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Fixed and common fixed point theorems in partially ordered quasi-metric spaces

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Abstract

In this paper, we prove some new fixed and common fixed point results in the framework of partially ordered quasi-metric spaces under linear and nonlinear contractions. Also we obtain some fixed point results in the framework of G-metric spaces.

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1. Introduction and preliminaries

The fixed point theory is considered as a basic and very simple mathematical setting, since it has some applications in many interesting fields such as differential equations, economics and engineering. The existence of a fixed point is a pivotal property of a function. Many necessary or sufficient conditions for the presence of such points are considered in many areas in mathematics.

The Banach contraction theorem [4] is considered as a fundamental theorem concerning fixed point theorem in a complete metric space which is appeared in 1922 and rise for its elegant and simple proof which it is known later as *Banach contraction principle*. Subsequently, a large number

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of generalizations of Banach contraction principle were obtained, for example in 2008, Agarwal et al. [1] introduced and proved the following theorem.

Theorem 1.1 ([1, Theorem 2.3]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Assume $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function with $\psi(t) < t$ for each $t > 0$. Moreover, suppose that f is a nondecreasing mapping satisfying the following form*

$$d(f(x), f(y)) \leq \psi(\max\{fd(x, y), d(x, f(x)), d(y, f(y))\})$$

for all $x \geq y$. Also assume either f is continuous or if $(x_n) \subseteq X$ is a nondecreasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n holds. If there exists an $x_0 \in X$ with $x_0 \leq f(x_0)$ then f has a fixed point.

The concept of quasi-metric spaces was generated by Wilson [19] in 1931 as the following:

Definition 1.2. Let X be a non empty set and $d : X \times X \to [0, \infty)$ be a given function which satisfies

(1) $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) \leq d(x, z) + d(z, y)$ for any points $x, y, z \in X$.

Then d is called a quasi metric on X and the pair (X, d) is called a quasi metric space.

It is clear that every metric space is a quasi metric space, but the reverse is not necessarily true.

Jleli and Samet [5] and Samet et al. [16] utilized the notion of quasi-metric space to obtain some fixed point theorems. In their interesting papers, they pointed out that some fixed point results in G-metric space in sense of Mustafa and Sims [14] can be obtained from quasi-metric space. Agarwal et. al [2] showed that many fixed point theorems in G-metric spaces can be derived from known existing results if all arguments are not distinct. For some results in G-metric space, we refer the reader to [7–18].

The convergence and completeness in a quasi-metric space are defined as follows:

Definition 1.3 ([5]). Let (X, d) be a quasi-metric space, (x_n) be a sequence in X, and $x \in X$. Then the sequence (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

Definition 1.4 ([5]). Let (X, d) be a quasi-metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is left-Cauchy if for every $\epsilon > 0$ there is positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leq \epsilon$ for all $n \geq m > N$.

Definition 1.5 ([5]). Let (X, d) be a quasi-metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is right-Cauchy if for every $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leq \epsilon$ for all $m \geq n > N$.

Definition 1.6 ([5]). Let (X, d) be a quasi-metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is Cauchy if for every $\epsilon > 0$ there is positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leq \epsilon$ for all $m, n > N$.

Definition 1.7 ([5]). Let (X, d) be a quasi-metric space. We say that

(1) (X, d) is left-complete if and only if every left-Cauchy sequence in X is convergent;
(2) (X, d) is right-complete if and only if every right-Cauchy sequence in X is convergent;
(3) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Mustafa and Sims [14] introduced the notion of G-metric spaces as follows:

Definition 1.8 ([14])**.** Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,

(G4) $G(x, y, z) = G(p\{x, y, z\})$, where $p\{x, y, z\}$ is the all possible permutations of $\{x, y, z\}$ (symmetry),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)\ \forall\, x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a *generalized metric*, or more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.9 ([14])**.** Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X. Then we say that (x_n) is G-convergent to x if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq k$.

