The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .2 No.3 (2011) 546-558

Finite Volume Methods for Fuzzy Parabolic Equations*

Mahmoud Mohseni Moghadam, Iman Jalal

SBUK Center of Excellence in Linear Algebra and Optimization, Kerman, Iran, mohseni@mail.uk.ac.ir
Faculty of Mathematics and Computer Science, Kerman Shahid Bahonar University, imanjalal@gmail.com

Received: July 2010, Revised: November 2010
Online Publication: January 2010

Abstract
In this paper a numerical method for solving “fuzzy partial differential equation” (FPDE) is considered. We present finite volume method that solves some FPDEs such as fuzzy hyperbolic equations, fuzzy parabolic equations and fuzzy elliptic equations. We obtain explicit, implicit and Crank–Nicolson schemes for solving fuzzy heat equation and then see if stability and consistency of these methods exist, and conditions for stability and consistency are given. These methods are illustrated by solving some examples.

Keywords: Fuzzy Partial Differential Equations, Finite Volume Methods.

1. Introduction

Partial differential equations form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. Knowledge about dynamical systems modeled by differential equations is often incomplete or vague. For example, for parametric quantities, functional relationships, or initial conditions, the well-known methods of solving FPDE analytically or numerically can only be used for finding the selected system behavior, e.g., by fixing unknown parameters to some plausible values. However, in this way it is not possible to characterize the whole set of system behaviors compatible with our partial knowledge. It

* This work has been partially supported by the Center of Excellence in Linear Algebra and Optimization, Shahid Bahonar University of Kerman, Kerman, IRAN.

motivates us all such systems as Fuzzy Input-Fuzzy Output (FIFO) systems. To investigate the predictions of FPDE models of such phenomena it is often necessary to approximate their solutions numerically, since the exact solutions for these kinds of problems are almost impossible. In this work we will present the finite volume method that is a discretization method which is well suited for the numerical solution of various types of fuzzy partial differential equations (elliptic, parabolic or hyperbolic, for instance). The method might be called the conservation law approach in that develops the difference scheme using a physical conservation law. In [2], J. Buckley and T. Feuring proposed a method to solve elementary fuzzy partial differential equations. In [1] T. Allahviranloo used a numerical method to solve FPDE that was based on the Seikala derivative. In this paper, our purpose is to solve fuzzy parabolic equations using the finite volume method.

2. Preliminaries

We begin this section with defining the notation we will use in the paper. We place a ~ sign over a letter to denote a fuzzy subset of the real numbers. We write $\tilde{A}(x)$, a number in [0,1], for the membership function of $\tilde{A}$ evaluated at x. An α-cut of $\tilde{A}$, written $\tilde{A}[\alpha]$, is defined as $\{x|\ \tilde{A}(x) \geq \alpha\}$ for $0 < \alpha \leq 1$. Since the α-cuts of fuzzy numbers are always closed and bounded, the intervals we write, $\tilde{N}[\alpha]=[\underline{N}(\alpha),\overline{N}(\alpha)]$ for all α.

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(\alpha), \bar{u}(\alpha))$, $0 \leq \alpha \leq 1$, which satisfies the following requirements:
1. $\underline{u}(\alpha)$ is a bounded left continuous non decreasing function over [0, 1].
2. $\bar{u}(\alpha)$ is a bounded left continuous non increasing function over [0, 1].
3. $\underline{u}(\alpha) \leq \bar{u}(\alpha)$, $0 \leq \alpha \leq 1$.

A crisp number a is simply represented by $\underline{u}(\alpha) = \bar{u}(\alpha) = a$, $0 \leq \alpha \leq 1$. The set of all the fuzzy numbers is denoted by E^1. A popular fuzzy number is the triangular fuzzy number $u = (m,r,\beta)$ which

$$u(x)=\begin{cases}\dfrac{x-m}{r}+1 & m-r\leq x\leq m\\ \dfrac{m-x}{\beta}+1 & m\leq x\leq m+\beta\\ 0 & otherwise.\end{cases}$$

Its parametric form is

$$\underline{u}(\alpha)=m+r(\alpha-1),\quad \bar{u}(\alpha)=m+\beta(\alpha-1).$$

Lemma 2.1. Let $v,w \in E^1$ and s be real number. Then for $0 \leq \alpha \leq 1$

$$u=v \text{ if and only if } \underline{u}(\alpha)=\underline{v}(\alpha) \text{ and } \underline{u}(\alpha)=\underline{v}(\alpha),$$

$$u + v = (\underline{v}(\alpha) + \underline{w}(\alpha), \bar{v}(\alpha) + \bar{w}(\alpha))$$
$$u - v = (\underline{v}(\alpha) - \bar{w}(\alpha), \bar{v}(\alpha) + \underline{w}(\alpha))$$
$$v.w = (\min\{\underline{v}(\alpha).\underline{w}(\alpha), \underline{v}(\alpha).\bar{w}(\alpha), \bar{v}(\alpha).\underline{w}(\alpha), \bar{v}(\alpha).\bar{w}(\alpha)\},$$
$$\max\{\underline{v}(\alpha).\underline{w}(\alpha), \underline{v}(\alpha).\bar{w}(\alpha), \bar{v}(\alpha).\underline{w}(\alpha), \bar{v}(\alpha).\bar{w}(\alpha)\})$$
$$sv = s(\underline{v}(\alpha), \bar{v}(\alpha)).see\,[10]$$

E^1 with addition and multiplication as defined by Lemma 2.1 is a convex cone which is then embedded isomorphically and isometrically in to a Banach space.

It is shown [7] that (E^1, D) is a complete metric space.

Definition 2.1. For arbitrary fuzzy numbers $u = (\underline{u}, \bar{u})$ and the quantity

$$D(u,v) = \sup_{0\le\alpha\le1}\{\max(|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|)\}$$

is the Hausdorff distance between u and v.

