J. Math. Computer Sci., 34 (2024), 350–360

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On a class of piece-wise fractional order derivative delay differential equation with integral type condition

Kamal Shah[a,b,*], Muhammad Sher[b], Muhammad Sarwar[a,b], Thabet Abdeljawad[a]

[a]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia.

[b]Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan.

Abstract

In this paper, we present a detailed study of a class of fractional-order delay differential equations, highlighting that many real-world problems exhibit multifaceted behaviors in their dynamical interpretations. To capture the aforementioned behavior in a more realistic way, the use of piecewise derivatives of fractional orders has increasingly been applied. Given the significant role of delay differential equations in modeling various real-world scenarios, this work specifically addresses a type of delay differential equation with a proportional delay term. Employing piecewise fractional derivatives and Ulam-Hyers (U-H) type stability analysis, we explore the qualitative theory of the analyzed problem. Utilizing fixed-point theory and techniques from functional analysis, we aim to achieve the desired outcomes. To demonstrate our findings, several illustrative examples are provided.

Keywords: Piecewise operator, theoretical results, Ulam type stability.

2020 MSC: 26A33, 34A08, 35A09.

1. Introduction

The use of mathematics and its applications in recent years has led to the development of a significant scenario that has been designed to forecast the behavior of complicated physical problems. It has been demonstrated in the past that mathematics plays a significant part in the process of solving difficulties that arise in a variety of scientific subfields. In particular, the features of the differentiable functions that are found in calculus are utilized to estimate the integer order derivatives of one quantity with respect to another at a point that is located in its infinitesimal neighborhood. This helps to determine the rate of change between the two quantities. As a consequence of this reality, the traditional derivative over time is unable to adequately capture the dynamics of a phenomenon that is associated with long memory. It is fair to apply the theory and properties of fractional calculus, which is the extension of integer order derivative to arbitrary real numbers, in order to characterize the dynamics of memory difficulties. In addition, the aforementioned field is applicable in a wide range of other scientific subfields,

*Corresponding author

Email addresses: kamalshah408@gmail.com (Kamal Shah), msherpak55@gmail.com (Muhammad Sher), sawarswati@gmail.com (Muhammad Sarwar), tabdeljawad@psu.edu.sa (Thabet Abdeljawad)

doi: 10.22436/jmcs.034.04.03

Received: 2023-10-27 *Revised:* 2024-01-19 *Accepted:* 2024-02-19

such as physics [12, 22], Biology [18, 20], Chemistry [23], and a variety of other fields [9]. At the same time as integer order was being developed, the idea of any arbitrary order derivative was also being developed. It has not been deemed the answer to problems that occur in the actual world due to the fact that it is quite complicated. In later years, as technology continued to evolve and new definitions of complex functions were developed, the field received the attention it deserved. The idea has currently been effectively used to address a wide range of complex scientific problems. Numerous real-world issues have been resolved by using fractional derivatives instead of integer order. The State-space model with non-integer order derivatives for lithium-ion batteries is one example of these issues [21], the Caputo fractional order derivative (CFD) model of Zika virus transmission dynamics [19], current empiricism, and classical science are all examples of fractional-order modeling of electric circuits [28], etc. On the other hand, the phrase "fractional derivative" does not have a single, widely accepted definition. Numerous definitions, such as the Riemann-Liouville (R-L) and Caputo definitions, can be found in the literature [10], Caputo-Fabrizio definition [8], Atangana-Baleanu definition [7], etc. These ideas have recently been used as the foundation for breaking down numerous mathematical problems. See, for instance [2, 15, 36] and the references contained therein. For more sophisticated applications of dynamical systems studied using fractional calculus, we refer to [4, 14, 24, 33].

The traditional idea of derivatives has primarily been put to use for the purpose of analyzing the dynamics of real-world problems. On the other hand, dynamical issues can occasionally display multi behaviors. This kind of situation is referred to as crossover behavior or multi-behavior phenomena. It has been discovered that the basic fractional order derivative or the integer order derivative are unable to adequately explain the crossover behavior of a physical phenomenon. These concepts are not adequately explained by standard fractional order derivatives involving exponential, Mittag-Leffler, or power law kernels. For instance, earthquakes, the swinging of a pendulum, and economic fluctuations in countries with less developed economies all suffer sudden alterations in the state that their dynamics are in. The use of piece-wise derivatives, as opposed to continuous derivatives, has been demonstrated to be the most effective method of illustrating this crossover occurrence. In this context, the most recent investigation of the aforementioned derivative's most important properties may be found in [3]. Fractional derivatives also take into account the memory and genetic impacts, which are both crucial aspects to take into account, when researching a range of dynamical challenges that arise in real-world situations. The situation becomes more plausible in particular when fractional order derivatives are taken into account in mathematical models of infectious disease transmission. Long-term memory effects associated with the conventional fractional calculus approach make it difficult to solve problems involving long-term calculations. Additionally, the traditional fractional calculus's mathematical tools can be used to characterize the power-law long memory. In a few recent research, the idea of piecewise derivatives of fractional orders has also been applied. For instance, piecewise fractional differential equations have been utilized by several writers to explore dynamical systems. Unquestionably, these operators will be employed to find answers to issues that demonstrate crossover behavior. Researchers' perspectives on the matter are quickly becoming more optimistic. This method is currently being used by many researchers in their studies on models of infectious illnesses, such as the SARS-2 disease model for CAT-T cells, which was studied by applying the previously mentioned idea in [31]. Using the notion described in [35], the COVID-19 model has been the subject of research. Some recent qualitative research has focused on the existence and uniqueness of solutions to use derivative of fractional order for studying system of nonlinear Cauchy problems, according to the authors [27].