Proposition 1.10 ([14])**.** *Let (X, G) be a G-metric space. Then the following assertions are equivalent*

(1) *(x_n) is G-convergent to x;*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$;*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 1.11 ([14])**.** Let (X, G) be a G-metric space. A sequence (x_n) in X is said to be G-Cauchy if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$G(x_n, x_m, x_l) < \epsilon, \quad \forall n, m, l \geq k.$$

Proposition 1.12 ([14])**.** *In a G-metric space, the following are equivalent*

(1) *the sequence (x_n) is G-Cauchy;*

(2) *for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq k$.*

Definition 1.13 ([14])**.** A G-metric space (X, G) is said to be G-complete or complete G-metric space if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

The following theorem is a relation between G-metric spaces and quasi metric spaces.

Theorem 1.14 ([5])**.** *Let (X,G) be a G-metric space and let $d : X \times X \to [0, \infty)$ defined by $d(x, y) = G(x, y, y)$. Then*

(1) *(X, d) is a quasi metric space;*

(2) *$(x_n) \subset X$ is G-convergent to $x \in X$ iff (x_n) is convergent in (X, d);*

(3) *$(x_n) \subset X$ is G-Cauchy iff (x_n) is Cauchy in (X, d);*

(4) *$(x_n) \subset X$ is G-complete iff (x_n) is complete in (X, d).*

2. Main result

We start with the following definitions.

Definition 2.1 ([3])**.** Let $(X, \preceq)$ be a partially ordered set. Two mappings F, $G : X \to X$ are said to be weakly increasing if $Fx \preceq GFx$ and $Gx \preceq FGx$, for all $x \in X$.

Definition 2.2 ([18])**.** Let $(X, \preceq)$ be a partially ordered set and A, B be closed subsets of X with $X = A \cup B$. Let $f, g : X \to X$ be two mappings. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \preceq gfx$ for all $x \in A$ and $gx \preceq fgx$ for all $x \in B$.

Definition 2.3 ([6])**.** The function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied.
(1) ϕ is continuous and nondecreasing.
(2) $\phi(t) = 0$ if and only if $t = 0$.

Our main result in this section is the following theorem.

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let A,B be two nonempty closed subsets of X with respect to the topology induced by d with $X = A \cup B$ and $A \cap B \neq \phi$. Let $f, g : A \cup B \to A \cup B$ such that the pair (f,g) is (A,B)-weakly increasing with $f(A) \subseteq B$, $g(B) \subseteq A$. Let ϕ, ψ be altering distance functions. Moreover, suppose that*

$$\phi d(fx, gy) \leq \phi \max\{d(x, y), d(fx, x), d(gy, y)\} - \psi \max\{d(x, y), d(fx, x), d(gy, y)\} \tag{2.1}$$

for all comparative $x, y \in X$ with $x \in A$, $y \in B$, and

$$\phi d(gx, fy) \leq \phi \max\{d(x, y), d(gx, x), d(fy, y)\} - \psi \max\{d(x, y), d(gx, x), d(fy, y)\} \tag{2.2}$$

for all comparative $x, y \in X$ with $x \in B$, $y \in A$. Also,

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. From 1. there exists $x_0 \in A$ such that $x_0 \preceq fx_0$. Since $f(A) \subseteq B$, then $x_1 = fx_0 \in B$. Also, since $g(B) \subseteq A$, then $x_2 = gx_1 \in B$. By continuing this way, we construct a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$, $x_{2n} \in A$, $gx_{2n+1} = x_{2n+2}$ and $x_{2n+1} \in B$, $n \in \mathbb{N} \cup \{0\}$. Since (f, g) is (A, B)-weakly increasing, then $x_0 \preceq fx_0 = x_1 \preceq gfx_0 = gx_1 = x_2 \preceq fgx_1 = fx_2 = x_3 \cdots$. Thus $x_n \preceq x_{n+1}$ for all $n \geq 0$. If there exists some $k \in \mathbb{N}$ such that $x_{2k} = x_{2k+1}$, then x_{2k} is a fixed point for f in $A \cap B$. To show that x_{2k} is also a fixed point for g it is equivalent to show that $x_{2k} = x_{2k+1} = x_{2k+2}$. Since $x_{2k} \preceq x_{2k+1}$, then by (2.2) we have