3. Fuzzy partial differential equations

Consider the FPDE

$$\phi(D_x, D_y)\tilde{U}(x, y) = \tilde{F}(x, y, \tilde{K}), \tag{3.1}$$

subject to certain boundary conditions where the operator $\phi(D_x, D_y)$ will be a polynomial, with a constant coefficient, in D_x and D_y, where $D_x(D_y)$ stands for the partial differential with respect to $x(y)$. The boundary conditions can be of the form $\tilde{U}(0, y) = \tilde{C}_1$, $\tilde{U}(x, 0) = \tilde{C}_2$, $\tilde{U}(M_1, y) = \tilde{C}_3, \ldots, \tilde{U}(0, y) = \tilde{C}_1, \tilde{U}(0, y) = g_1(y; \tilde{C}_4)$, $\tilde{U}(x, 0) = f_1(x; \tilde{C}_5), \ldots .$ $\tilde{F}(x, y, \tilde{K})$ is the fuzzy function which has $\tilde{K} = (\tilde{K}_1, \ldots, \tilde{K}_n)$ for $\tilde{K}_i$ a triangular fuzzy number in J_i, $1 \le i \le n$. Let $I_1 = [0, M_1]$, $I_2 = [0, M_2]$. The fuzzy function $\tilde{U}$ maps $I_1 \times I_2$ into fuzzy numbers. Also let $\tilde{C} = (\tilde{C}_1, \ldots, \tilde{C}_m)$ with $\tilde{C}_i$ being triangular fuzzy numbers in the intervals L_i, $1 \le i \le m$. Let

$$\tilde{K}[\alpha] = \prod_{i=1}^{n}\tilde{K}_i[\alpha], \quad \tilde{C}[\alpha] = \prod_{i=1}^{n}\tilde{C}_i[\alpha].$$

Suppose $\tilde{U}(x, y)[\alpha] = [\underline{U}(x, y; \alpha), \bar{U}(x, y; \alpha)]$. We assume that the $\underline{U}(x, y; \alpha)$ and $\bar{U}(x, y; \alpha)$ have continuous partial derivatives so that $\phi(D_x, D_y)U(x, y; \alpha)$ and $\phi(D_x, D_y)U(x, y; \alpha)$ are continuous for all $(x, y) \in I_1 \times I_2$ and all α. Define

$$\Gamma(x, y; \alpha) = \phi(D_x, D_y)\tilde{U}(x, y)[\alpha] = [\phi(D_x, D_y)\underline{U}(x, y; \alpha), \phi(D_x, D_y)\bar{U}(x, y; \alpha)], \tag{3.2}$$

for all $(x, y) \in I_1 \times I_2$ and all α.

Definition 3.1 If for each fixed $(x, y) \in I_1 \times I_2$, $\Gamma(x, y; \alpha)$ defines the α-cut of a fuzzy number, then we will say that $\tilde{U}(x, y)$ is differentiable. (See [2]).

Sufficient conditions for $\Gamma(x, y; \alpha)$ to define α-cuts of a fuzzy number are:
1. $\phi(D_x, D_y)\underline{U}(x, y; \alpha)$ is an increasing function of α for each $(x, y) \in I_1 \times I_2$;
2. $\phi(D_x, D_y)\bar{U}(x, y; \alpha)$ is an decreasing function of α for each $(x, y) \in I_1 \times I_2$;
3. $\phi(D_x, D_y)\underline{U}(x, y; 1) \le \phi(D_x, D_y)\bar{U}(x, y; 1)$ for all $(x, y) \in I_1 \times I_2$.

Consider the system of partial differential equations

$$\phi\,(D_x,D_y)\underline{U}(x, y; \alpha) = \underline{F}(x, y; \alpha), \tag{3.3}$$
$$\phi\,(D_x,D_y)\bar{U}(x, y; \alpha) = \overline{F}\,(x, y; \alpha) \tag{3.4}$$

for all $(x, y) \in I_1 \times I_2$ and all $\alpha \in [0, 1]$, where

$$\underline{F}(x, y; \alpha) = min\,\{F(x, y, k)\ /k \in \underline{K}[\alpha]\}, \tag{3.5}$$
$$\overline{F}\,(x, y; \alpha) = max\,\{F(x, y, k)\ /k \in \overline{K}\,[\alpha]\}. \tag{3.6}$$

We append to equations (3.3) and (3.4) any boundary conditions, for example, if they were $\tilde{U}\,(0, y) = \tilde{C}_1$ and $\tilde{U}\,(M_1, y) = C_2$, then we add

$$\underline{U}\,(0, y; \alpha) = \underline{C}_1(\alpha),\ \underline{U}(M_1, y; \alpha) = \underline{C}_1(\alpha)$$

to equation (3.3) and

$$\bar{U}(0, y; \alpha) = C_1(\alpha),\ \bar{U}(M_1, y; \alpha) = \underline{C}_2\,(\alpha)$$

to equation (3.4) where $C_i[\alpha] = [C_1(\alpha),C_2(\alpha)]$, $i = 1, 2$. Let $\underline{U}(x, y; \alpha)$ and $\bar{U}(x, y; \alpha)$ solves equations (3.3) and (3.4), plus the boundary equations, respectively.

Definition 3.2 If for all $(x, y) \in I_1{\times}I_2$, $\tilde{U}\,(x, y)\,[\alpha] = [\underline{U}(x, y; \alpha), \bar{U}(x, y; \alpha)]$ defines the α-cut of a fuzzy number, then $\tilde{U}\,(x, y)$ is the solution for (3.1). (See [2]).