It is also very important to note that delay-type problems have received a lot of attention from researchers working in fractional calculus. This is a significant subfield in the context of this sector. Many scholars published large amount of research devoted to study solutions to delay-type problems (see [5, 11, 30]). Additionally, a delay type problems have numerous applications in investigation of epidemiological problems of infectious diseases. As we have already said, models utilizing ordinary fractional order derivatives are subject to the following limitations: need a lot of storage space and result in inefficient operation. As a result of this, researchers have developed a short-memory fractional order modeling process in addition to short-memory fractional derivatives in order to overcome the problem. You can get qualitative analysis of delay differential equations in [6, 29]. Fractional differential is currently commonly employed to describe many complex physical and mechanical phenomena. When the fractional differential term is added to the integer-order dynamic system, the entire system will be altered. The fractional differential term is a useful tool for characterizing the physical transformation process with memory properties. These kinds of items, or dynamic features, are ubiquitous in the real system.

Inspired from the above discussion, we consider the given problem as

$$\begin{cases} {}_{0}^{PC}\mathbf{D}_{\theta}^{\delta}\mathbf{U}(\theta) = \Phi(\theta, \mathbf{U}(\theta), \mathbf{U}(\lambda\theta), \mathbf{H}\mathbf{U}(\theta)), \quad \theta \in [0, \mathbf{T}], \\ \mathbf{U}(0) = \mathbf{U}_0 + \displaystyle\int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi, \end{cases} \tag{1.1}$$

where $0 < \delta \leqslant 1$, $0 < \lambda < 1$, $\Phi \in C[I \times \mathbb{R}^3, \mathbb{R}]$, $\Psi \in C[I, \mathbb{R}]$ and ${}^{PC}{}_{0}\mathbf{D}_{\theta}^{\delta}$ denotes piece-wise derivative describe as:

$${}_{0}^{PC}\mathbf{D}_{\theta}^{\delta} = \begin{cases} \frac{d\mathbf{U}}{d\theta}, & \text{when } \theta \in [0, \theta_1], \\ {}_{0}^{C}\mathbf{D}_{\theta}^{\delta}, & \text{when } \theta \in (\theta_1, \mathbf{T}], \end{cases} \tag{1.2}$$

and $\mathbf{H}$ is a control function define as $\mathbf{H}\mathbf{U}(\theta) = \int_0^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\mathbf{U}(\xi)d\xi$. Here it should be kept in mind that a steady power supply and best performance, proportional delay involving insulators must be used to maintain continuous electrical conductivity between the overhead wires and the train. In our considered problem $\lambda \in (0, 1)$ makes the problem well posed but if $\lambda > 1$, then the problem (1.1) becomes ill-posed. Extending the nonlinear functional analysis tools, we can establish the mathematical analysis for such problems. Fixed point theories which have already described for various spaces can be here extended to deduce the results. Applications for integral boundary conditions can be found in many applied domains, including population dynamics, chemical engineering, subterranean water flow, and blood flow issues (see [32]).

To the best of our knowledge, the piecewise fractional order derivative with delay term has not been used to investigate the problem we have presented. Establishing the qualitative analysis for such problems is important from numerical and optimization point of views. Therefore, scientists are seeking to come up with numerical answers to such problems. The system needs to be stable before a numerical solution can be found. There are various types of stability, but the one that has been discussed here is very important for the numerical aspect of differential equations. There is a plethora of material related to the stability that was explored by [16, 17]. In order to address the problem of (1.1), we are particularly interested in developing results in existence theory and U-H-type stability analysis. U-H stability for a problem is studied around the best or exact solution of the problem. In order to get the results, we use fixed-point theory and tools of nonlinear functional analysis. For stability analysis we follow the methodology presented in [13, 25, 34]. Some examples will be given at the end to support our conclusion. The U-H stability is important from the optimization and numerical perspectives for the piecewise fractional order differentiations and proportional delay terms, where we need best approximate solution instead of exact or analytical solution. Because in such nonlinear problems prior we do not know the exact solution or impossible to find exact solution.

Here we state that differential equations with integral conditions and controllability terms have many applications in modeling various process of electromagnetic theory, fluid mechanics, and hydro dynamics [26]. Further problems with proportional delay terms are called pantographs equations which mainly are used in modeling the phenomenon related to electrodynamics (see details in [1]).

2. Preliminaries

In this section of our manuscript, we review some key findings from [3, 10, 27, 35]. Let, $\mathbf{J} = [0, \mathbf{T}]$, and $\Omega = C(\mathbf{J}, \mathbb{R})$, then for any $\mathbf{U} \in \Omega$ the supremum norm $\|\cdot\|$ on Ω is defined as

$$\|\mathbf{U}\| = \sup_{\theta \in \mathbf{J}} |\mathbf{U}(\theta)|.$$

Thus, Ω is a Banach space with the above norm defined on it. Also, we use $\mathbf{J}_1 = [0,\ \theta_1],\ \mathbf{J}_2 = (\theta_1, \mathbf{T}]$.

Definition 2.1. For a continuous function $\mathbf{U}$, the R-L integral of fractional order with $\delta \in (0, 1]$ is defined as:

$$^{RL}\mathrm{I}^{\delta}\mathbf{U}(\theta) = \int_0^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\mathbf{U}(\xi)d\xi, \quad \theta \in \mathbf{J}.$$

Definition 2.2. For a continuous function $\mathbf{U}$, the piecewise form of R-L fractional integral with $\delta \in (0, 1]$ is defined as

$$^{PRL}\mathrm{I}^{\delta}\mathbf{U}(\theta) = \begin{cases} \int_0^{\theta_1} \mathbf{U}(\xi)d\xi, & \text{if } \ \theta \in \mathbf{J}_1, \\ \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}\mathbf{U}(\xi)}{\Gamma(\delta)}d\xi, & \text{if } \ \theta \in \mathbf{J}_2. \end{cases}$$

Definition 2.3. The CFD with $\delta(0 < \delta \leqslant 1)$ is defined as follows

$$^{C}\mathbf{D}^{\delta}_{\theta_0}\mathbf{U}(\theta) = \int_0^{\theta} \frac{(\theta-\xi)^{\delta-1}\mathbf{U}'(\xi)}{\Gamma(\delta)}d\xi.$$

Definition 2.4. For a continuous function $\mathbf{U}$, the CFD in piecewise form is defined as

$$^{PC}\mathrm{D}^{\delta}\mathbf{U}(\theta) = \begin{cases} \frac{d\mathbf{U}}{d\theta}, & \text{if } \ \theta \in \mathbf{J}_1, \\ {}^{C}\mathrm{D}^{\delta}\mathbf{U}(\theta), & \text{if } \ \theta \in \mathbf{J}_2, \end{cases}$$

where $^{PC}\mathrm{D}^{\delta}$ represents classical derivative on $[0,\ \theta_1]$ and the CFD on $\mathbf{J}_2$.