$$\begin{aligned}
\phi d(x_{2k+2}, x_{2k+1}) &= \phi d(gx_{2k+1}, fx_{2k}) \\
&\leq \phi \max\{d(x_{2k+1}, x_{2k}), d(x_{2k+2}, x_{2k+1}), d(x_{2k+1}, x_{2k})\} \\
&\quad - \psi \max\{d(x_{2k+1}, x_{2k}), d(x_{2k+2}, x_{2k+1}), d(x_{2k+1}, x_{2k})\} \\
&\leq \phi d(x_{2k+2}, x_{2k+1}) - \psi d(x_{2k+2}, x_{2k+1}).
\end{aligned}$$

Therefore, $\psi d(x_{2k+2}, x_{2k+1}) = 0$, and so $d(x_{2k+2}, x_{2k+1}) = 0$. Hence $x_{2k+2} = x_{2k+1}$. Thus x_{2k} is a common fixed point for f and g in $A \cap B$.

Now, assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Let $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned}
\phi d(x_{n+1}, x_n) = \phi d(x_{2t+1}, x_{2t}) &= \phi d(fx_{2t}, gx_{2t-1}) \\
&\leq \phi \max\{d(x_{2t}, x_{2t-1}), d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\} \\
&\quad - \psi \max\{d(x_{2t}, x_{2t-1}), d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\} \\
&\leq \phi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\} \\
&\quad - \psi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\}.
\end{aligned}$$

If $\max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\} = d(x_{2t+1}, x_{2t})$, then

$$\phi d(x_{2t+1}, x_{2t}) \leq \phi d(x_{2t+1}, x_{2t}) - \psi d(x_{2t+1}, x_{2t}).$$

Therefore, $\psi d(x_{2t+1}, x_{2t}) = 0$ and so $d(x_{2t+1}, x_{2t}) = 0$. Thus $x_{2t+1} = x_{2t}$ is a contradiction. Hence $\max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t-1})\} = d(x_{2t}, x_{2t-1})$. Therefore

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \tag{2.3}$$

and

$$\phi d(x_{n+1}, x_n) \leq \phi d(x_n, x_{n-1}) - \psi d(x_n, x_{n-1}). \tag{2.4}$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}
\phi d(x_{n+1}, x_n) = \phi d(x_{2t+2}, x_{2t+1}) &= \phi d(gx_{2t+1}, fx_{2t}) \\
&\leq \phi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1}), d(x_{2t+1}, x_{2t})\} \\
&\quad - \psi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1}), d(x_{2t+1}, x_{2t})\} \\
&\leq \phi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1})\} \\
&\quad - \psi \max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1})\}.
\end{aligned}$$

If $\max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1})\} = d(x_{2t+2}, x_{2t+1})$, then $\phi d(x_{2t+2}, x_{2t+1}) \leq \phi d(x_{2t+2}, x_{2t+1}) - \psi d(x_{2t+2}, x_{2t+1})$. Therefore, $\psi d(x_{2t+2}, x_{2t+1}) = 0$, and so $d(x_{2t+2}, x_{2t+1}) = 0$. Thus $x_{2t+2} = x_{2t+1}$ is a contradiction. Hence $\max\{d(x_{2t+1}, x_{2t}), d(x_{2t+2}, x_{2t+1})\} = d(x_{2t+1}, x_{2t})$. Therefore,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \tag{2.5}$$

and

$$\phi d(x_{n+1}, x_n) \leq \phi d(x_n, x_{n-1}) - \psi d(x_n, x_{n-1}). \tag{2.6}$$

From (2.3) and (2.5), we have for all $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}). \tag{2.7}$$