4. Fuzzy parabolic equations

Consider the fuzzy heat equation which is an example of the fuzzy parabolic equations.

$$\tilde{U}_t\,(x,t) = \beta^2\tilde{U}_{xx}\,(x,t), \quad 0 < x < 1, \quad t > 0, \tag{4.1}$$

where

$$\tilde{U}\,(0,t) = \tilde{K}_1, \quad \tilde{U}\,(1,t) = \tilde{K}_2, \quad t > 0$$
$$\tilde{U}\,(x,0) = \tilde{f}\,(x), \qquad 0 \le x \le 1.$$

Since any fuzzy number u can be written as $u= (\underline{u}(\alpha), \bar{u}(\alpha))$, $0 \le \alpha \le 1$ (see[3]) thus If $\tilde{U}_{xx} \in E$ and $\tilde{U}_t \in E$ then we have

$$\underline{U}_t\,(x,t;\alpha) = \beta^2\underline{U}_{xx}\,(x,t;\alpha) \tag{4.2}$$
$$\underline{U}\,(0,t;\alpha) = \underline{K}_1(\alpha), \quad \underline{U}\,(1,t;\alpha) = \underline{K}_2(\alpha), \quad t > 0,$$
$$\underline{U}\,(x,0;\alpha) = \underline{f}\,(x;\alpha), \qquad 0 \le x \le 1, \quad \alpha \in [0,1],$$
$$\bar{U}_t\,(x,t;\alpha) = \beta^2\bar{U}_{xx}\,(x,t;\alpha) \tag{4.3}$$
$$\bar{U}\,(0,t;\alpha) = \bar{K}_1(\alpha), \quad \bar{U}\,(1,t;\alpha) = \bar{K}_2(\alpha), \quad t > 0,$$
$$\bar{U}\,(x,0;\alpha) = \bar{f}\,(x;\alpha), \qquad 0 \le x \le 1, \quad \alpha \in [0,1].$$

5. Finite volume method

In this section we shall solve problems (4.2), (4.3) numerically. Our method is to reduce the problem above to a discrete problem that we are able to solve. First we consider the grid is placed on the interval (0,1) given in Figure 1.

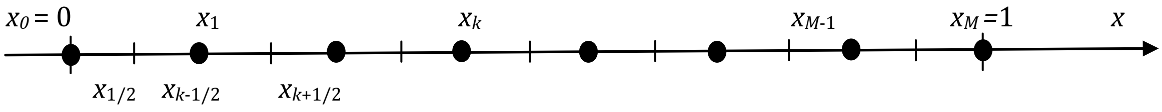

Figure 1. Grid placed on the interval (0,1) whit blocks centered at the grid points.

If we wish to refer to one of the points in the grid, we shall call the points x_k, $k = 0, \ldots M$ where $x_k = k\,\Delta x$, $\Delta x=1/M$, $k = 0, \ldots M$. We shall refer to endpoint of the interval about the point x_k as $x_{k\pm 1/2}$. This interval is referred to as the control volume associated with the *kth* grid point. To derive a difference equation associated with the *kth* grid point above we integrate of the equations (4.2), (4.3) over each interval $(x_{k-1/2}, x_{k+1/2})$ (which we refer to as the *kth* cell). Assuming that β is constant, we get

$$\int_{x_{k-1/2}}^{x_{k+1/2}} \underline{U}_t(x,t;\alpha)dx = \beta^2 \int_{x_{k-1/2}}^{x_{k+1/2}} \underline{U}_{xx}(x,t;\alpha)dx, \quad \int_{x_{k-1/2}}^{x_{k+1/2}} \overline{U}_t(x,t;\alpha)dx = \beta^2 \int_{x_{k-1/2}}^{x_{k+1/2}} \overline{U}_{xx}(x,t;\alpha)dx \quad (5.1)$$

We note that both sides of equations (4.1) are still functions of t. If we integrate from $t_n = n\Delta t$ to $t_{n+1} = (n+1)\,\Delta t$, we get

$$\int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} \underline{U}_t(x,t;\alpha)dxdt = \beta^2 \int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} \underline{U}_{xx}(x,t;\alpha)dxdt,$$
$$\int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} \overline{U}_t(x,t;\alpha)dxdt = \beta^2 \int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} \overline{U}_{xx}(x,t;\alpha)dxdt. \quad (5.2)$$

From the above calculation, we get

$$\int_{x_{k-1/2}}^{x_{k+1/2}} (\underline{U}(x,t_{n+1};\alpha) - \underline{U}(x,t_n;\alpha))dx = \beta^2 \int_{t_n}^{t_{n+1}} (\underline{U}_x(x_{k+1/2},t;\alpha) - \underline{U}_x(x_{k-1/2},t;\alpha))dt,$$
$$\int_{x_{k-1/2}}^{x_{k+1/2}} (\overline{U}(x,t_{n+1};\alpha) - \overline{U}(x,t_n;\alpha))dx = \beta^2 \int_{t_n}^{t_{n+1}} (\overline{U}_x(x_{k+1/2},t;\alpha) - \overline{U}_x(x_{k-1/2},t;\alpha))dt \quad (5.3)$$

which we call equations (5.3) integral form of the conservation law. Hence, equations (5.3) are exact equations. We now proceed to obtain an approximation of fuzzy partial differential equation (4.1). We do this approximating the integral form of conservation law given in equations (5.3). We begin by approximating the integrals on the left by midpoint rule with respect to x. we get

$$\int_{x_{k-1/2}}^{x_{k+1/2}} (\underline{U}(x,t_{n+1};\alpha) - \underline{U}(x,t_n;\alpha))dx = \Delta x\,(\underline{U}(x_k,t_{n+1};\alpha) - \underline{U}(x_k,t_n;\alpha)) + O(\Delta t\,\Delta x^3),$$
$$\int_{x_{k-1/2}}^{x_{k+1/2}} (\overline{U}(x,t_{n+1};\alpha) - \overline{U}(x,t_n;\alpha))dx = \Delta x\,(\overline{U}(x_k,t_{n+1};\alpha) - \overline{U}(x_k,t_n;\alpha)) + O(\Delta t\,\Delta x^3). \quad (5.4)$$

Where the Δt term is due to the fact that the functions we are integrating are in the forms $\underline{U}(x,t_{n+1};\alpha) - \underline{U}(x,t_n;\alpha)$ and $\overline{U}(x,t_{n+1};\alpha) - \overline{U}(x,t_n;\alpha)$. We approximate the integrals on the right by the lower rectangular rule with respect to t (evaluating the function at t_n). We get

$$\int_{t_n}^{t_{n+1}} (\underline{U}_x(x_{k+1/2},t;\alpha)-\underline{U}_x(x_{k-1/2},t;\alpha))dt = \Delta t(\underline{U}_x(x_{k+1/2},t_n;\alpha)-\underline{U}_x(x_{k-1/2},t_n;\alpha))+O(\Delta t^2\Delta x),$$
$$\int_{t_n}^{t_{n+1}} (\bar{U}_x(x_{k+1/2},t;\alpha)-\bar{U}_x(x_{k-1/2},t;\alpha))dt = \Delta t(\bar{U}_x(x_{k+1/2},t_n;\alpha)-\bar{U}_x(x_{k-1/2},t_n;\alpha))+O(\Delta t^2\Delta x). \quad (5.5)$$