Lemma 2.5. *For* $\mathbf{Y} \in \mathrm{L}[0, \mathbf{T}]$, *assume* $\mathbf{Y}(0) = 0$, *then solution of*

$$\begin{aligned} {}^{PC}\mathrm{D}^{\delta}\mathbf{U}(\theta) &= \mathbf{Y}(\theta), \ \textit{with} \ \ \delta \in (0, 1], \\ \mathbf{U}(0) &= \mathbf{U}_0, \end{aligned}$$

is given by

$$\mathbf{U}(\theta) = \begin{cases} \mathbf{U}_0 + \int_0^{\theta_1} \mathbf{Y}(\xi)d\xi, & \text{if } \ \theta \in \mathbf{J}_1, \\ \mathbf{U}(\theta_1) + \frac{1}{\Gamma(\delta)}\int_{\theta_1}^{\theta}(\theta-\xi)^{\delta-1}\mathbf{Y}(\xi)d\xi, & \text{if } \ \theta \in \mathbf{J}_2. \end{cases}$$

3. Existence theory

This section of our paper will examine the existence theory's consider problem (1.1). To achieve the intended outcomes, the idea of fixed-point theory will be used. First, the necessary conditions will be created for at least one possible solution to the issue. Then, a novel solution will be discussed using the Banach principle. The integral form of the issue (1.1) is given as follows using Lemma 2.5:

$$\begin{aligned} \mathbf{U}(\theta) &= \begin{cases} \mathbf{U}(0) + \int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_1, \\ \mathbf{U}(\theta_1) + \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_2, \end{cases} \\ &= \begin{cases} \mathbf{U}_0 + \int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi + \int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_1, \\ \mathbf{U}(\theta_1) + \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}\Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))}{\Gamma(\delta)}d\xi, & \text{when } \theta \in \mathbf{J}_2. \end{cases} \end{aligned}$$

Let Ω be the space of continuous functions, define operator $\mathbf{Z} : \Omega \to \Omega$ by:

$$\mathbf{Z}(\mathbf{U}) = \begin{cases} \mathbf{U}_0 + \displaystyle\int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi + \int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi, \\ \mathbf{U}(\theta_1) + \displaystyle\int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi. \end{cases} \tag{3.1}$$

Divide the above operator (3.1) into two sub operators as

$$\mathbf{Z_1}(\mathbf{U}) = \begin{cases} \mathbf{U}_0 + \int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_1, \\ \mathbf{U}(\theta_1), & \text{when } \theta \in \mathbf{J}_2, \end{cases}$$

and

$$\mathbf{Z_2}(\mathbf{U}) = \begin{cases} \int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{HU}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_1, \\ \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{HU}(\xi))d\xi, & \text{when } \theta \in \mathbf{J}_2. \end{cases}$$

For further analysis, we need the following assumptions.

(A_1) If there exist positive constants $\mathbb{B}$, $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$ such that:

$$\|\Phi(\theta, \mathbf{U}, \mathbf{V}, \mathbf{W})\| \leqslant \mathbb{B}_1\|\mathbf{U}\| + \mathbb{B}_2\|\mathbf{V}\| + \mathbb{B}_3\|\mathbf{W}\| + \mathbb{B}.$$

(A_2) If there exist a positive constant $\mathbb{G}$ such that:

$$\|\Psi(\mathbf{U}) - \Psi(\mathbf{V})\| \leqslant \mathbb{G}\|\mathbf{U} - \mathbf{V}\|.$$

Theorem 3.1. *Under the assumptions* (A_1)-(A_2), *our proposed problem* (1.1) *has at least one solution if* $\mathbb{G}' = \mathbb{G}\theta_1 < 1$.

Proof. To prove the above statement, we just need to show the operator $\mathbf{Z_1}$ is a contraction and operator $\mathbf{Z_2}$ is equi-continuous. For this define $\mathbf{D} = \{\mathbf{U} \in B : \|\mathbf{U}\| \leqslant r\}$ be closed, convex, and bounded subset of B. Clearly, $\mathbf{Z_1}$ is continuous. Let $\mathbf{U}_1, \mathbf{U}_2 \in \mathbf{D}$, then

$$\|\mathbb{Z}_1(\mathbf{U}_1) - \mathbb{Z}_1(\mathbf{U}_2)\| = \begin{cases} \mathbb{G}\theta_1\|\mathbf{U}_1 - \mathbf{U}_2\|, & \theta \in \mathbf{J}_1, \\ 0, & \theta \in \mathbf{J}_2, \end{cases} \leqslant \mathbb{G}'\|\mathbf{U}_1 - \mathbf{U}_2\|, \quad \text{where } \mathbb{G}' = \mathbb{G}\theta_1.$$

Hence, $\mathbf{Z_1}$ is a contraction. Now we show $\mathbf{Z_2}$ is equi-continuous. For this take, $\mathbf{U} \in \mathbf{D}$ and consider