Thus $(d(x_{n+1}, x_n) : n \in \mathbb{N})$ is a nonnegative decreasing sequence, so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Also, from (2.4) and (2.6), we have for all $n \in \mathbb{N}$

$$\phi d(x_{n+1}, x_n) \leq \phi d(x_n, x_{n-1}) - \psi d(x_n, x_{n-1}). \tag{2.8}$$

By taking the limit as $n \to \infty$ in (2.8), we get $\phi r \leq \phi r - \psi r$ which implies that $\psi r = 0$. Therefore, $r = 0$. Thus

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.9}$$

Again, let $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}\phi d(x_n, x_{n+1}) = \phi d(x_{2t}, x_{2t+1}) &= \phi d(gx_{2t-1}, fx_{2t}) \\ &\leq \phi \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1}), d(x_{2t+1}, x_{2t})\} \\ &\quad - \psi \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1}), d(x_{2t+1}, x_{2t})\}.\end{aligned}$$

From (2.7), we have $d(x_{2t}, x_{2t-1}) > d(x_{2t+1}, x_{2t})$. Thus

$$\begin{aligned}\phi d(x_{2t}, x_{2t+1}) &\leq \phi \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\} \\ &\quad - \psi \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\} \\ &\leq \phi \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\}.\end{aligned} \tag{2.10}$$

Since ϕ is an altering distance function, then

$$d(x_{2t}, x_{2t+1}) \leq \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\}. \tag{2.11}$$

From (2.7) we have

$$d(x_{2t+1}, x_{2t}) \leq d(x_{2t}, x_{2t-1}) \leq \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\}. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t+1})\} \leq \max\{d(x_{2t-1}, x_{2t}), d(x_{2t}, x_{2t-1})\}. \tag{2.13}$$

Similarly, we can show that

$$\max\{d(x_{2t+1}, x_{2t+2}), d(x_{2t+2}, x_{2t+1})\} \leq \max\{d(x_{2t+1}, x_{2t}), d(x_{2t}, x_{2t+1})\}. \tag{2.14}$$

From (2.13) and (2.14), we get that

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} \leq \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_n)\} \text{ holds for all n} \in \mathbb{N}.$$

So $(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\})$ is a nonnegative decreasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} = r.$$

From (2.9), we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

From (2.10), we get

$$\phi(r) \leq \phi(r) - \psi(r).$$

So $\psi(r) = 0$, and hence $r = 0$. Therefore, for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, our claim is to show that (x_n) is Cauchy sequence. To show that (x_n) is a Cauchy sequence it is sufficient to show that (x_{2n}) is a Cauchy sequence; that is (x_{2n}) is left-Cauchy and right-Cauchy. Suppose to the contrary that (x_{2n}) is not left-Cauchy. Then there is $\epsilon > 0$ and two subsequences (x_{2n_k}) and (x_{2m_k}) such that (x_{2n_k}) chosen to be the smallest index for which

$$d(x_{2n_k}, x_{2m_k}) \geq \epsilon \quad 2n_k > 2m_k > k. \tag{2.15}$$

This means that

$$d(x_{2n_k-2}, x_{2m_k}) < \epsilon.$$

From (2.15), we get

$$\begin{aligned}\epsilon \le d(x_{2n_k}, x_{2m_k}) &\le d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2m_k})\\ &\le d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2m_k}).\end{aligned}$$

Taking the limit as $k \to \infty$ and using (2.9), we conclude

$$\lim_{k\to\infty} d(x_{2n_k-1}, x_{2m_k}) = \epsilon. \tag{2.16}$$

Again, from (2.15), we obtain

$$\epsilon \le d(x_{2n_k}, x_{2m_k}) \le d(x_{2n_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2m_k}).$$

Taking the limit as $k \to \infty$ and using (2.9), we see that

$$\epsilon \le \lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}). \tag{2.17}$$

The contraction condition (2.2) yields

$$\begin{aligned}\phi d(x_{2n_k}, x_{2m_k+1}) &= \phi d(gx_{2n_k-1}, fx_{2m_k})\\ &\le \phi \max\{d(x_{2n_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k-1}), d(x_{2m_k+1}, x_{2m_k})\}\\ &\quad - \psi \max\{d(x_{2n_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k-1}), d(x_{2m_k+1}, x_{2m_k})\}.\end{aligned}$$