Where the Δx term in the order of approximation is due to the fact that the integrands are a difference in k. We are then left to approximate the terms $\underline{U}_x(x_{k+1/2},t_n;\alpha)-\underline{U}_x(x_{k-1/2},t_n;\alpha)$ and $\bar{U}_x(x_{k+1/2},t_n;\alpha)-\bar{U}_x(x_{k-1/2},t_n;\alpha)$. If we expand Taylor series about x_k, we get

$$\underline{U}_x(x_{k+1/2},t_n;\alpha)-\underline{U}_x(x_{k-1/2},t_n;\alpha)=\frac{\delta_x^{\,2}\underline{U}}{\Delta x}(x_k,t_n;\alpha)+O(\Delta x^3),$$
$$\bar{U}_x(x_{k+1/2},t_n;\alpha)-\bar{U}_x(x_{k-1/2},t_n;\alpha)=\frac{\delta_x^{\,2}\bar{U}}{\Delta x}(x_k,t_n;\alpha)+O(\Delta x^3) \quad (5.6)$$

where $\delta_x^{\,2}\underline{U}_{k\,n}=\underline{U}_{k+1n}-2\underline{U}_{k\,n}+\underline{U}_{k-1n}$, $\delta_x^{\,2}\bar{U}_{k\,n}=\bar{U}_{k+1n}-2\bar{U}_{k\,n}+\bar{U}_{k-1n}$, $\underline{U}_{k\,n}=\underline{U}(x_k,t_n;\alpha)$ and $\bar{U}_{k\,n}=\bar{U}(x_k,t_n;\alpha)$. We then combine equations (5.4)-(5.6) with (5.3) to get

$$\Delta x\,\delta_t\underline{U}_{k\,n}+O(\Delta t\Delta x^3)=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\underline{U}_{k\,n}+O(\Delta t\Delta x^3)+O(\Delta t^2\Delta x),$$
$$\Delta x\,\delta_t\bar{U}_{k\,n}+O(\Delta t\Delta x^3)=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\bar{U}_{k\,n}+O(\Delta t\Delta x^3)+O(\Delta t^2\Delta x) \quad (5.7)$$

Where $\delta_t\underline{U}_{k\,n}=\underline{U}_{k\,n+1}-\underline{U}_{k\,n}$, $\delta_t\bar{U}_{k\,n}=\bar{U}_{k\,n+1}-\bar{U}_{k\,n}$.And, finally, we note that if we replace the functions evaluation $\underline{U}_{k\,n}$ and $\bar{U}_{k\,n}$ by the approximations $\underline{u}_{k\,n}$ and $\bar{u}_{k\,n}$ and approximate equations (5.7) by dropping the O terms, we obtain difference equations

$$\Delta x\,\delta_t\underline{u}_{k\,n}=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\underline{u}_{k\,n}, \qquad \Delta x\,\delta_t\bar{u}_{k\,n}=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\bar{u}_{k\,n}, \quad (5.8)$$
$$\underline{u}_{0n}=\underline{K}_1(\alpha),\quad \underline{u}_{M\,n}=\underline{K}_2(\alpha),\quad n=0,1,2,\ldots N \qquad \bar{u}_{0n}=\bar{K}_1(\alpha),\quad \bar{u}_{M\,n}=\bar{K}_2(\alpha),\quad n=0,1,2,\ldots N$$
$$\underline{u}_{k\,0}=\underline{f}(x_k;\alpha),\quad k=1,2,\ldots,M-1, \qquad \bar{u}_{k\,0}=\bar{f}(x_k;\alpha),\quad k=1,2,\ldots,M-1.$$

Clearly each value at time level t_{n+1} can be independently calculated from values at time level t_n; for this reason this is called an explicit difference scheme.

The stability limit $\Delta t \le \frac{(\Delta x)^2}{2\beta^2}$ is a very severe restriction, and implies that very many time steps will be necessary to follow the solution over a reasonably large time interval. We shall now show how the use of an upper rectangular rule with respect to t gives a difference scheme which avoids this restriction, but at the cost of a slightly more sophisticated calculation. If we replace the integrals on the right (4.3) by the upper rectangular rule with respect to t (evaluating the function at t_{n+1}), the integrals on the left remaining the same, we obtain the scheme

$$\Delta x\,\delta_t\underline{U}_{k\,n}+O(\Delta t\Delta x^3)=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\underline{U}_{k\,n+1}+O(\Delta t\Delta x^3)+O(\Delta t^2\Delta x),$$
$$\Delta x\,\delta_t\bar{U}_{k\,n}+O(\Delta t\Delta x^3)=\frac{\beta^2\Delta t}{\Delta x}\delta_x^{\,2}\bar{U}_{k\,n+1}+O(\Delta t\Delta x^3)+O(\Delta t^2\Delta x) \quad (5.9)$$

instead of (5.7). Now if we replace the function evaluation $\underline{U}_{k\,n}$ and $\bar{U}_{k\,n}$ by the approximations $\underline{u}_{k\,n}$ and $\bar{u}_{kn}$ and approximate equations (5.9) by dropping the O terms, we obtain difference equations

$$\Delta x\,\delta_t \underline{u}_{k\,n} = \frac{\beta^2 \Delta t}{\Delta x}\delta_x{}^2 \underline{u}_{k\,n+1}, \quad \Delta x\,\delta_t \bar{u}_{k\,n} = \frac{\beta^2 \Delta t}{\Delta x}\delta_x{}^2 \bar{u}_{k\,n+1}. \tag{5.10}$$