$$\begin{aligned}
\|\mathbf{Z_2}(\mathbf{U})\| &= \sup_{\theta\in[0,\mathbf{T}]} \begin{cases} \left|\int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{HU}(\xi))d\xi\right|, & \text{when } \theta \in \mathbf{J}_1, \\ \left|\int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{HU}(\xi))d\xi\right|, & \text{when } \theta \in \mathbf{J}_2, \end{cases} \\
&\leqslant \sup_{\theta\in[0,\mathbf{T}]} \begin{cases} (\mathbb{B}_1|\mathbf{U}(\theta)| + \mathbb{B}_2|\mathbf{U}(\lambda\theta)| + \mathbb{B}_3|\mathbf{HU}(\theta)| + \mathbb{B})\,\theta_1, & \text{when } \theta \in \mathbf{J}_1, \\ \Big(\mathbb{B}_1|\mathbf{U}(\theta)| + \mathbb{B}_2|\mathbf{U}(\lambda\theta)| + \mathbb{B}_3|\mathbf{HU}(\theta)| + \mathbb{B}\Big)\frac{(\theta-\theta_1)^{\delta}}{\Gamma(\delta)}, & \text{when } \theta \in \mathbf{J}_2, \end{cases} \\
&\leqslant \begin{cases} \left(\theta_1\mathbb{B}_1 + \theta_1\mathbb{B}_2 + \frac{\theta_1\mathbf{T}^{\delta}}{\Gamma(\delta)}\mathbb{B}_3|\right)\|\mathbf{U}\| + \theta_1\mathbb{B}, & \text{when } \theta \in \mathbf{J}_1, \\ \left(\frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}\mathbb{B}_1 + \frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}\mathbb{B}_2 + \frac{(\mathbf{T}-\theta_1)^{\delta}\mathbf{T}^{\delta}}{[\Gamma(\delta)]^2}\mathbb{B}_3\right)\|\mathbf{U}\| + \mathbb{B}\frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}, & \text{when } \theta \in \mathbf{J}_2, \end{cases} \\
&\leqslant \begin{cases} \mathbb{B}^*\|\mathbf{U}\| + \mathbb{C}^*, & \text{when } \theta \in \mathbf{J}_1, \\ \mathbb{B}'\|\mathbf{U}\| + \mathbb{C}', & \text{when } \theta \in \mathbf{J}_2, \end{cases} \leqslant \mathbb{M}\|\mathbf{U}\| + \mathbb{N},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{B}^* &= \left(\theta_1\mathbb{B}_1 + \theta_1\mathbb{B}_2 + \frac{\theta_1\mathbf{T}^{\delta}}{\Gamma(\delta)}\mathbb{B}_3\right), & \mathbb{C}^* &= \theta_1\mathbb{B}, \\
\mathbb{B}' &= \left(\frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}\mathbb{B}_1 + \frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}\mathbb{B}_2 + \frac{(\mathbf{T}-\theta_1)^{\delta}\mathbf{T}^{\delta}}{[\Gamma(\delta)]^2}\mathbb{B}_3\right), & \mathbb{C}' &= \mathbb{B}\frac{(\mathbf{T}-\theta_1)^{\delta}}{\Gamma(\delta)}, \\
\mathbb{M} &= \max\{\mathbb{B}^*, \mathbb{B}'\}, & \mathbb{N} &= \max\{\mathbb{C}^*, \mathbb{C}'\}.
\end{aligned}$$

Thus, $\mathbf{Z_2}(\mathbf{U})$ is bounded. As Φ is continuous so is $\mathbf{Z_2}(\mathbf{U})$. Further, let $\xi_1 < \xi_2 \in \mathbf{J}$, then

$$\|\mathbf{Z_2}\mathbf{U}(\xi_2) - \mathbf{Z_2}\mathbf{U}(\xi_1)\|$$

$$
\begin{aligned}
&\leqslant \sup_{\{\xi_1,\xi_2\in\mathbf{J}\}} \begin{cases} \left|\int_0^{\xi_2}\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi)-\int_0^{\xi_1}\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi))\right|d\xi, & \text{when } \theta\in\mathbf{J}_1,\\ \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\left|\int_{\theta_1}^{\xi_2}\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi)-\int_{\theta_1}^{\xi_1}\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi))\right|d\xi, & \text{when } \theta\in\mathbf{J}_2,\end{cases}\\
&\leqslant \sup_{\{\xi_1,\xi_2\in\mathbf{J}\}} \begin{cases} \left|\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi)\right|(\xi_2-\xi_1),\\ \frac{1}{\Gamma(\delta)}\left|\Phi(\xi,\mathbf{U}(\xi),\mathbf{U}(\lambda\xi),\mathbf{HU}(\xi)\right|[(\xi_2-\theta_1)^{\delta}-(\xi_1-\theta_1)^{\delta}],\end{cases}\\
&\leqslant \sup_{\{\xi_1,\xi_2\in\mathbf{J}\}} \begin{cases} (\mathbb{B}_1|\mathbf{U}(\theta)|+\mathbb{B}_2|\mathbf{U}(\lambda\theta)|+\mathbb{B}_3|\mathbf{HU}(\theta)|+\mathbb{B})\,(\xi_2-\xi_1),\\ \frac{1}{\Gamma(\delta)}\left(\mathbb{B}_1|\mathbf{U}(\theta)|+\mathbb{B}_2|\mathbf{U}(\lambda\theta)|+\mathbb{B}_3|\mathbf{HU}(\theta)|+\mathbb{B}\right)[(\xi_2-\theta_1)^{\delta}-(\xi_1-\theta_1)^{\delta}],\end{cases}\\
&\leqslant \begin{cases} \mathbb{M}^*(\xi_2-\xi_1),\\ \mathbb{N}^*[(\xi_2-\theta_1)^{\delta}-(\xi_1-\theta_1)^{\delta}],\end{cases}\\
&\to 0 \ \text{ as } \ \xi_1\to\xi_2,
\end{aligned}
$$

here,

$$\mathbb{M}^*=\left(\left[\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3\frac{\mathbf{T}^{\delta}}{\Gamma(\delta)}\right]r+\mathbb{B}\right)$$

and

$$\mathbb{N}^*=\frac{1}{\Gamma(\delta)}\left(\left[\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3\frac{\mathbf{T}^{\delta}}{\Gamma(\delta)}\right]r+\mathbb{B}\right).$$

As a result, $\mathbf{Z}_2$ is bounded and $\mathbf{Z}_2$ is equi-continuous. Hence, we conclude that the proposed problem has at least one solution. □

Assume the following assumption.