Taking the limit as $k \to \infty$ and using the continuity of ϕ, ψ and using (2.9),(2.16), and (2.17), we get

$$\phi\epsilon \le \phi \lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) \le \phi\epsilon - \psi\epsilon.$$

Therefore, $\psi\epsilon = 0$, and hence $\epsilon = 0$ which is 1a contradiction since $\epsilon > 0$. Hence (x_{2n}) is a left-Cauchy sequence. In a similar manner we can prove that (x_{2n}) is a right-Cauchy sequence.

Since (X, d) is a complete quasi metric space, then (x_n) converges to some element $u \in X$. Therefore any subsequence of (x_n) also converges to u. Thus the subsequences (x_{2n}) and (x_{2n+1}) also converge to u. Since (x_{2n}) is a sequence in A, A is a closed subset of X and $\lim_{n\to\infty} x_{2n} = u$, then $u \in A$. Also, since (x_{2n+1}) is a sequence in B, B is a closed subset of X and $\lim_{n\to\infty} x_{2n+1} = u$, then $u \in B$.

By using the continuity of f, we get

$$\lim_{n\to\infty} d(x_n, fu) = \lim_{n\to\infty} d(fx_{n-1}, fu) = 0 \quad\text{and}\quad \lim_{n\to\infty} d(fu, x_n) = \lim_{n\to\infty} d(fu, fx_{n-1}) = 0.$$

Hence

$$\lim_{n\to\infty} d(fu, x_n) = \lim_{n\to\infty} d(x_n, fu) = 0.$$

Thus (x_n) converges to fu. By uniqueness of the limit, we have $fu = u$. So u is a fixed point of f in $A \cap B$.

Now, since $u \preceq u$, then from (2.2), we get

$$\phi d(gu, u) = \phi d(gu, fu)$$

$$\leq \phi \max\{d(u,u), d(gu,u), d(fu,u)\} - \psi \max\{d(u,u), d(gu,u), d(fu,u)\}.$$

Thus $\phi d(gu,u) \leq \phi d(gu,u) - \psi d(gu,u)$. Hence $\psi d(gu,u) = 0$, and so $d(gu,u) = 0$. Therefore $gu = u$. Hence u is a common fixed point for f and g in $A \cap B$. □

Remark 2.5. The previous theorem is still correct if we choose the function $\psi : [0,\infty) \to [0,\infty)$ just as a continuous function.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let A,B be two nonempty closed subsets of X with respect to the topology induced by d with $X = A \cup B$ and $A \cap B \neq \phi$. Let $f : A \cup B \to A \cup B$ such that $fx \preceq f^2x$ for all $x \in X$ with $f(A) \subseteq B,\ f(B) \subseteq A$. Let ϕ, ψ be altering distance functions. Also suppose that*

$$\phi d(fx,fy) \leq \phi \max\{d(x,y), d(fx,x), d(fy,y)\} - \psi \max\{d(x,y), d(fx,x), d(fy,y)\}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$ or $x \in B,\ y \in A$. Also,

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. It follows from Theorem 2.4 by taking $g = f$. □

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let $f, g : X \to X$ such that the pair f and g are weakly increasing mappings. Let ϕ, ψ be an altering distance functions. Moreover, suppose that*

$$\phi d(fx,gy) \leq \phi \max\{d(x,y), d(fx,x), d(gy,y)\} - \psi \max\{d(x,y), d(fx,x), d(gy,y)\}$$

for all comparative $x, y \in X$, and

$$\phi d(gx,fy) \leq \phi \max\{d(x,y), d(gx,x), d(fy,y)\} - \psi \max\{d(x,y), d(gx,x), d(fy,y)\}$$

for all comparative $x, y \in X$. Also,

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. It follows from Theorem 2.4 by taking $A = B = X$. □

By replacing g by f and taking $A = B = X$ in Theorem 2.4 we get the following result.