This is an example of an implicit scheme, which is not so easy to use as the explicit scheme described earlier. The schemes (5.10) involve six unknown values of $\underline{u}$ and $\bar{u}$ on the new time level $n+1$; we cannot immediately calculate the values of $\underline{u}_{kn}$, $\bar{u}_{kn}$ since the equations involve the four neighbouring values $\underline{u}_{k+1n+1}$, $\underline{u}_{k-1n-1}$, $\bar{u}_{k+1n+1}$ and $\bar{u}_{k-1n-1}$ which are also unknown. We must now write the equations in the form

$$\underline{u}_{k\,n} = -r\underline{u}_{k+1n+1} + (1+2r)\underline{u}_{k\,n+1} - r\underline{u}_{k-1\,n+1},$$
$$\bar{u}_{k\,n} = -r\bar{u}_{k+1n+1} + (1+2r)\bar{u}_{k\,n+1} - r\bar{u}_{k-1\,n+1}, \quad r = \frac{\beta^2 \Delta t}{\Delta x^2}. \tag{5.11}$$

Giving k the values $1, 2, \ldots, (M-1)$ we thus obtain a system of $2(M-1)$ linear equations in the $2(M-1)$ unknowns $\underline{u}_{k\;n+1}$ and $\bar{u}_{k\;n+1}$, $k = 1, 2, \ldots, M-1$. Instead of calculating each of these unknowns by a separate trivial formula, we must now solve this system of equations to give the values simultaneously. Note that in the first and last of these equations, corresponding to $k = 1$ and $k = M-1$, we incorporate the known values of $\underline{u}_{0\;n+1}$, $\underline{u}_{M\;n+1}$, $\bar{u}_{0\;n+1}$ and $\bar{u}_{M\;n+1}$ given by the boundary conditions.

We have now considered two finite volume methods, which differ only in that one approximates the integrals on the right (5.3) by the lower rectangular rule (evaluating the function at t_n) and the other uses from the upper rectangular rule (evaluating the function at t_{n+1}). If we replace the integrals on the right (5.3) by the trapezoidal rule with respect to t (average of the lower rectangular rule and the upper rectangular rule), the integrals on the left remaining the same, we obtain the scheme

$$\Delta x\,\delta_t \underline{U}_{k\,n} + O(\Delta t\,\Delta x^3) = \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \underline{U}_{k\,n} + O(\Delta t\,\Delta x^3) + \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \underline{U}_{k\,n+1} + O(\Delta t\,\Delta x^3) + O(\Delta t^3 \Delta x),$$
$$\Delta x\,\delta_t \bar{U}_{k\,n} + O(\Delta t\,\Delta x^3) = \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \bar{U}_{k\,n} + O(\Delta t\,\Delta x^3) + \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \bar{U}_{k\,n+1} + O(\Delta t\,\Delta x^3) + O(\Delta t^3 \Delta x). \tag{5.12}$$

Now if we replace the function evaluation $\underline{U}_{k\,n}$ and $\bar{U}_{k\,n}$ by the approximations $\underline{u}_{k\,n}$ and $\bar{u}_{kn}$ and approximate equations (5.12) by dropping the O terms, we obtain difference equations

$$\Delta x\,\delta_t \underline{u}_{k\,n} = \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \underline{u}_{k\,n} + \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \bar{u}_{k\,n+1}, \quad \Delta x\,\delta_t \bar{u}_{k\,n} = \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \bar{u}_{k\,n} + \frac{\beta^2 \Delta t}{2\Delta x}\delta_x{}^2 \bar{u}_{k\,n+1}. \tag{5.13}$$

This is well known and popular Crank–Nicolson scheme. Where we must solve a system of equations in the form

$$-\frac{r}{2}\underline{u}_{k+1n} + (1+r)\underline{u}_{k\,n} - \frac{r}{2}\underline{u}_{k-1\,n} = -\frac{r}{2}\underline{u}_{k+1n+1} + (1+r)\underline{u}_{k\,n+1} - \frac{r}{2}\underline{u}_{k-1\,n+1},$$
$$-\frac{r}{2}\bar{u}_{k+1n} + (1+r)\bar{u}_{k\,n} - \frac{r}{2}\bar{u}_{k-1\,n} = -\frac{r}{2}\bar{u}_{k+1n+1} + (1+r)\bar{u}_{k\,n+1} - \frac{r}{2}\bar{u}_{k-1\,n+1}, \quad r = \frac{\beta^2 \Delta t}{\Delta x^2}. \tag{5.14}$$

A natural generalization is to take weighted average of the two formulae (explicit and implicit). Since the time difference on the left-hand sides is the same, we obtain difference equations

$$\Delta x\, \delta_t \underline{u}_{k\,n} = \frac{\beta^2 \Delta t}{\Delta x}(\theta\delta_x{}^2\underline{u}_{k\,n} + (1-\theta)\delta_x{}^2\underline{u}_{k\,n+1}), \quad \Delta x\, \delta_t \overline{u}_{k\,n} = \frac{\beta^2 \Delta t}{\Delta x}(\theta\delta_x{}^2\overline{u}_{k\,n} + (1-\theta)\delta_x{}^2\overline{u}_{k\,n+1}). \quad (5.15)$$

We shall assume that we are using an average with nonnegative weights, so that $0 \le \theta \le 1$; $\theta = 0$ gives the explicit scheme, $\theta = 1$ the fully implicit scheme and $\theta = \frac{1}{2}$ Crank–Nicolson scheme.

6. Consistency and stability

Definition 6.1 At any point away from the boundary we can define the truncation error $[\underline{T}_{k\,n}, \overline{T}_{k\,n}]$

$$\underline{T}_{k\,n} := \Delta x\, \delta_t \underline{U}_{k\,n} - \frac{\beta^2 \Delta t}{\Delta x}(\theta\delta_x{}^2\underline{U}_{k\,n} + (1-\theta)\delta_x{}^2\underline{U}_{k\,n+1})$$
$$\overline{T}_{k\,n} := \Delta x\, \delta_t \overline{U}_{k\,n} - \frac{\beta^2 \Delta t}{\Delta x}(\theta\delta_x{}^2\overline{U}_{k\,n} + (1-\theta)\delta_x{}^2\overline{U}_{k\,n+1}). \quad (6.16)$$

Definition 6.2 the difference scheme is consistence with fuzzy partial differential equation (4.1) if
$\underline{T}_{k\,n} \to 0, \overline{T}_{k\,n} \to 0 \quad as\ \Delta x, \Delta t \to 0 \quad \forall (x,t) \in (0,1)\times(0,t_0)$ (for some $t_0 > t$).