(A_3) Let positive values $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$ exist, such that

$$\|\Phi(\theta_1,\mathbf{U}_1,\mathbf{V}_1,\mathbf{W}_1)-\Phi(\theta_2,\mathbf{U}_2,\mathbf{V}_2,\mathbf{W}_2)\|\leqslant\mathbb{C}_1\|\mathbf{U}_1-\mathbf{U}_2\|+\mathbb{C}_2\|\mathbf{V}_1-\mathbf{V}_2\|+\mathbb{C}_3\|\mathbf{W}_1-\mathbf{W}_2\|.$$

Theorem 3.2. *Let* (A_2)-(A_3) *hold and* $\mathbb{D}^*=\max\{\mathbb{D}_1,\mathbb{D}_2\}<1$, *where* $\mathbb{D}_1=\left[\mathbb{G}+\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3\frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right]\theta_1$ *and* $\mathbb{D}_2=\left[\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3\frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right]\frac{(\mathbf{T}-\theta_1)^{\delta}}{\delta\Gamma(\delta)}$, *then proposed problem has a unique solution.*

Proof.

$$
\begin{aligned}
&|\mathbf{Z}(\mathbf{U}_2)-\mathbf{Z}(\mathbf{U}_1)|\\
&=\begin{cases} \int_0^{\mathbf{T}}|\Psi(\mathbf{U}_2(\xi))-\Psi(\mathbf{U}_1(\xi)|d\xi+\int_0^{\theta_1}\left|\Phi(\xi,\mathbf{U}_2(\xi),\mathbf{U}_2(\lambda\xi),\mathbf{HU}_2(\xi))-\Phi(\xi,\mathbf{U}_1(\xi),\mathbf{U}_1(\lambda\xi),\mathbf{HU}_1(\xi))\right|d\xi, & \text{when } \theta\in\mathbf{J}_1,\\ \int_{\theta_1}^{\theta}\frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)}\left|\Phi(\xi,\mathbf{U}_2(\xi),\mathbf{U}_2(\lambda\xi),\mathbf{HU}_2(\xi))-\Phi(\xi,\mathbf{U}_1(\xi),\mathbf{U}_1(\lambda\xi),\mathbf{HU}_1(\xi))\right|d\xi, & \text{when } \theta\in\mathbf{J}_2,\end{cases}\\
&\leqslant\begin{cases} \left[\mathbb{G}+\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3\frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right]\theta_1\|\mathbf{U}_2-\mathbf{U}_1\|,\\ \left[\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3\frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right]\frac{(\mathbf{T}-\theta_1)^{\delta}}{\delta\Gamma(\delta)}\|\mathbf{U}_2-\mathbf{U}_1\|,\end{cases}\\
&\leqslant\begin{cases} \mathbb{D}_1\|\mathbf{U}_2-\mathbf{U}_1\|,\\ \mathbb{D}_2\|\mathbf{U}_2-\mathbf{U}_1\|,\end{cases}\\
&\leqslant\mathbb{D}^*\|\mathbf{U}_2-\mathbf{U}_1\|.
\end{aligned}
$$

□

4. U-H Stability

We shall create the necessary circumstances for U-H type stability in this part. This kind of stability is crucial for a problem's numerical solution. The fundamental query regarding an approximation is "under what circumstances the approximation of a problem is nearly equal to the exact solution?" Ulam provided a comprehensive response to this crucial subject in 1940's (see [34]). Hyers and Rassiass expanded on the previously mentioned stability and generalized it to become generalized U-H, U-H-Rassias stability (see [13, 25]). You may read the definition and a discussion of the asserted stability in [16, 17]. Consider the corresponding perturb problem of (1.2) as

$$\begin{cases} {}_0^{PC}\mathbf{D}_\theta^\delta \mathbf{U}(\theta) = \Phi(\theta, \mathbf{U}(\theta), \mathbf{U}(\lambda\theta), \mathbf{H}\mathbf{U}(\theta)) + \varphi(\theta), \quad \theta \in I = [0, \mathbf{T}], \\ \mathbf{U}(0) = \mathbf{U}_0 + \int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi. \end{cases} \tag{4.1}$$

Here, $\varphi(\theta) \in C([\theta_0, \mathbf{T}], R)$ with property $|\varphi(\theta)| \leqslant \epsilon, \quad \text{for} \quad \epsilon > 0$. Also for any solution, $\mathbf{U} \in B$ of perturb problem (4.1) such that,

$$\mathbf{Z}\mathbf{U}(\theta) = \mathbf{U}(\theta) + \varphi(\theta).$$

The solution of above perturb problem is given by:

$$\mathbf{U}(\theta) = \begin{cases} \mathbf{U}_0 + \int_0^{\mathbf{T}} \Psi(\mathbf{U}(\xi))d\xi + \int_0^{\theta_1} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi + \int_0^{\theta_1} \varphi(\xi)d\xi, & \text{when } \theta \in \mathbf{J}_1, \\ \mathbf{U}(\theta_1) + \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)} \Phi(\xi, \mathbf{U}(\xi), \mathbf{U}(\lambda\xi), \mathbf{H}\mathbf{U}(\xi))d\xi + \int_{\theta_1}^{\theta} \frac{(\theta-\xi)^{\delta-1}}{\Gamma(\delta)} \varphi(\xi)d\xi, & \text{when } \theta \in \mathbf{J}_2. \end{cases} \tag{4.2}$$

From (4.2) and (3.1), we can write

$$\|\mathbf{Z}\mathbf{U}(\theta) - \mathbf{U}(\theta)\| \leqslant \begin{cases} \epsilon.\theta_1, \\ \frac{\epsilon.(\mathbf{T}-\theta_1)^\delta}{\delta\Gamma(\delta)}, \end{cases} \leqslant \Theta.\epsilon, \quad \text{where} \quad \Theta = \max\left\{\epsilon.\theta_1, \frac{\epsilon.(\mathbf{T}-\theta_1)^\delta}{\delta\Gamma(\delta)}\right\}.$$

Theorem 4.1. *The solution of system* (1.2) *is U-H and generalized U-H stable if* $\mathbf{D}^* < 1$.