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let $f : X \to X$ such that $fx \preceq f^2x$. Let ϕ, ψ be an altering distance functions. Moreover, suppose that*

$$\phi d(fx,fy) \leq \phi \max\{d(x,y), d(fx,x), d(fy,y)\} - \psi \max\{d(x,y), d(fx,x), d(fy,y)\}$$

for all comparative $x, y \in X$. Also,

1. *suppose that there exists* $x_0 \in A$ *such that* $x_0 \preceq fx_0$,
2. *if* f *or* g *is continuous.*

Then f *and* g *have a common fixed point in* $A \cap B$.

If we define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$ and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = (1-k)t,\ k \in [0,1)$, then we get the following result.

Theorem 2.9. *Let* $(X,\preceq)$ *be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let A,B be two nonempty closed subsets of X with respect to the topology induced by d with* $X = A \cup B$ *and* $A \cap B \neq \phi$. *Let* $f, g : A \cup B \to A \cup B$ *such that the pair (f,g) is (A,B)-weakly increasing with* $f(A) \subseteq B$, $g(B) \subseteq A$. *Suppose that*

$$d(fx, gy) \le k \max\{d(x,y), d(fx,x), d(gy,y)\}$$

for all comparative $x, y \in X$ *with* $x \in A,\ y \in B$, *and*

$$d(gx, fy) \le k \max\{d(x,y), d(gx,x), d(fy,y)\}$$

for all comparative $x, y \in X$ *with* $x \in B,\ y \in A$. *Also,*

1. *suppose that there exists* $x_0 \in A$ *such that* $x_0 \preceq fx_0$,
2. *if* f *or* g *is continuous.*

Then f *and* g *have a common fixed point in* $A \cap B$.

Corollary 2.10. *Let* $(X,\preceq)$ *be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let A,B be two nonempty closed subsets of X with respect to the topology induced by d with* $X = A \cup B$ *and* $A \cap B \neq \phi$. *Let* $f : A \cup B \to A \cup B$ *such that* $fx \preceq f^2x$ *forall* $x \in X$ *with* $f(A) \subseteq B,\ f(B) \subseteq A$. *Suppose that*

$$d(fx, fy) \le k \max\{d(x,y), d(fx,x), d(fy,y)\}$$

for all comparative $x, y \in X$ *with* $x \in A,\ y \in B$ *or* $x \in B,\ y \in A$. *Also,*

1. *suppose that there exists* $x_0 \in A$ *such that* $x_0 \preceq fx_0$,
2. *if* f *or* g *is continuous.*

Then f *and* g *have a common fixed point in* $A \cap B$.

Proof. The proof follows from Theorem 2.9 by taking $g = f$. □

Corollary 2.11. *Let* $(X,\preceq)$ *be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let* $f, g : X \to X$ *such that the pair f and g are weakly increasing. Suppose that*

$$d(fx, gy) \le k \max\{d(x,y), d(fx,x), d(gy,y)\}$$

for all comparative $x, y \in X$, *and*

$$d(gx, fy) \le k \max\{d(x,y), d(gx,x), d(fy,y)\}$$

for all comparative $x, y \in X$. *Also,*

1. *suppose that there exists* $x_0 \in A$ *such that* $x_0 \preceq fx_0$,

2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. It follows from Theorem 2.9 by taking $A = B = X$ □

If we take $g = f$ and $A = B = X$ in Theorem 2.9, then we get the following result.