If we now use Taylor expansions to calculate the truncation error we obtain

$$\begin{aligned}\underline{T}_{k\,n} &= [\Delta x\, \Delta t \underline{U}_t - \beta^2 \Delta x\, \Delta t \underline{U}_{xx}] + [\beta^2 \Delta x\, (\Delta t)^2 (\tfrac{1}{2} - \theta)\underline{U}_{xxt} - \tfrac{1}{12}(\Delta x)^3 \Delta t \underline{U}_{xxxx}] \\ &+ [\tfrac{1}{24}\Delta x\, (\Delta t)^3 \underline{U}_{ttt} - \tfrac{\beta^2}{8}\Delta x\, (\Delta t)^3 \underline{U}_{xxtt}] \\ &+ [\tfrac{\beta^2}{12}(\Delta x)^3 (\Delta t)^2 (\tfrac{1}{2} - \theta)\underline{U}_{xxxxt} - \tfrac{2\beta^2}{6!}(\Delta x)^5 (\Delta t)\underline{U}_{xxxxxx}],\end{aligned}$$

$$\begin{aligned}\overline{T}_{k\,n} &= [\Delta x\, \Delta t \overline{U}_t - \beta^2 \Delta x\, \Delta t \overline{U}_{xx}] + [\beta^2 \Delta x\, (\Delta t)^2 (\tfrac{1}{2} - \theta)\overline{U}_{xxt} - \tfrac{1}{12}(\Delta x)^3 \Delta t \overline{U}_{xxxx}] \\ &+ [\tfrac{1}{24}\Delta x\, (\Delta t)^3 \overline{U}_{ttt} - \tfrac{\beta^2}{8}\Delta x\, (\Delta t)^3 \overline{U}_{xxtt}] \\ &+ [\tfrac{\beta^2}{12}(\Delta x)^3 (\Delta t)^2 (\tfrac{1}{2} - \theta)\overline{U}_{xxxxt} - \tfrac{2\beta^2}{6!}(\Delta x)^5 (\Delta t)\overline{U}_{xxxxxx}].\end{aligned} \quad (6.17)$$

Theorem 6.1 the difference scheme **(**5.15) is consistence for all values θ if $\underline{U}_{ttt}, \underline{U}_{xxxx}, \underline{U}_{xxxxxx}$ and $\overline{U}_{ttt}, \overline{U}_{xxxx}, \overline{U}_{xxxxxx}$ be uniformly bounded on (0,1)×(0,t_0) (for some $t_0 > t$).

Proof: See [6],[8],[11].

One interpretation of stability of a difference scheme is that for a stable difference scheme small errors in the initial conditions cause small errors in the solution.

Theorem 6.2 equation **(**5.15) stable for $0 \le \theta < \frac{1}{2}$ if and only if $r \le \frac{1}{2}(1 - 2\theta)^{-1}$, and stable for all r when $\frac{1}{2} \le \theta \le 1$.

Proof: See [6],[8],[11].

7. Examples

Example 7.1 Consider the fuzzy parabolic equation

$$\tilde{U}_t(x,t) = \tilde{U}_{xx}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{7.1}$$

with the boundary conditions

$$\tilde{U}(0,t) = \tilde{U}(1,t) = 0, \quad t > 0$$
$$\tilde{U}(x,0) = \tilde{f}(x) = \tilde{k}\sin(\pi x), \; 0 \le x \le 1.$$

and $\tilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [\alpha - 1, 1 - \alpha]$.
The exact solution for

$$\underline{U}_t(x,t;\alpha) - \beta^2 \underline{U}_{xx}(x,t;\alpha) = 0,$$
$$\overline{U}_t(x,t;\alpha) - \beta^2 \overline{U}_{xx}(x,t;\alpha) = 0$$

for 0<x<1,t>0 are $\underline{U}(x,y;\alpha) = \underline{k}(\alpha)e^{-\pi^2 t}\sin(\pi x)$ and $\overline{U}(x,y;\alpha) = \overline{k}(\alpha)e^{-\pi^2 t}\sin(\pi x)$. We use the explicit, implicit and Crank–Nicolson schemes approximate the exact solution at the point (0.1, 0.0001) with Δx = 0.1, Δt = 0.00001, N=10, at the point (0.1, 0.1) with Δx = 0.1, Δt = 0.01, N=10 and at the point (0.1, 0.1) with Δx = 0.1, Δt = 0.00001, N=10000. Figure. 2, Figure. 3 and Figure. 4 show the exact and approximate solutions for each $\alpha \in [0,1]$. Table 1 shows the Hausdorff distance between the solutions.

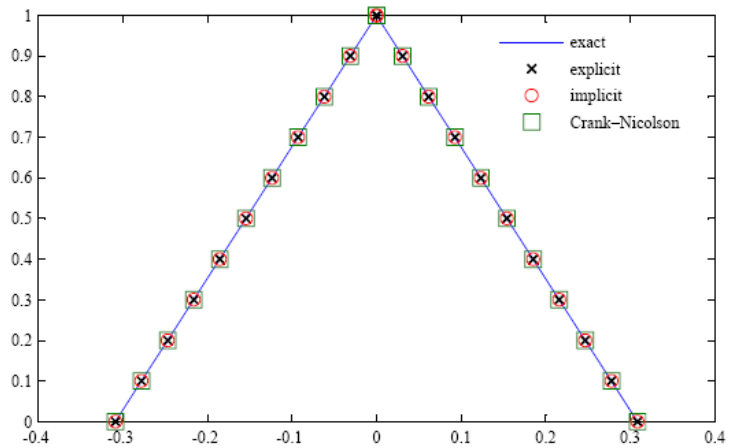

Figure 2. Results obtained at the point (0.1, 0.0001) with explicit, implicit

and Crank–Nicolson schemes; Δx = 0.1, Δt = 0.00001, N=10(Example 7.1)