Proof. Let $\bar{\mathbf{U}}(\theta) \in B$ be any solution of problem (4.1) and $\mathbf{U} \in \mathbf{X}$ represent unique solution to problem (1.2), then

$$|\mathbf{U}(\theta) - \bar{\mathbf{U}}(\theta)| = |\mathbf{U}(\theta) - \mathbf{Z}\bar{\mathbf{U}}(\theta)| \leqslant |\mathbf{U}(\theta) - \mathbf{Z}\mathbf{U}(\theta)| + |\mathbf{Z}\mathbf{U}(\theta) - \mathbf{Z}\bar{\mathbf{U}}(\theta)| \leqslant \Theta\epsilon + \mathbf{D}^*|\mathbf{U}(\theta) - \bar{\mathbf{U}}(\theta)| \leqslant \frac{\Theta\epsilon}{1-\mathbf{D}^*}.$$

Hence, problem (1.2) is U-H and generalized U-H type stable. On the same way the remaining types of the stated stability can be studied. □

5. Illustrative examples

Here we demonstrate our results by some examples.

Example 5.1.

$${}_0^{PC}\mathbf{D}_\theta^{\frac{1}{2}} \mathbf{U}(\theta) = \beta\mathbf{U}(\theta) + \gamma\mathbf{U}(\lambda\theta) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\xi)^{-\frac{1}{2}}\mathbf{U}(\xi)d\xi, \quad \mathbf{U}(0) = \mathbf{U}_0 + \int_0^1 \mu\mathbf{U}(\xi)d\xi. \tag{5.1}$$

From (5.1) we see that

$$\Phi(\theta, \mathbf{U}(\theta), \mathbf{U}(\lambda\theta), \mathbf{H}\mathbf{U}(\theta)) = \beta\mathbf{U}(\theta) + \gamma\mathbf{U}(\lambda\theta) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\xi)^{-\frac{1}{2}}\mathbf{U}(\xi)d\xi$$

and

$$\Psi(\mathbf{U}(\theta)) = \mu\mathbf{U}(\theta).$$

Now, $\|\Phi\| \leqslant \beta\|\mathbf{U}\| + \gamma\|\mathbf{U}\| + \frac{2}{\Gamma(\frac{1}{2})}\|\mathbf{U}\| + 0$, hence Φ satisfied assumption (A_1) with $\mathbb{B} = 0$, $\mathbb{B}_1 = \beta$, $\mathbb{B}_2 = \gamma$, and $\mathbb{B}_3 = \frac{2}{\Gamma(\frac{1}{2})}$. Also, assumption ($A_2$) is hold since $\|\Psi(\mathbf{U}) - \Psi(\mathbf{V})\| \leqslant \mu\|\mathbf{U} - \mathbf{V}\|$, and finally:

$$\|\Phi(\theta_1, \mathbf{U}_1, \mathbf{V}_1, \mathbf{W}_1) - \Phi(\theta_2, \mathbf{U}_2, \mathbf{V}_2, \mathbf{W}_2)\| \leqslant \beta\|\mathbf{U}_1 - \mathbf{U}_2\| + \gamma\|\mathbf{U}_1 - \mathbf{U}_2\| + \frac{2}{\Gamma(\frac{1}{2})}\|\mathbf{U}_1 - \mathbf{U}_2\|.$$

This means that assumption (A_3) holds for $\mathbb{C}_1 = \beta$, $\mathbb{C}_2 = \gamma$, and $\mathbb{C}_3 = \frac{2}{\Gamma(\frac{1}{2})}$. For multiple choices of parameters "β, γ, and μ" the statements of Theorems 3.1, 3.2, and 4.1 obviously hold.

The considerations, also known as pantographs problems, are increasingly used to model various processes in electrodynamics. Here it should be kept in mind that pantographs require direct contact with overhead power lines to generate electricity for the running trains. Because of the train's speed and the outside weather, their usage environment is always changing. For details theory and applications, we refer to [1].

Example 5.2. Consider a pantograph type problem involves trigonometric functions as

$$\begin{cases} {}^{PC}_{\ 0}\mathbf{D}^{0.9}_{\theta}\mathbf{U}(\theta) = \frac{1}{10}\mathbf{U}(\theta) + \sin\left(\frac{\mathbf{U}(\lambda\theta)}{5}\right) + \frac{1}{50\Gamma(0.9)}\int_0^1 (1-\xi)^{-0.1}\cos(\mathbf{U}(\xi))d\xi, \\ \mathbf{U}(0) = \mathbf{U}_0 + \frac{1}{10}\int_0^1 \mathbf{U}(\xi)d\xi. \end{cases} \tag{5.2}$$

From (5.2), we see that

$$\Phi\left(\theta, \mathbf{U}(\theta), \mathbf{U}(\lambda\theta), \mathbf{H}\mathbf{U}(\theta)\right) = \frac{1}{10}\mathbf{U}(\theta) + \sin\left(\frac{\mathbf{U}(\lambda\theta)}{5}\right) + \frac{1}{50\Gamma(0.9)}\int_0^1 (1-\xi)^{-0.1}\cos(\mathbf{U}(\xi))d\xi,$$

and

$$\Psi(\mathbf{U}(\theta)) = \frac{\mathbf{U}(\theta)}{10}.$$

Consider

$$\begin{aligned} \|\Phi\| &= \sup_{\theta\in[0,1]}\left\{\left|\frac{1}{10}\mathbf{U}(\theta) + \sin\left(\frac{\mathbf{U}(\lambda\theta)}{5}\right) + \frac{1}{50\Gamma(0.9)}\int_0^1 (1-\xi)^{-0.1}\cos(\mathbf{U}(\xi))d\xi\right|\right\}, \\ &\leqslant \frac{1}{10}\|\mathbf{U}\| + \frac{1}{5}\|\mathbf{U}\| + \frac{1}{50\Gamma(0.9)}\|\mathbf{U}\| + 0 \end{aligned}$$

and

$$\|\Psi(\mathbf{U}) - \Psi(\mathbf{V})\| = \sup_{\theta\in[0,1]}\left\{\frac{1}{10}|\mathbf{U} - \mathbf{V}|\right\} \leqslant \frac{1}{10}\|\mathbf{U} - \mathbf{V}\|.$$