Corollary 2.12. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X,d) is a complete quasi-metric space. Let $f : X \to X$ such that $fx \preceq f^2x\ \forall x \in X$. Suppose that*

$$d(fx, fy) \leq k \max\{d(x, y), d(fx, x), d(fy, y)\}$$

for all comparative $x, y \in X$, and

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

3. Common fixed point theorems in G-metric spaces

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X,G) is a complete G-metric space. Let A,B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$. Let $f, g : A \cup B \to A \cup B$ be two mappings such that the pair (f, g) is (A, B)-weakly increasing with $f(A) \subseteq B$, $g(B) \subseteq A$. Let ϕ and ψ be an altering distance functions. Moreover, suppose that*

$$\begin{aligned}\phi G(fx, gy, gy) \leq\ & \phi \max\{G(x, y, y), G(fx, x, x), G(gy, y, y)\} \\ & - \psi \max\{G(x, y, y), G(fx, x, x), G(gy, y, y)\}\end{aligned}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$, and

$$\begin{aligned}\phi G(gx, fy, fy) \leq\ & \phi \max\{G(x, y, y), G(gx, x, x), G(fy, y, y)\} \\ & - \psi \max\{G(x, y, y), G(gx, x, x), G(fy, y, y)\}\end{aligned}$$

for all comparative $x, y \in X$ with $x \in B,\ y \in A$. Also

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. Let $d : X \times X \to [0, \infty)$ defined by $d(x, y) = G(x, y, y)$ for all comparative $x, y \in X$ with $x \in A,\ y \in B$ and $d(y, x) = G(y, x, x)$ for all comparative $x, y \in X$ with $x \in A,\ y \in B$. Then by Theorem 1.14, (X, d) is a quasi metric space. From the contractive conditions we have

$$\phi d(fx, gy) \leq \phi \max\{d(x, y), d(fx, x), d(gy, y)\} - \psi \max\{d(x, y), d(fx, x), d(gy, y)\}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$, and

$$\phi d(gx, fy) \leq \phi \max\{d(x, y), d(gx, x), d(fy, y)\} - \psi \max\{d(x, y), d(gx, x), d(fy, y)\}$$

for all comparative $x, y \in X$ with $x \in B,\ y \in A$. By Theorem 2.4, f and g have a common fixed point in $A \cap B$. □

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X,G) is a complete G-metric space. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$. Let $f, g : A \cup B \to A \cup B$ be two mappings such that the pair (f, g) is (A, B)-weakly increasing with $f(A) \subseteq B$ and $g(B) \subseteq A$. Suppose that there exists $r \in [0, 1)$ such that*

$$G(fx, gy, gy) \le k \max\{G(x, y, y), G(fx, x, x), G(gy, y, y)\}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$, and

$$G(gx, fy, fy) \le k \max\{G(x, y, y), G(gx, x, x), G(fy, y, y)\}$$

for all comparative $x, y \in X$ with $x \in B,\ y \in A$. Also,

1. *suppose that there exists $x_0 \in A$ such that $x_0 \preceq fx_0$,*
2. *if f or g is continuous.*

Then f and g have a common fixed point in $A \cap B$.

Proof. As in the proof of Theorem 3.1, we consider the function $d : X \times X \to [0, \infty)$ such that $d(x, y) = G(x, y, y)$ for all comparative $x, y \in X$ with $x \in A,\ y \in B$ and $d(y, x) = G(y, x, x)$ for all comparative $x, y \in X$ with $x \in A,\ y \in B$. Then by Theorem 1.14, (X, d) is a quasi metric space. From the contractive conditions we have

$$d(fx, gy) \le k \max\{d(x, y), d(fx, x), d(gy, y)\}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$, and

$$d(gx, fy) \le k \max\{d(x, y), d(gx, x), d(fy, y)\}$$

for all comparative $x, y \in X$ with $x \in B,\ y \in A$. By Theorem 2.9, f and g have a common fixed point in $A \cap B$. □

Remark 3.3. We can prove Theorem 3.2 from Theorem 3.1 by choosing $\phi t = t$ and $\psi t = (1 - k)t$, where $0 \le k < 1$.

Next, we introduce an example to support our result.

Example 3.4. Let $X = \{0, 1, 2, 3, \cdots\}$ and define a relation $\preceq$ on X by $a, b \in X,\ a \preceq b$ iff $a - b \ge 0$ and let A and B be two subsets of X such that $A = \{0, 2, 4, 6, \cdots\},\ B = \{0, 1, 3, 5, \cdots\}$.