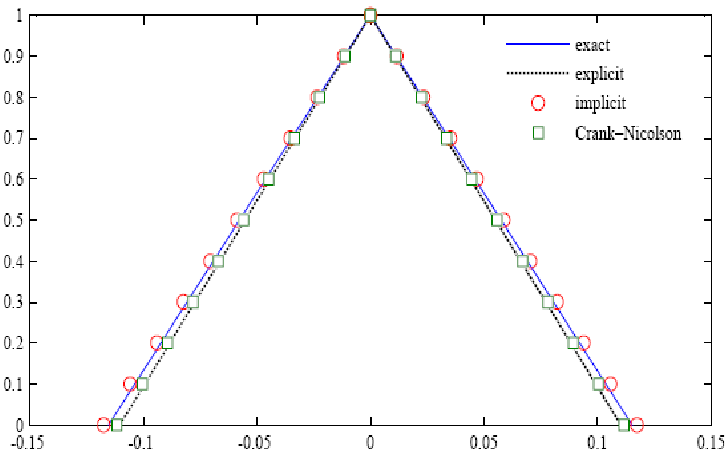

Figure 3. Results obtained at the point (0.1, 0.1) with explicit, implicit and Crank–Nicolson schemes; Δx = 0.1, Δt = 0.01, N=10 (Example 7.1)

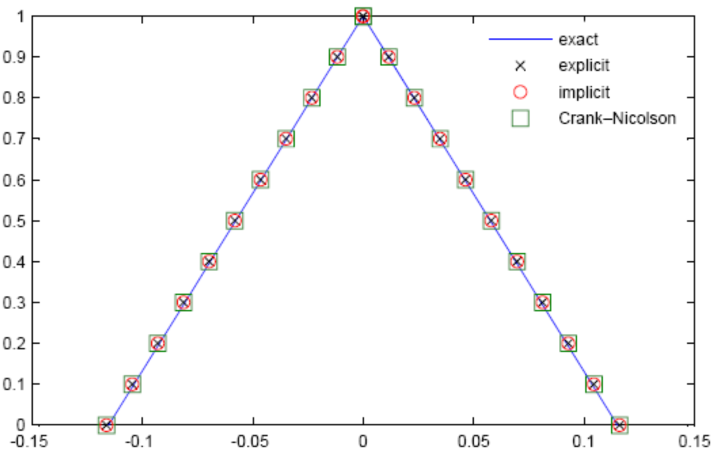

Figure 4. Results obtained at the point (0.1, 0.1) with explicit, implicit and Crank–Nicolson schemes; Δx = 0.1, Δt = 0.00001, N=10000 (Example 7.1)

Table 1. Hausdorff distance between the solutions (Example 7.1)

(x, t)	(Δx, Δt)	N	(0.1,0.0001)	(0.1,0.00001)	10	(0.1,0.1)	(0.1,0.01)	10	(0.1,0.1)	(0.1,0.00001)	10000
Hausdorff distance between the exact solution and the explicit scheme solution			2.4×10^{-6}			4.8689×10^{-3}			9.300×10^{-4}		
Hausdorff distance between the exact solution and the implicit scheme solution			2.5×10^{-6}			2.2793×10^{-3}			9.411×10^{-4}		
Hausdorff distance between the exact solution and the Crank–Nicolson scheme solution			2.5×10^{-6}			3.5755×10^{-3}			9.356×10^{-4}		

Example 7.2 Consider the previous example with $\tilde{k}[\alpha]=[\underline{k}(\alpha),\ \overline{k}(\alpha)]=[0.5\alpha+0.5, 1.5-0.5\alpha]$.
We use the equations explicit, implicit, and Crank–Nicolson approximate the exact solution at the point (0.4, 0.5) with Δx = 0.1, Δt = 0.0005, N=1000 and Δx = 0.1, Δt = 0.01, N =50. Figure.5, Figure.6 and Figure. 7 show the exact and approximate solutions for each $\alpha \in [0,1]$.Table 2 shows the Hausdorff distance between the solutions.

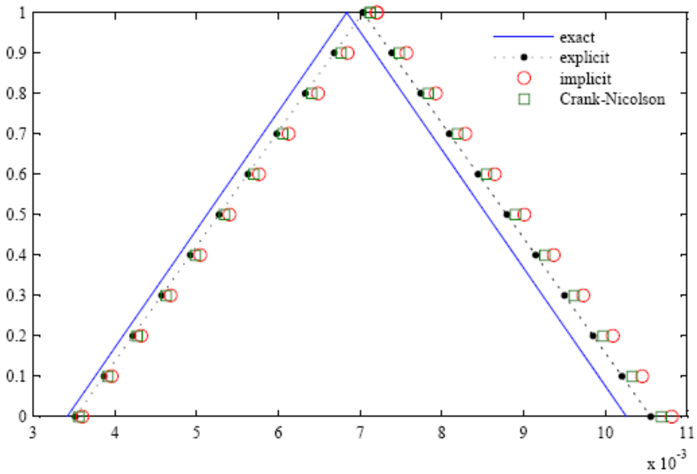

Figure 5. Results obtained at the point (0.4, 0.5) with explicit, implicit and Crank–Nicolson schemes; Δx = 0.1, Δt = 0.0005, N=1000 (Example 7.2)

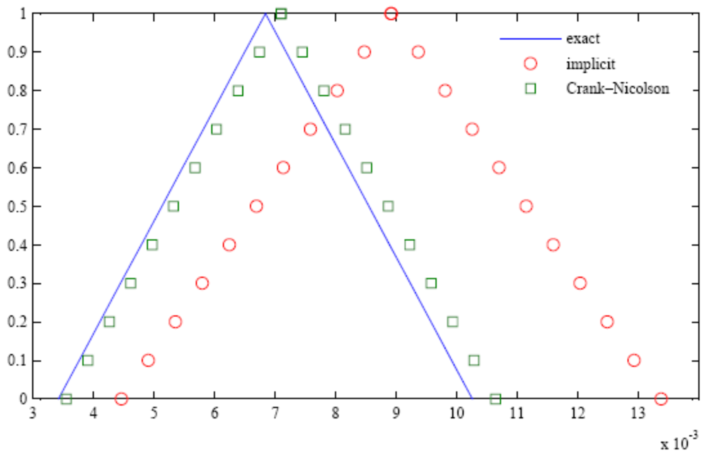

Figure 6. Results obtained at the point (0.4, 0.5) with implicit and Crank–Nicolson schemes; Δx = 0.1, Δt = 0.01, N=50(Example 7.2)

