) z^n = \frac{z + xz^3}{2 + 3kyz - (9y^2 - x(k^2+4)) z^2 - 3kxyz^3 + 2x^2z^4},$$

$$\sum_{n=0}^{\infty} J_{k,-n} m_n(x,y) z^n = \frac{3yz + 2kxz^2 - 3xyz^3}{2 + 3kyz - (9y^2 - x(k^2+4)) z^2 - 3kxyz^3 + 2x^2z^4},$$

$$\sum_{n=0}^{\infty} j_{k,-n} M_n(x,y) z^n = \frac{-kz + 6yz^2 + kxz^3}{2 + 3kyz - (9y^2 - x(k^2+4)) z^2 - 3kxyz^3 + 2x^2z^4},$$

$$\sum_{n=0}^{\infty} j_{k,-n} m_n(x,y) z^n = \frac{8 + 9kyz + \left(2x\left(k^2+4\right) - 18y^2\right) z^2 - 3kxyz^3}{2 + 3kyz - (9y^2 - x(k^2+4)) z^2 - 3kxyz^3 + 2x^2z^4}.$$

Putting $k = 1$ in Corollary 3.21, we obtain Table 6.

Table 6: New generating functions of the products of Jacobsthal and Jacobsthal Lucas numbers at positive and negative indices with bivariate Mersenne and bivariate Mersenne Lucas polynomials.

Coefficient of z^n	Generating function	Coefficient of z^n	Generating function
$J_n M_n(x,y)$	$\frac{z+4xz^3}{1-3yz-(18y^2-10x)z^2+12xyz^3+16x^2z^4}$	$J_{-n} M_n(x,y)$	$\frac{z+xz^3}{2+3yz-(9y^2-5x)z^2-3xyz^3+2x^2z^4}$
$J_n m_n(x,y)$	$\frac{3yz-4xz^2-12xyz^3}{1-3yz-(18y^2-10x)z^2+12xyz^3+16x^2z^4}$	$J_{-n} m_n(x,y)$	$\frac{3yz+2xz^2-3xyz^3}{2+3yz-(9y^2-5x)z^2-3xyz^3+2x^2z^4}$
$j_n M_n(x,y)$	$\frac{z+12yz^2-4xz^3}{1-3yz-(18y^2-10x)z^2+12xyz^3+16x^2z^4}$	$j_{-n} M_n(x,y)$	$\frac{-z+6yz^2+xz^3}{2+3yz-(9y^2-5x)z^2-3xyz^3+2x^2z^4}$
$j_n m_n(x,y)$	$\frac{4-9yz+\left(20x-36y^2\right)z^2+12xyz^3}{1-3yz-(18y^2-10x)z^2+12xyz^3+16x^2z^4}$	$j_{-n} m_n(x,y)$	$\frac{8+9yz+\left(10x-18y^2\right)z^2-3xyz^3}{2+3yz-(9y^2-5x)z^2-3xyz^3+2x^2z^4}$

4. Conclusion

In this paper, we studied the generalized (p,q)-numbers given in [13]. We gave some new results of this generalized (p,q)-numbers, including the explicit formula and the negative extension of them. Moreover, by using the symmetric functions we obtained the new generating functions for the products of bivariate Mersenne and bivariate Mersenne Lucas polynomials with (p,q)-numbers at positive and negative indices.

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References

[1] F. R. V. Alves, *Bivariate Mersenne polynomials and matrices*, Notes Number Theory Discrete Math., **26** (2020), 83–95. 1

[2] S. Boughaba, A. Boussayoud, *On some identities and generating function of both k-Jacobsthal numbers and symmetric functions in several variables*, Konuralp J. Math., **7** (2019), 235–242.

[3] A. Boussayoud, A. Abderrezzak, *Complete homogeneous symmetric functions and Hadamard product*, Ars Combin., **144** (2019), 81–90.

[4] A. Boussayoud, M. Kerada, R. Sahali, *Symmetrizing operations on some orthogonal polynomials*, Int. Electron. J. Pure Appl. Math., **9** (2015), 191–199. 1, 2.3

[5] P. Catarino, H. Campos, P. Vasco, *On the Mersenne sequence*, Ann. Math. Inform., **46** (2016), 37–53. 1

[6] H. H. Gulec, N. Taskara, *On the* (s,t)*-Pell and* (s,t)*-Pell-Lucas sequences and their matrix representations*, Appl. Math. Lett., **25** (2012), 1554–1559. 1

[7] M. Merca, *A generalization of the symmetry between complete and elementary symmetric functions*, Indian J. Pure Appl. Math., **45** (2014), 75–89. 1, 1.2, 1.6

[8] N. Saba, A. Boussayoud, *New theorem on symmetric functions and their applications on some* (p,q)*-numbers*, J. Appl. Math. Inform., **40** (2022), 243–257. 1

[9] N. Saba, A. Boussayoud, *Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials*, Open J. Math. Sci., **4** (2020), 179–185. 1.8

[10] N. Saba, A. Boussayoud, *Ordinary generating functions of binary products of (p, q)-modified Pell numbers and k-numbers at positive and negative indices*, J. Sci. Arts, **3** (2020), 627–646. 1

[11] N. Saba, A. Boussayoud, *On the bivariate Mersenne Lucas polynomials and their properties*, Chaos Solitons Fractals, **146** (2021), 6 pages. 1

[12] N. Saba, A. Boussayoud, A. Abderrezzak, *Complete homogeneous symmetric functions of third and second-order linear recurrence sequences*, Electron. J. Math. Anal. Appl., **9** (2021), 226–242. 1

[13] N. Saba, A. Boussayoud, A. Abderrezzak, *Symmetric and generating functions of generalized* (p,q)*-numbers*, Kuwait J. Sci., **48** (2021), 1–15. 1, 1.1, 2.2, 4

[14] N. Saba, A. Boussayoud, S. Araci, M. Kerada, M. Acikgoz, *Construction of a new class of symmetric function of binary products of* (p,q)*-numbers with 2-orthogonal Chebyshev polynomials*, Bol. Soc. Mat. Mex. (3), **27** (2021), 26 pages. 1, 1

[15] N. Saba, A. Boussayoud, M. Chelgham, *Symmetric and generating functions for some generalized polynomials*, Malaya J. Mat., **8** (2020), 1756–1765. 1

[16] N. Saba, A. Boussayoud, K. V. V. Kanuri, *Mersenne Lucas numbers and complete homogeneous symmetric functions*, J. Math. Comput. Sci., **24** (2022), 127–139. 1, 1, 1.5, 1.9

[17] N. Saba, A. Boussayoud, M. Kerada, *Generating functions of even and odd Gaussian numbers and polynomials*, J. Sci. Arts, **21** (2021), 125–144.

[18] A. Suvarnamani, *Some properties of* (p,q)*-Lucas number*, Kyungpook Math. J., **56** (2016), 367–370. 1, 1

[19] A. Suvarnamani, M. Tatong, *Some properties of* (p,q)*-Fibonacci numbers*, Prog. Appl. Sci. Tech., **5** (2015), 17–21. 1, 1

[20] S. Uygun, *The (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas sequences*, Appl. Math. Sci., **9** (2015), 3467–3476. 1