Define $d : X \times X \to [0, \infty)$ by $d(x, y) = \begin{cases} 0, & x = y; \\ x + 2y, & x \ne y. \end{cases}$

Let $f, g : A \cup B \to A \cup B$ be defined by $fx = \begin{cases} 0, & x = 0, 1, 2; \\ x - 3, & x \ge 3. \end{cases}$ $\quad gx = \begin{cases} 0, & x = 0, 1; \\ x - 1, & x \ge 2. \end{cases}$

Also, define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\phi t = t^2,\ \psi t = t$. Then

(1) $(X, d, \preceq)$ is a partially ordered complete quasi metric space;
(2) A and B are closed subsets of X with respect to the topology induced by d;
(3) the pare (f, g) is (A, B)-weakly increasing with $f(A) \subseteq B,\ g(B) \subseteq A$;
(4) ϕ and ψ are altering distance functions;
(5) there is $x_0 \in X$ such that $x_0 \preceq fx_0$;

(6)

$$\phi d(fx, gy) \leq \phi \max\{d(x, y), d(fx, x), d(gy, y)\} - \psi \max\{d(x, y), d(fx, x), d(gy, y)\} \quad (3.1)$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$, and

$$\phi d(gx, fy) \leq \phi \max\{d(x, y), d(gx, x), d(fy, y)\} - \psi \max\{d(x, y), d(gx, x), d(fy, y)\} \quad (3.2)$$

for all comparative $x, y \in X$ with $x \in B,\ y \in A$.

Proof. The proofs of (1), (2), (3), (4), and (5) are clear. We show (6).

Let $x \in A,\ y \in B$. Then we have the following cases:

Case (I): If $x \in \{0, 1, 2\}$ and $y \in \{0, 1\}$, then $fx = 0$ and $gy = 0$. Hence the left hand side of (3.1) is equal to 0 and so (3.1) is satisfied.

Case (II): If $x \geq 3$ and $y \geq 2$, then

Subcase (1): If $x - 3 = y - 1$, then $\phi d(fx, gy) = [d(x - 3, y - 1)]^2 = [0]^2 = 0$ and so (3.1) is satisfied.

Subcase (2): If $x - 3 \neq y - 1$, then

$$\phi d(fx, gy) = [d(x - 3, y - 1)]^2 = [x + 2y - 5]^2 = x^2 + 4y^2 + 25 + 4xy - 10x - 20y.$$

On the other hand

$$\begin{aligned}
&[\max\{d(x, y), d(fx, x), d(gy, y)\}]^2 - \max\{d(x, y), d(fx, x), d(gy, y)\} \\
&\quad = [\max\{d(x, y), d(x - 3, x), d(y - 1, y)\}]^2 - \max\{d(x, y), d(x - 3, x), d(y - 1, y)\} \\
&\quad = [\max\{x + 2y, 2x - 3, 2y - 1\}]^2 - \max\{x + 2y, 2x - 3, 2y - 1\}.
\end{aligned}$$

If $\max\{x + 2y, 2x - 3, 2y - 1\} = x + 2y$, then the right hand side is $x^2 + 4y^2 + 4xy - x - 2y$. Assume to the contrary that $x^2 + 4y^2 + 25 + 4xy - 10x - 20y > x^2 + 4y^2 + 4xy - x - 2y$. Then we have $9x + 18y < 25$ a contradiction since $x \geq 3$ and $y \geq 2$. Thus we have $x^2 + 4y^2 + 25 + 4xy - 10x - 20y \leq x^2 + 4y^2 + 4xy - x - 2y$. If $\max\{x + 2y, 2x - 3, 2y - 1\} = 2x - 3$ or $2y - 1$, then the result is clear since if $a, b \in \mathbb{N}$ with $a < b$, then $a^2 - a < b^2 - b$. Thus (3.1) is satisfied. In a similar manner we can show that (3.2) is satisfied. Hence all hypothesis of Theorem 3.1 hold true. Therefore f and g have a common fixed point in $A \cap B$. In this example the common fixed point of f and g in $A \cap B$ is 0. □

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