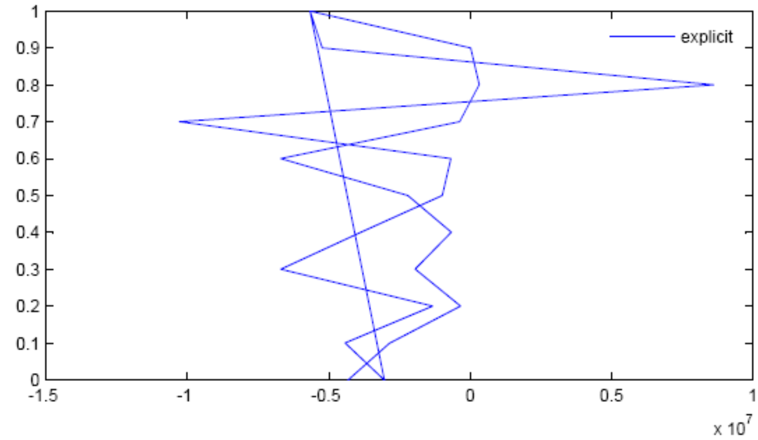

Figure 7. Results obtained at the point (0.4, 0.5) with explicit scheme; Δx = 0.1, Δt = 0.01, N=50 (Example 7.2)

Table 2. Hausdorff distance between the solutions (Example 7.2)

(x, t)	$(\Delta x, \Delta t)$	N	(0.4,0.5)	(0.1,0.0005)	1000	(0.4,0.5)	(0.1,0.01)	50
Hausdorff distance between the exact solution and the explicit scheme solution			2.960×10^{-4}			1.02798621×10^{7}		
Hausdorff distance between the exact solution and the implicit scheme solution			5.519×10^{-4}			3.119×10^{-3}		
Hausdorff distance between the exact solution and the Crank–Nicolson scheme solution			4.235×10^{-4}			3.819×10^{-4}		

We see that at the point (0.4, 0.5) whit Δx = 0.1, Δt = 0.01, N=50, r =0.01/ $(0.1)^2$ =1 therefore explicit scheme in this point is an unstable mode (by theorem6.2) that Figure 7 shows clearly this instability. Furthermore Table 3 shows at the point (0.4, 0.5) whit Δx = 0.1, Δt = 0.01, N=50 the explicit scheme solutions don't define the α-cuts of a fuzzy number for some α.

Table 3. Results obtained at the point (0.4, 0.5) with Explicit scheme; Δx = 0.1, Δt = 0.01, N=50 (Example 7.2)

α	$\underline{u}(x_4,t_{50};\alpha)$	$\bar{u}(x_4,t_{50};\alpha)$
0	-4.309453×10^{6}	-3.035247×10^{6}
0.1	-2.830447×10^{6}	-4.418238×10^{6}
0.2	-3.38239×10^{5}	-1.327518×10^{6}
0.3	-1.936428×10^{6}	-6.688252×10^{6}
0.4	-6.63759×10^{5}	-3.872857×10^{6}
0.5	-2.209119×10^{6}	-9.84858×10^{5}
0.6	-6.692739×10^{6}	-6.76479×10^{4}
0.7	-3.69562×10^{5}	-1.0279862×10^{7}
0.8	3.26223×10^{5}	-8.618906×10^{6}
0.9	2.1003×10^{4}	-5.233417×10^{6}
1	-5.660894×10^{6}	-5.660894×10^{6}

8. Conclusions

We presented difference methods for solving Fuzzy parabolic equations. These numerical solutions are based on the seikkala derivative. If these solutions define α-cuts of a fuzzy number, then the solutions of FPDE, would exist, which has been concluded from the numerical values.

References

[1] Allahviranloo, T., "Difference methods for fuzzy partial differential equations", *Computational Methods in Appliead Mathematics*, No.3, Vol.2, pp. 233-242, 2002.
[2] Buckley, J. J., and Feuring, T. "Introduction to fuzzy partial differential equations", *Fuzzy Sets and Systems*, Vol.105, pp. 241-248, 1999.
[3] Burden, S.L.R.L., and Faires, J. D., "*Numerical Analysis*", 1997.
[4] Chen, M., Wu, C., Xue,X., and Liu,G., "On fuzzy boundary value problems", *information science,*Vol.178, pp.1877-1892, 2008.
[5] Eymard, R., Gallouet, T., and Herbin, R., "*Finite volume methods*", Handbook of Numerical Analysis, Ph. Ciarlet J.L. Lions eds, North Holland, Vol.7, pp. 713-1022, 2000.
[6] Leveque,R.J., "*Finite Volume Methods for Hyperbolic Problems*",Cambridge,2004.
[7] Matloka,M., "On fuzzy integrals", *Proceedings of the Second Polish Symposium on Interval and Fuzzy mathematics,* Polite chnika poznansk, pp167–170,1987.
[8] Morton, K.W.,and Mayers, D.F., "*Numerical Solution of Partial Differential Equations*", Cambridge, 2005.
[9] Patankar, S.V., "*Numerical heat transfer and fluid flow*", Series in Computational Methods in Mechanics and Thermal Sciences, Minkowycz and Sparrow Eds, McGraw Hill,1980.
[10] Seikkala.S, "on the fuzzy initial value problem", *Fuzzy sets and systems* Vol.24, pp. 319-330, 1987.
[11] Thomas, J.W., "*Numerical Partial Differential Equations: Finite Difference Methods*", Springer-Verlag, 1995.