Thus, Φ Satisfies assumption (A_1) with constants $\mathbb{B} = 0$, $\mathbb{B}_1 = \frac{1}{10}$, $\mathbb{B}_2 = \frac{1}{5}$, and $\mathbb{B}_3 = \frac{1}{50\Gamma(0.9)}$. Also, Ψ satisfies assumption (A_2) with constant $\mathbb{G} = \frac{1}{10} = \mathbb{G}'$, hence all conditions of Theorem 3.1 hold. Consequently problem (5.2) has at least one solution. For unique solution and stability consider:

$$|\Phi\left(\theta, \mathbf{U}(\theta), \mathbf{U}(\lambda\theta), \mathbf{H}\mathbf{U}(\theta)\right) - \Phi\left(\theta, \mathbf{V}(\theta), \mathbf{V}(\lambda\theta), \mathbf{H}\mathbf{V}(\theta)\right)| \leqslant \frac{1}{10}\|\mathbf{U} - \mathbf{V}\| + \frac{1}{5}\|\mathbf{U} - \mathbf{V}\| + \frac{1}{50\Gamma(0.9)}\|\mathbf{U} - \mathbf{V}\|.$$

From above inequality assumption (A_3) holds with constants $\mathbb{C}_1 = \frac{1}{10}$, $\mathbb{C}_2 = \frac{1}{5}$, and $\mathbb{C}_3 = \frac{1}{50\Gamma(0.9)}$. Take $\theta_1 = 0.1$, then

$$\mathbb{D}_1 = \left[\mathbb{G} + \mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3\frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right]\theta_1 = \left[\frac{1}{10} + \frac{1}{10} + \frac{1}{5} + \frac{1}{50\Gamma(0.9)} \times \frac{(0.1)^{0.9}}{0.9\Gamma(0.9)}\right]0.1 = 0.04028034,$$

$$\mathbb{D}_2 = \left[\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3 \frac{\theta_1^{\delta}}{\delta\Gamma(\delta)}\right] \frac{(\mathbf{T} - \theta_1)^{\delta}}{\delta\Gamma(\delta)} = \left[\frac{1}{10} + \frac{1}{5} + \frac{1}{50\Gamma(0.9)} \times \frac{0.1^{0.9}}{0.9\Gamma(0.9)}\right] \frac{0.9^{0.9}}{0.9\Gamma(0.9)} = 0.26552557.$$

As, $\mathbb{D}^* = \max\{\mathbb{D}_1, \mathbb{D}_2\} = 0.26552557 < 1$, thus all the conditions of Theorems 3.2 and 4.1 hold. Consequently, problem (5.2) has a unique solution and Ulam-Hyer's type stable.

6. Conclusion and discussion

Recently problems under piecewise fractional order derivatives have been increasingly studied for numerical purposes. But the area devoted to establish mathematical analysis for such problems is very rarely considered. Also, problems with integral boundary conditions under piecewise derivatives of fractional orders have been considered very rarely in recent literature. Therefore, it is important to investigate the aforesaid area from different aspects of analysis including the existence theory and stability analysis. Because the mentioned analysis is essential for optimizations and numerical analysis of those problems involving piecewise fractional order differentiations and integrations. Therefore, we have considered a class of nonlinear differential equations with piecewise derivative of fractional order and proportional type delay terms. For the sake of existence theory and U-H type stability, we thoroughly investigated a generalized fractional differential equation within the context of this study. Regarding the suggested model, both proportional delay and controllability terms have been investigated. In the end, concrete examples were used to support the results. In the future, the above analysis we have developed can be extended to more complex and generalized systems with various kinds of delay terms and different kinds of piecewise fractional order derivatives. Also we can extend for multiple delay problems like housefly model where bi delay terms involved etc. We can study by the said analysis. We pointed out following.

- Dealing such problems it should be kept in mind the portioned delay terms must lies $\lambda \in (0, 1)$ and never greater than unity because if $\lambda > 1$, the problem becomes ill posed. Also for the controllability term the control function should be specified and bounded.
- The dependency on nonlinearity dictates the complexity; more complex nonlinearities require additional assumptions and hypotheses for analysis.
- The proportional delay problems also called pantograph problems have significant applications in electric locomotive and book land problems.
- Here the time domain we have divided into two interval and we have described the solutions in two branches, where separately for each Branch we have deduced the existence theory using fixed point approach. The complexities are those dealing two branch problems with fractional order integrals terms. We have stated some growth assumptions through which we reduced the high nonlinearity and in this way we obtained sufficient conditions for the existence theory.
- Here we have divided time domain into two pieces, and if divide in multiple sub intervals more interesting behaviors of dynamics are possible to appear.
- If we consider some practical problems like logistic equation with proportional delay, this analysis can be implemented there easily.
- If the value of contraction parameter $\mathbf{D}^*$ is less than unity, system achieves stability otherwise will be unstable in U-H sense. Hence the Lipchitz constant plays key role in this regards.

Acknowledgment

Authors are grateful to Prince Sultan University for APC and support via the TAS research lab.

References

[1] S. M. Ali, *On Pantograph Problems Involving Weighted Caputo Fractional Operators with Respect to Another Function*, Fractal Fract., **7** (2023), 1–14. 1, 5

[2] Y. Alruwaily, L. Almaghamsi, K. Karthikeyan, E.-S. El-hady, *Existence and uniqueness for a coupled system of fractional equations involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions*, AIMS Math., **8** (2023), 10067–10094. 1

[3] A. Atangana, S. İ. Araz, *New concept in calculus: Piecewise differential and integral operators*, Chaos Solitons Fractals, **145** (2021), 11 pages. 1, 2

[4] M. Awadalla, J. Alahmadi, K. R. Cheneke, S. Qureshi, *Fractional Optimal Control Model and Bifurcation Analysis of Human Syncytial Respiratory Virus Transmission Dynamics*, Fractal Fract., **8** (2024), 1–19. 1

[5] B. Balachandran, T. Kalmár-Nagy, D. E. Gilsinn, *Delay differential equations*, Springer New York, (2009). 1

[6] K. Balachandran, S. Kiruthika, J. J. Trujillo, *Existence of solutions of nonlinear fractional pantograph equations*, Acta Math. Sci. Ser. B (Engl. Ed.), **33** (2013), 712–720. 1

[7] S. Bushnaq, T. Saeed, D. F. M. Torres, A. Zeb, *Control of COVID-19 dynamics through a fractional-order model*, Alex. Eng. J., **60** (2021), 3587–3592. 1

[8] M. Caputo, M. Fabrizio, *On the notion of fractional derivative and applications to the hysteresis phenomena*, Meccanica, **52** (2017), 3043–3052. 1

[9] M. Dalir, M. Bashour, *Applications of fractional calculus*, Appl. Math. Sci., **4** (2010), 1021–1032. 1

[10] E. C. de Oliveira, J. A. Tenreiro Machado, *A review of definitions for fractional derivatives and integral*, Math. Probl. Eng., **2014** (2014), 6 pages. 1, 2

[11] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, Kluwer Academic Publishers Group, Dordrecht, (1992). 1

[12] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co., River Edge, (2000). 1

[13] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224. 1, 4

[14] M. D. Johansyah, A. Sambas, S. Qureshi, S. Zheng, T. M. Abed-Elhameed, S. Vaidyanathan, I. M. Sulaiman, *Investigation of the hyperchaos and control in the fractional order financial system with profit margin*, Partial Differ. Equ. Appl. Math., **9** (2024), 1–15. 1

[15] M. Khan, Z. Ahmad, F. Ali, N. Khan, I. Khan, K. S. Nisar, *Dynamics of two-step reversible enzymatic reaction under fractional derivative with Mittag-Leffler Kernel*, Plos One, **18** (2023). 1

[16] H. Khan, S. Ahmed, J. Alzabut, A. T. Azar, *A generalized coupled system of fractional differential equations with application to finite time sliding mode control for Leukemia therapy*, Chaos Solitons Fractals, **174** (2023), 12 pages. 1, 4

[17] A. Khan, J. F. Gómez-Aguilar, T. Saeed Khan, H. Khan, *Stability analysis and numerical solutions of fractional order HIV/AIDS model*, Chaos Solitons Fractals, **122** (2019), 119–128. 1, 4

[18] S. Kumar, R. Kumar, R. P. Agarwal, B. Samet, *A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods*, Math. Methods Appl. Sci., **43** (2020), 5564–5578. 1

[19] K. J. Latawiec, R. Stanislawsk, M. Lukaniszyn, W. Czuczwara, M. Rydel, *Fractional-order modeling of electric circuits: modern empiricism vas classical science*, In: 2017 Progress in Applied Electrical Engineering (PAEE), IEEE, (2017), 1–4. 1

[20] F. Liu, K. Burrage, *Novel techniques in parameter estimation for fractional dynamical models arising from biological systems*, Comput. Math. Appl., **62** (2011), 822–833. 1

[21] B. Maayah, S. Bushnaq, S. Momani, O. Abu Arqub, *Iterative multistep reproducing kernel Hilbert space method for solving strongly nonlinear oscillators*, Adv. Math. Phys., **2014** (2014), 7 pages. 1

[22] M. Magdziarz, A. Weron, K. Burnecki, J. Klafter, *Fractional Brownian motion versus the continuous-time random walk: A simple test for subdiffusive dynamics*, Phy. Rev. Lett., **103** (2009), 4 pages. 1

[23] K. B. Oldham, *Fractional differential equations in electrochemistry*, Adv. Eng. Softw., **41** (2010), 9–12. 1

[24] A. Padder, L. Almutairi, S. Qureshi, A. Soomro, A. Afroz, E. Hincal, A.Tassaddiq, *Dynamical Analysis of Generalized Tumor Model with Caputo Fractional-Order Derivative*, Fractal Fract., **7** (2023), 19 pages. 1

[25] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1, 4

[26] V. Rokhlin, *Rapid solution of integral equations of classical potential theory*, J. Comput. Phys., **60** (1985), 187–207. 1

[27] K. Shah, T. Abdeljawad, A. Ali, *Mathematical analysis of the Cauchy type dynamical system under piecewise equations with Caputo fractional derivative*, Chaos Solitons Fractals, **161** (2022), 8 pages. 1, 2

[28] W. Shatanawi, A. Boutiara, M. S. Abdo, M. B. Jeelani, K. Abodayeh, *Nonlocal and multiple-point fractional boundary value problem in the frame of a generalized Hilfer derivative*, Adv. Difference Equ., **2021** (2021), 19 pages. 1

[29] M. Sher, K. Shah, J. Rassias, *On qualitative theory of fractional order delay evolution equation via the prior estimate method*, Math. Methods Appl. Sci., **43** (2020), 6464–6475. 1

[30] H. Smith, *An introduction to delay differential equations with applications to the life sciences*, Springer, New York, (2011). 1

[31] A. Sohail, Z. Yu, R. Arif, A. Nutini, T. A. Nofal, *Piecewise differentiation of the fractional order CAR-T cells-SARS-2 virus model*, Results Phys., **33** (2022), 1–7. 1

[32] S. Song, Y. Cui, *Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance*, Bound. Value Probl., **2020** (2020), 12 pages. 1

[33] A. Tassaddiq, S. Qureshi, A. Soomro, O. Abu Arqub, M. Senol, *Comparative analysis of classical and Caputo models for COVID-19 spread: vaccination and stability assessment*, Fixed Point Theory Algorithms Sci. Eng., **2024** (2024), 23 pages. 1

[34] S. M. Ulam, *Problems in modern mathematics*, Courier Corporation, (2004). 1, 4

[35] A. Zeb, A. Atangana, Z. A. Khan, S. Djillali, *A robust study of a piecewise fractional order COVID-19 mathematical model*, Alex. Eng. J., **61** (2022), 5649–5665. 1, 2

[36] T. Zhang, Y. Li, *Global exponential stability of discrete-time almost automorphic Caputo-Fabrizio BAM fuzzy neural networks via exponential Euler technique*, Knowl.-Based Syst., **246** (2022). 1